# EXAM SOLUTIONS

Let $k$ be an algebraically closed field.

**Problem 1** (Problem 1.1)**.** Let $\mathcal{F}$ be a sheaf on a topological space $X$. Suppose that for every $\mathfrak{p} \in X$, $\mathcal{F}_{\mathfrak{p}} = 0$. Show that $\mathcal{F} = 0$.

*Solution:* Let $W \subseteq X$ be an open set and take $t \in \mathcal{F}(W)$. For each $\mathfrak{p} \in W$, the image of $t$ $\mathcal{F}_{\mathfrak{p}}$ is 0, so there exists a neighbourhood $U_{\mathfrak{p}} \subseteq W$ of $\mathfrak{p}$ such that $t \mid_{U_{\mathfrak{p}}} = 0$. Since the $\{U_{\mathfrak{p}}\}_{\mathfrak{p} \in W}$ is an open cover of $W$, $t = 0$. □

**Problem 2** (Problem 2.2)**.** Show that every point in $\mathbb{P}^2_k$ is an algebraic set.

*Solution:* Let $\mathfrak{p} = [a_0 : a_1 : a_2]$. Assume without loss of generality that $a_0 \neq 0$. Let $I = (a_0x_1 - a_1x_0, a_0x_2 - a_1x_0)$. We claim that $\{\mathfrak{p}\} = \mathcal{Z}(I)$. Clearly, $\mathfrak{p} \in \mathcal{Z}(I)$. To see reverse inclusion, suppose $\mathfrak{q} = [b_0 : b_1 : b_2] \in \mathcal{Z}(I)$ This means that

$$a_0b_1 - a_1b_0 = 0 \qquad a_0b_2 - a_2b_0 = 0.$$

Set $\lambda = b_0/a_0$. The above relations then imply that

$$b_1 = \lambda a_1 \qquad b_2 = \lambda a_2.$$

Since $b_0 = \lambda a_0$, we conclude that $\mathfrak{p} = \mathfrak{q}$. □

**Problem 3** (Problem 3.4)**.** Let $V$ be an irreducible projective variety over $k$. Let $S$ be the singular points of $V$. Show that $S$ is closed.

*Solution:* Let $V \subseteq \mathbb{P}^n$, and suppose that $V = V^+(f_1, \ldots, f_m)$ with $f_i \in k[x_0, \ldots, x_n]$. A point $\mathfrak{p} \in V$ is singular if and only if the Jacobian matrix

$$M(\mathfrak{p}) = \left[\frac{\partial f_i}{\partial x_j}(\mathfrak{p})\right]$$

has rank less than $n - \dim V$. Now $M(\mathfrak{p})$ has rank less than $n - \dim V$ if and only if the $n - \dim V \times n - \dim V$ minors of $M(\mathfrak{p})$ vanish for every. Therefore, the singular locus is the vanishing locus of all $n - \dim V \times n - \dim V$ minors of the Jacobian matrix. Since the each minor is a polynomial, the Singular locus is thus algebraic. □

**Problem 4** (Problem 3.6)**.** Let $\mathcal{F}$ be a coherent sheaf over a variety $V$. Prove that (a) implies (b)

(a) Each stalk $\mathcal{F}_{\mathfrak{p}}$ is a free $\mathcal{O}_{V,\mathfrak{p}}$-module for every $\mathfrak{p} \in V$.
(b) There is an open cover $V = \bigcup U_i$ such that for each $i$,

$$\mathcal{F} \mid_{U_i} \cong \mathcal{O}_V^{n_i} \mid_{U_i} .$$

*Solution:* Take $\mathfrak{p} \in V$. We claim there exists a neighbourhood $U \subseteq V$ of $\mathfrak{p}$ such that $\mathcal{F} \mid_U \cong \mathcal{O}_V^n \mid_U$. Let $W \subseteq V$ be an affine open set with $\mathfrak{p} \in W$. Let $R = \mathcal{O}_V(W)$ and $M = \mathcal{F}(W)$. Note that $M$ is a finitely generated $R$-module since $\mathcal{F}$ is coherent. Now $\mathfrak{p} \in \operatorname{spec} R$ and $M_{\mathfrak{p}} = \mathcal{F}_{\mathfrak{p}}$, and $R_{\mathfrak{p}} = \mathcal{O}_{V,\mathfrak{p}}$.

Let $\frac{e_1}{f_1}, \ldots, \frac{e_n}{f_n}$ be an $R_{\mathfrak{p}}$ basis of $M_{\mathfrak{p}}$ with $e_i \in M$ and $f_i \in R \backslash \mathfrak{p}$. Set $t = f_1 \ldots f_n$. Now each $\frac{e_i}{f_i}$ exists in $M_t$, so we can define a map

$$\varphi \colon R_t^n \to M_t.$$

The map $\varphi_{\mathfrak{p}}$ is an isomorphism, and so $(\ker \varphi)_{\mathfrak{p}} = (\operatorname{coker} \varphi_{\mathfrak{p}}) = 0$. Since $\ker \varphi$ and $\operatorname{coker} \varphi$ are finitely generated $R_t$-modules, there exists an element $a \in R_t$ that annihilates these modules. In particular

$$\varphi_a \colon (R_t^n)_a \to (M_t)_a$$

is an isomorphism. Let $U = D(a) \subseteq \operatorname{spec} R_t \subseteq \operatorname{spec} R \subseteq V$. It follows that $\varphi_a$ induces an isomorphism between $\mathcal{O}_V \mid U \cong \mathcal{F} \mid U$. □

**Problem 5** (Problem 5.1)**.** Let $k$ be an algebraically closed field, and let $V \subseteq \mathbb{P}^d_k$ be a projective hypersurface. This means that $V = V^+(f)$ with homogenous $f \in k[x_0, \ldots, x_d]$ and the ideal $(f)$ is radical. Compute

$$H^i(\mathcal{O}_V(n))$$

for all $i, n \in \mathbb{Z}$. To make life easier, you may suppose that $d \geq 2$.

*Solution:* Let $t = \deg f$. Let $X = \mathbb{P}^d$ and let $S = k[x_0, \ldots, x_d]$. For every $n$, there is a short exact sequence of modules

$$0 \to S(-t) \xrightarrow{f} S \to S/fS \to 0.$$

This yields a short exact sequence of sheaves

$$0 \to \mathcal{O}_X(n-t) \xrightarrow{f} \mathcal{O}(n) \to \mathcal{O}_V \to 0.$$

Note that we are abusing notation: above we consider $\mathcal{O}_V = \widetilde{k[V]}$ as a sheaf over $X$. The sheaf cohomology is the same though if we consider it as a sheaf over $V$. We now look at the associated long exact sequence, and then look at the various cases.

For $i = 0$, since $H^0(\mathcal{O}_X(n)) = S_n$ and $H^1(\mathcal{O}_X(n)) = 0$ we have

$$0 \to S_{n-t} \xrightarrow{f} S_n \to H^0(\mathcal{O}_V(n)) \to 0$$

which tells us that

$$H^0(\mathcal{O}_V(n)) \cong S_n/fS_{n-t} = (S/fS)_n.$$

Next, recall that

$$H^d(\mathcal{O}_X(n)) = S^*_{-d-n-1} \qquad H^{d-1}(\mathcal{O}_X(n)) = H^{d+1}(\mathcal{O}_X(n)) = 0.$$

Applying this to the long exact sequence in cohomology

$$0 \to H^{d-1}(\mathcal{O}_V(n)) \to H^d(\mathcal{O}_X(n-t)) \xrightarrow{f} H^d(\mathcal{O}_X(n)) \to H^d(\mathcal{O}_V(n)) \to 0$$

yields

$$0 \to H^{d-1}(\mathcal{O}_V(n)) \to S^*_{-d-n+t-1} \xrightarrow{f} S^*_{-d-n-1} \to H^d(\mathcal{O}_V(n)) \to 0.$$

We now look at the $k$-dual of this sequence.

$$0 \to H^d(\mathcal{O}_V(n))^* \to S_{-d-n-1} \xrightarrow{f} S_{-d-n+t-1} \to H^{d-1}(\mathcal{O}_V(n))^* \to 0.$$

We conclude that

$$0 = H^d(\mathcal{O}_V(n))^* = H^d(\mathcal{O}_V(n)) \qquad H^{d-1}(\mathcal{O}_V(n))^* \cong (S/fS)_{-d-n+t-1}.$$

and so we conclude that

$$H^d(\mathcal{O}_V(n)) \cong (S/fS)^*_{-d-n+t-1}.$$

Lastly, $H^i(\mathcal{O}_V(n)) = 0$ for all $n$ and $i \neq 0, d-1$. □

*Alternative Solution:* Using the same notation as above, there is a short exact sequence of modules

$$0 \to S(-t) \xrightarrow{f} S \to S/fS \to 0.$$

Let $\mathfrak{m} = S_+$. This gives rise to a long exact sequence of local cohomology modules

$$\cdots \to H^i_{\mathfrak{m}}(S(-t)) \xrightarrow{f} H^i_{\mathfrak{m}}(S) \to H^i_{\mathfrak{m}}(S/fS) \to \cdots.$$

Recall that $H^i_{\mathfrak{m}}(S) = 0$ for all $i \neq d+1$ and

$$H^{d+1}_{\mathfrak{m}}(S)_n = S^*_{-d-n-1}$$

and so the same arguments as above, give us the that $H^i_{\mathfrak{m}}(S/f) = 0$ for all $i \neq d$ and

$$H^d_{\mathfrak{m}}(S/fS)_n = (S/fS)_{-d-n+t-1}.$$

Using the isomorphism

$$H^{i+1}_{\mathfrak{m}}(S/fS) \cong \bigoplus_{n \in \mathbb{Z}} H^i(\mathcal{O}_V(n))$$

for all $i \geq 1$.

All that remains is the $i = 0$ case. Here, recall that we have a short exact sequence

$$0 \to H^0_{\mathfrak{m}}(S/fS)_n \to (S/fS)_n \to \mathcal{O}_V(n)(V) \to H^1_{\mathfrak{m}}(S/fS)_n \to 0.$$

Since we shows that $H^0_{\mathfrak{m}}(S/fS) = H^1_{\mathfrak{m}}(S/fS) = 0$, the result is clear. □

**Problem 6** (Problem 6.3)**.** Let $V$ be a nonsingular projective curve over $\mathbb{C}$. Suppose that every invertible sheaf over $V$ is isomorphic to $\mathcal{O}_V(n)$ for some $n \in \mathbb{Z}$. Show that $V$ is birational to $\mathbb{P}^1_{\mathbb{C}}$.

*Solution:* We claim that there are to points $\mathfrak{p}, \mathfrak{q} \in V$ with $\mathfrak{p} \neq \mathfrak{q}$ and $\mathfrak{p} \sim \mathfrak{q}$. Given points $\mathfrak{p} \sim \mathfrak{q}$. Using the fact that $\mathbb{C}$ is uncountable, we know that $V$ is also uncountable. By assumption, there are only countable isomorphism classes of invertible sheaves. Thus there exist distinct $\mathfrak{p}, \mathfrak{q} \in V$, such that $\mathcal{L}(\mathfrak{p}) \cong \mathcal{L}(\mathfrak{q})$. Since the Picard group and the class group of $V$ are isomorphic, we conclude that $\mathfrak{p} \sim \mathfrak{q}$. □