1 Commutative algebra

Theorem 1.1: Problem 15

If $\mu_i(x_i) = 0$, there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$.

Proof. Suppose $\mu_i(x_i) = 0$ for some $i \in I$. Then x_i is an element of $M_i \cap D$ and hence is a finite sum of generators of D in M_i, i.e.

$$x_i = \sum_{s=1}^{k} x_{(s)} - \mu_{i_s j_s}(x_{(s)}). \tag{1.1}$$

Here we have $i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k \in I$ where $i_r \leq j_r$ for $r \in 1, \ldots, k$ and $x_{(1)} \in M_{i_1}, x_{(2)} \in M_{i_2}, \ldots, x_{(k)} \in M_{i_k}$.
Since all our i's and j's make a finite subset of I, call this J, we can find another element $t \in I$ such that $i \leq t$ for all $i \in J \cup i$.
Now, for all elements $l \in I$ we define π_l to be the projection homomorphism, i.e.

$$\pi_l(x_i) = \begin{cases} x_i, & \text{if } l = i \\ 0, & \text{else.} \end{cases} \tag{1.2}$$

By the first part of the problem, we have

$$\pi_l(x_i) = \sum_{i_b = l} x_{(b)} - \sum_{j_c = l} \mu_{i_b j_c}(x_{(c)}). \tag{1.3}$$

We can now apply the function μ_{lt}, and we get

$$\mu_{lt}(\pi_l(x_i)) = \sum_{i_b = l} \mu_{i_b t} x_{(b)} - \sum_{j_c = l} \mu_{j_c t} \mu_{i_b j_c}(x_{(c)}). \tag{1.4}$$

Now, summing over all of the l's we get

$$\sum_{l \in I} \mu_{lt}(\pi_l(x_i)) = \sum_{s=1}^{k} \mu_{i_s t} x_{(s)} - \sum_{s=1}^{k} \mu_{j_s t} \mu_{i_s j_s}(x_{(s)}) = 0. \tag{1.5}$$

Hence, since the left side is equal to just $\mu_{it}(\pi_i(x_i)) = \mu_{it}(x_i)$ we get our final result,

$$\mu_{it}(x_i) = 0. \tag{1.6}$$

□