We then have:

$$\begin{tikzcd} F \arrow[r, "\varphi"] \arrow[d, "\alpha"', "\cong"] & M \arrow[d, hook, "\beta"] \\ \hat{F} \arrow[r, two heads, "\hat{\varphi}"'] & \hat{M} \end{tikzcd}$$

- α is iso as F is free A-mod and $A \cong \hat{A}$.
- As $\bigcap_{n \geq 1} M_n = 0$, we get β injective.
- surj of $\hat{\varphi} \Rightarrow \beta$, and hence also φ, are surjective.

Thus $x_1, \dots, x_r$ generate M as A-module. ∎

Cor 10.25. In the setup of 10.24, if $G(M)$ is Noetherian as a $G(A)$-mod, then M is Noeth. A-mod.

Pf: Let $M' \subseteq M$ be submod.

Let $M_n' = M' \cap M_n$, so (M_n') is $\mathfrak{a}$-filt. of M'

and $M_n'/M_{n+1}' \hookrightarrow M_n/M_{n+1}$ inj hence

we have an embedding of $G(M')$ into $G(M)$.

As $G(M)$ Noeth (by assumption), $G(M')$ is f.g.

Also, $\bigcap M_n' \subseteq \bigcap M_n = 0$

So $(10.24) \Rightarrow M'$ is f.g. A-mod

$\Rightarrow M$ is Noeth A-mod. ∎

Theorem 10.26

If A is Noetherian ring, $\mathfrak{a} \subseteq A$, then $\mathfrak{a}$-adic completion $\hat{A}$ of A is Noeth.

Pf:

By (10.22), $G_\mathfrak{a}(A)$ is Noetherian and $G_\mathfrak{a}(A) \cong G_{\hat{\mathfrak{a}}}(\hat{A})$ is Noetherian.

By (10.25), taking $M = \hat{A}$ (filtered by $\hat{\mathfrak{a}}^n$ hence $\bigcap \hat{\mathfrak{a}}^n = 0$)

we get $\hat{A}$ is Noetherian (as a module over itself, hence as a ring.). ∎

Cor 10.27:

If A is Noetherian ring, then $A[[x_1, \dots, x_n]]$ is Noetherian.

Pf: $A[x_1, \dots, x_n]$ is Noetherian by HBT, and $A[[x_1, \dots, x_n]]$ is the completion of $A[x_1, \dots, x_n]$ for the $(x_1, \dots, x_n)$-adic topology (exercise). ∎