MA8109 Stochastic processes and differential equations

Fall 2015 – Synopsis

This note is intended to provide a *synopsis* of the course: What has been covered, basic definitions and important results, etc.

The note will keep on growing as the lectures move ahead. Ideally, a new version will be posted every week.

Notation

Here I summarize some notation used throughout.

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are the sets of natural numbers (starting at 1), integers, rational numbers, real numbers, and complex numbers respectively. Also, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, and $\overline{\mathbb{R}} = [-\infty, \infty]$.
- I write $\underline{\lim}$ and $\overline{\lim}$ instead of the more common $\liminf$ and $\limsup$.
- $\mathcal{B}$ is the σ-algebra of *Borel sets* on $\mathbb{R}$ or $\overline{\mathbb{R}}$ (depending on context).
- I use := to mean "is defined as", and =: if the term being defined is on the right.
- A^{c} is the *complement* $\Omega \setminus A$. The "universal" set Ω needs to be understood.
- $A \sqcup B$ is the union $A \cup B$ of two *disjoint* sets A and B.
- $\bigsqcup_{n=1}^{\infty} A_n$ is the union of a sequence of *pairwise disjoint* sets.
- Y^X, where X and Y are sets, is the set of functions $X \to Y$.
- As a special case, $Y^{\mathbb{N}}$ is the set of all sequences $(y_1, y_2, \ldots)$ in Y.
- $[S]$ equals 1 if the statement S is true, 0 otherwise (*indicator bracket*).
- $[A]$ is the *indicator function* of the set A, defined by $[A](x) = [x \in A]$.
- If $a \in \overline{\mathbb{R}}$ we write $a^+ := \max(a, 0)$ and $a^- := (-a)^+ = -\min(a, 0)$.
  Then $a^\pm \geq 0$, $a^+ a^- = 0$, $a = a^+ - a^-$, and $|a| = a^+ + a^-$.
- If f is a function, define $f^\pm$ by $f^\pm(x) = f(x)^\pm$.

First week (W34)

A recurring example is **coin tossing space** $\Omega = \{0,1\}^{\mathbb{N}}$, consisting of all infinite sequences of zeroes and ones, representing coin tosses (zero for tails, one for head) if you wish.

An *algebra on* Ω, (or perhaps more precisely, an algebra of subsets of Ω) is a set $\mathcal{A}$ of subsets of Ω so that

- $\emptyset \in \mathcal{A}$
- $A \in \mathcal{A}$ implies $A^{\mathsf{c}} \in \mathcal{A}$
- $A, B \in \mathcal{A}$ implies $A \cup B \in sA$

For each $n \in \mathbb{N}$, there is an algebra $\mathcal{F}_n$ of subsets of Ω, defined as the events *determined by* $(\omega_1, \dots, \omega_n)$: Thus $A \in \mathcal{F}$ if and only if whenever $\omega \in A$ and $\omega' \in \Omega$ $\omega_k = \omega'_k$ for $k = 1, \dots, n$ implies $\omega' \in A$. Or put differently, if $\pi_n \colon \Omega \to \{0,1\}^n$ is the projection map onto the first n coordinates, the members of $\mathcal{F}_n$ are the inverse images of sets $B \subseteq \{0,1\}^n$. Thus $\mathcal{F}_n$ has 2^{2^n} members.

If we think of independent coin tosses with an unbiased coin, elementary probability theory dictates a probability $P(\pi_n^{-1}(B)) = 2^{-n}\#B$ when $B \subseteq \{0,1\}^n$ (here $\#B$ is the number of members of B).

The algebras $\mathcal{F}_n$ form an increasing sequence of algebras, and so their union

$$\mathcal{F}_* := \bigcup_{n=1}^{\infty} \mathcal{F}_n$$

is an algebra too: It consists of all *finitely determined* events.

The *strong law of large numbers* implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \omega_k = \frac{1}{2} \quad \text{a. s.},$$

where “a. s.” stands for “*almost surely*”, meaning “with probability 1”.

Note that we are unable to even give this statement a precise meaning within our current framework so far, since it is a statement regarding an event not in $\mathcal{F}_*$ (worse, it is utterly independent of *any finite* number of cointosses ω_k).

Our next task is to remedy this.

σ-algebras and measures

1 Definition. A *σ-algebra on* Ω (or perhaps more precisely, a σ-algebra of subsets of Ω) is a set $\mathcal{F}$ of subsets of Ω so that
- $\emptyset \in \mathcal{F}$
- $A \in \mathcal{F}$ implies $A^{\mathsf{c}} \in \mathcal{F}$
- $A_n \in \mathcal{F}$ for $n = 1, 2, \ldots$ implies $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$

Because any intersection of σ-algebras is itself a σ-algebra, there exists a smallest σ-algebra $\mathcal{F} := \sigma(\mathcal{F}_*)$ containing $\mathcal{F}_*$, called the σ-algebra *generated by* $\mathcal{F}_*$.

We want to extend P to a *probability measure* on $\mathcal{F}$.

2 Definition. A *measure* on $\mathcal{F}$ is a map $\mu \colon \mathcal{F} \to [0, \infty]$ satisfying
– $\mu(\emptyset) = 0$
– $A_n \in \mathcal{F}$ pairwise disjoint for $n \in \mathbb{N}$ implies $\mu\Big(\bigsqcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n)$

3 Definition.
– A *measurable space* is a pair $(\Omega, \mathcal{F})$ where Ω is a set and $\mathcal{F}$ a σ-algebra on Ω.
– A *measure space* is a triple $(\Omega, \mathcal{F}, \mu)$ where $(\Omega, \mathcal{F})$ is a measurable space and μ a measure on $\mathcal{F}$.
– A *probability space* is a measure space $(\Omega, \mathcal{F},)$ where P is a probability measure.

4 Definition. A *monotone class* is a set $\mathcal{M}$ of subsets of Ω satisfying
– If $A_n \in \mathcal{M}$ and $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$,
– If $A_n \in \mathcal{M}$ and $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$.

5 Lemma (Monotone Class Lemma) *If $\mathcal{A}$ is an algebra on Ω and $\mathcal{M}$ is a montone class with $\mathcal{A} \subseteq \mathcal{M}$, then $\mathcal{A} \subseteq \sigma(\mathcal{M})$.*

From this we get

6 Theorem (Uniqueness of extension) *Let $\mathcal{A}$ be an algebra. Any two finite measures which agree on all members of $\mathcal{A}$, also agree on all members of $\sigma(\mathcal{A})$.*

Returning to cointossing space $(\Omega, \mathcal{F})$ with $\Omega = \{0, 1\}^{\mathbb{N}}$, we conclude that there cannot be more than one probability measure on this space extending the function P defined previously on $\mathcal{F}_*$.

That there in fact *exists* such a measure is non-trivial, but true. Thanks to the uniqueness theorem, we do not need worry too much about which of several possible methods of construction we use; they must all produce the same measure.

Lebesgue measure

This is another measure of great importance. It is defined on the σ-algebra $\mathcal{B}$ of *Borel subsets* of $\mathbb{R}$, which is the σ-algebra generated by the set of intervals (or equivalently, open intervals – or closed intervals – or half open intervals $(a, b]$ – or open sets – or closed sets – or ...). We shall write λ for Lebesgue measure. It is the unique Borel measure (meaning a measure on $\mathcal{B}$) so that $\lambda((a, b]) = b - a$ for all $a \leq b$. (These do not form an algebra, so the uniqueness theorem does not apply directly – but the set of all finite unions of such integrals does, if we also include intervals of the form $(-\infty, a]$ and (a, ∞).)

7 Definition. A *measurable function* on a measurable space $(\Omega, \mathcal{F})$ is a function $f\colon \Omega \to \overline{\mathbb{R}}$ so that $f^{-1}(-\infty, a[)-\infty, a] \in \mathcal{F}$ for all $a \in \overline{\mathbb{R}}$. (Then $f^{-1}(B) \in \mathcal{F}$ for all Borel sets B, because the sets B satisfying the condition is a σ-algebra.)

A *random variable* (R. V.) on a probability space $(\Omega, \mathcal{F}, P)$ is a measurable function on $(\Omega, \mathcal{F})$. (We usually use uppercase letters such as X for random variables.)

8 Lemma *If a sequence of measurable functions converges pointwise to some limit, then the limit is measurable.*

We can now define a random variable U on coin tossing space:

$$U(\omega) = \sum_{n=1}^{\infty} \omega_n 2^{-n}.$$

Think of it as using the coin tosses ω_n as the digits in the binary expansion of $U(\omega) \in [0,1]$.

We write $P(U \le u)$ as shorthand notation for $P(\{\omega \in \Omega\colon U(\omega) \le u\})$.

It turns out that $P(U \le u) = u$ for all $u \in [0,1]$. (Easily proved for *dyadic rational* u, that is, $u = m/2^k$ for integers m, k; then it follows for all u, beacuse $P(U \le u) = u$ is a monotone function of u.) In other words, U is *uniformly distributed* on the interval $[0,1]$. We shall call such a random variable a *standard uniform variable.* From it, we can build random variables of any desired distribution.

9 Definition. The *distribution* of a random variable X on $(\Omega, \mathcal{F}, P)$ is the Borel measure μ_X given by

$$\mu_x(B) = P(X \in B) = P(X^{-1}(B)).$$

It is uniquely determined by the *cumulative distribution function*

$$F_X(x) = \mu_X([-\infty, x]) = P(X \le x).$$

In particular, the distribution of a standard uniform variable U is Lebesgue measure on $[0,1]$:

$$\mu_U(B) = \lambda(B \cap [0,1]) \qquad (B \in \mathcal{B}).$$

Integration

We define the integral for certain *measurable* functions f on a measure space $(\Omega, \mathcal{F}, \mu)$:

10 Definition. A *simple function* is a measurable function which takes only a finite number of values. Such a function can be written

$$\varphi = \sum_{k=1}^{n} a_k\,[A_k]$$

with $a_k \in \overline{\mathbb{R}}$ and $A_k \in \mathcal{F}$. We can always choose the a_k to be distinct and nonzero and the A_k to be nonempty and mutually disjoint. This may be called the *canonical* representation of φ. It is unique up to permutation of the indices. For a non-canonical representation, we must take care not to subtract infinities. So we disallow $a_j = -\infty$, $a_k = \infty$ and $A_j \cap A_k \neq \emptyset$.

11 Definition. The *integral of a simple function* such as above is

$$\int_\Omega \varphi\,d\mu = \sum_{k=1}^{n} a_k \mu(A_k).$$

Here and elsewhere we use the convention that $0 \cdot (\pm\infty) = 0$. If the sum contains terms equal to both $-\infty$ and $+\infty$, we do not define the integral of φ. Note that the integral of a *nonnegative* simple function is *always* defined. Its value may be ∞.

12 Definition. The *integral of a nonnegative measurable function* f is

$$\int_\Omega f\,d\mu = \sup\Bigl\{\int_\Omega \varphi\,d\mu \colon \varphi \text{ is simple and } 0 \le \varphi \le f\Bigr\}.$$

13 Theorem (The Monotone Convergence Theorem (MCT)) *If f_n is a measurable function and $0 \le f_n \le f_{n+1}$ for $n \in \mathbb{N}$ then*

$$\int_\Omega \lim_{n\to\infty} f_n\,d\mu = \lim_{n\to\infty} \int_\Omega f_n\,d\mu.$$

(Note that both limits exist by monotonicity, and the limit function on the left is measurable.)

14 Theorem (Fatou's lemma) *If $f_n \ge 0$ is measurable for all $n \in \mathbb{N}$ then*

$$\int_\Omega \underline{\lim}_{n\to\infty} f_n\,d\mu \le \underline{\lim}_{n\to\infty} \int_\Omega f_n\,d\mu.$$

After showing that any nonnegative measurable function is a pointwise limit of a non-decreasing sequence of nonnegative simple function, we have no difficulty using MCT to show that the integral is *additive*, and in the end, we get an integral that is linear, given by

15 Definition. The *integral of a measurable function f* is defined to be

$$\int_\Omega f\,d\mu = \int_\Omega f^+\,d\mu - \int_\Omega f^-\,d\mu.$$

If both integrals on the right have infinite value, we do not define the integral. If they are both finite, we call f *integrable*.

16 Theorem (The Dominated Convergence Theorem (DCT)) *If f_n is measurable and $|f_n| \le g$ for all $n \in \mathbb{N}$ where g is integrable, and if the sequence converges pointwise, then*

$$\int_\Omega \lim_{n\to\infty} f_n\,d\mu = \lim_{n\to\infty} \int_\Omega f_n\,d\mu.$$

The Riemann integral (or the Darboux integral – the two are equivalent, even though they are constucted in slightly different ways) is the integral you learned in basic calculus based on Riemann sums. Any Riemann integrable function is Lebesgue integrable, and the Riemann integral equals the Lebesgue integral. (I am sure you are much relieved.) But it is trivial to find Lebesgue integrable functions which are not Riemann integrable: The indicator function $[\mathbb{Q}]$ of the rational numbers is one example. Note that $\mathbb{Q}$ is countable, and so $\lambda(\mathbb{Q}) = 0$, since the Lebesgue measure of any singleton set is zero. But $[\mathbb{Q}]$ is discontinuous everywhere, whereas Riemann integrable functions are continuous *almost everywhere* (i.e., except on a set of measure zero).