MA8109 Stochastic Prosesses and Differential Equations 2015-12-07

Solution

Problem 1

a. Since $\mathsf{E}(B_t) = 0$ for all t, we clearly get $\mathsf{E}(A_t) = 0$ as well. And since $\mathsf{E}(B_s B_t) = s \wedge t$, we find $\mathsf{E}(A_s A_t) = \mathsf{E}(B_s B_t) - s\,\mathsf{E}(B_1 B_t) - t\,\mathsf{E}(B_s B_1) + st\,\mathsf{E}(B_1 B_1) = s \wedge t - st - ts + st = s \wedge t - st$. Setting $s = t$ we get $\mathsf{E}(A_t^2) = t - t^2$ as claimed.

We could further rewrite the answer as $\mathsf{E}(A_s A_t) = \big(s \wedge t\big)\big(1 - s \vee t\big) = \big(s \wedge t\big)\big((1-s) \wedge (1-t)\big)$, which makes the symmetry between A_t and and A_{1-t} apparent.

b. Clearly, with the new definition $\mathsf{E}(A_t) = 0$. And since $t/(t-1)$ is a strictly increasing function of t, we get – assuming $s \le t$ for simplicity

$$\mathsf{E}(A_s A_t) = (1-s)(1-t)\,\mathsf{E}(B_{s/(1-s)} B_{t/(1-t)}) = (1-s)(1-t)\Big(\frac{s}{1-s} \wedge \frac{t}{1-t}\Big)$$
$$= (1-s)(1-t)\frac{s}{1-s} = s(1-t) = s - st = s \wedge t - st$$

Interchanging s and t yields the same result if $s \ge t$.

The distibution of variables within a Gaussian family is uniquely determined by expectations and covariances, so we conclude that (A_t) is a Brownian bridge.

For the converse, assume $(A_s)_{s\in[0,1]}$ is a Brownian bridge, and put $B_t = (1-s)^{-1} A_s$ where $t = s/(1-s)$. Solving for s we find $s = t/(1+t)$ and $(1-s)^{-1} = t + 1$, so we end up with the definition

$$B_t = (1+t) A_{t/(1+t)} \qquad t \ge 0.$$

Again, $\mathsf{E}(B_t) = 0$ follows from $\mathsf{E}(A_s) = 0$, and so – once more assuming $s \le t$ for simplicity –

$$\mathsf{E}(B_s B_t) = (1+s)(1+t)\,\mathsf{E}(A_{s/(1+s)} A_{t/(1+t)}) = (1+s)(1+t)\Big(\frac{s}{1+s} \wedge \frac{t}{1+t} - \frac{s}{1+s}\frac{t}{1+t}\Big)$$
$$= (1+s)(1+t)\Big(\frac{s}{1+s} - \frac{s}{1+s}\frac{t}{1+t}\Big) = s = s \wedge t.$$

It follows that (B_t) is a Brownian motion.

Problem 2

a. A martingale with respect to some filtration $(\mathcal{M}_t)$ is an $(\mathcal{M}_t)$-adapted process (M_t) with $\mathsf{E}(|M_t|) < \infty$ satisfying $\mathsf{E}(M_t \mid \mathcal{M}_s) = M_s$ when $s \le t$.

The martingale representation theorem states that any L^2 martingale (M_t) (i.e., $\mathsf{E}(M_t^2) < \infty$) with respect to the filtration $(\mathcal{F}_t)$ given by n-dimensional Brownian motion (B_t) is given by an Itô integral:

$$M_t = M_0 + \int_0^t X_s \, dB_s \tag{1}$$

for a unique $(\mathcal{F}_t)$-adapted process (X_t).

The converse follows from the basic theory of Itô integrals. Uniqueness follows directly from the Itô isometry.

b. A stopping time τ with respect to a filtration $(\mathcal{F}_t)$ is a random variable τ so that the event $\{\tau \le t\}$ is measurable for all t.

In colloquial terms, it will be known at time t whether $\tau \le t$ ("are we there yet?") has occured.

Now, for the stopped martingale $(M_{\tau\wedge t})$, we use the representation (1):

$$M_{\tau\wedge t} = M_0 + \int_0^{\tau\wedge t} X_s \, dB_s = M_0 + \int_0^t [s \le \tau] X_s \, dB_s,$$

where $[s \le \tau]$ equals 1 if $s \le \tau$, and 0 otherwise. Since the process $([t \le \tau] X_t)$ is clearly $(\mathcal{F}_t)$-adapted, the integral is a martingale, and we are done.

c. From the previous question, we get $\mathsf{E}(M_{\tau\wedge t}) = \mathsf{E}\big(\mathsf{E}(M_{\tau\wedge t}\,|\,\mathcal{F}_0)\big) = \mathsf{E}(M_{\tau\wedge 0}) = \mathsf{E}(M_0)$. Whenever $\tau < \infty$, we clearly have $M_{\tau\wedge n} \to M_\tau$ as $n \to \infty$, and so $M_{\tau\wedge n} \to M_\tau$ a. s. Thanks to the boundedness assumption on $M_{\tau\wedge n}$, we can now employ the dominated convergence theorem to conclude that $\mathsf{E}(M_\tau) = 0$.

d. Following the hint, we assume that $\mathsf{E}(\tau) < \infty$.

Clearly also $B_{\tau\wedge t} \le 1$, so $|M^\alpha_{\tau\wedge t}| \le e^\alpha$, and we may conclude from point **c** that $\mathsf{E}(M^\alpha_\tau) = E(M^\alpha_0) = 1$.

Since paths are continuous, $B_\tau = 1$, because the only way to exit $(-\infty, 1)$ is by passing through 1. Thus $M^\alpha_\tau = e^{\alpha-\alpha^2\tau/2}$, so we have in fact $\mathsf{E}(e^{\alpha-\alpha^2\tau/2}) = 1$. Now we estimate:

$$1 = \mathsf{E}(e^{\alpha-\alpha^2\tau/2}) \ge \mathsf{E}(1+\alpha-\alpha^2\tau/2) = 1+\alpha-\tfrac{1}{2}\alpha^2\,\mathsf{E}(\tau),$$

which implies $\alpha\,\mathsf{E}(\tau) \ge 2$ for any $\alpha > 0$. This contradicts the assumption $\mathsf{E}(\tau) < \infty$.

Problem 3

a. Formally, the given equation can be written as

$$\frac{d^2X_t}{dt^2} = -X_t + \frac{dB_t}{dt},$$

where the final term is a white noise term, which here plays the role of a random force acting on the mass.

Itô's formula yields

$$dU_t = X_t\,dX_t + Y_t\,dY_t + \tfrac{1}{2}\big((dX_t)^2 + (dY_t)^2\big) = X_tY_t\,dt - Y_tX_t\,dt + Y_t\,dB_t + \tfrac{1}{2}\big(0+dt\big) = Y_t\,dB_t + \frac{1}{2}\,dt.$$

Written on integral form:

$$U_t = U_0 + \frac{t}{2} + \int_0^t Y_s\,dB_s.$$

Since the expectation of an Itô integral is zero, we conclude $\mathsf{E}(U_t) = \mathsf{E}(U_0) + \frac{t}{2} = \frac{1}{2}(x^2+y^2+t)$.

b. We rewrite the equations as

$$dZ_t = HZ_t\,dt + \sigma\,dB_t, \qquad Z_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Using Itô's formula we get $d(e^{-tH}Z_t) = e^{-tH}(dZ_t - HZ_t\,dt) = e^{-tH}\sigma\,dB_t$, and so

$$e^{-tH}Z_t = Z_0 + \int_0^t e^{-sH}\sigma\,dB_s,$$

and therefore

$$Z_t = e^{tH}(x,y)^{\mathsf{T}} + \int_0^t e^{(t-s)H}\sigma\,dB_s. \tag{2}$$

Since the integrand in the Itô integral is deterministic, we could certainly do a partial integration to turn the integral into an ordinary integral with a stochastic integrand instead, but this seems unnecessary for our purposes.

We focus on the matrix exponential (after verifying that $H^2 = -I$):

$$\begin{aligned} e^{tH} &= \sum_{n=0}^\infty \frac{(tH)^n}{n!} = \sum_{k=0}^\infty \frac{(tH)^{2k}}{(2k)!} + \sum_{k=0}^\infty \frac{(tH)^{2k+1}}{(2k+1)!} = \sum_{k=0}^\infty \frac{(-1)^k t^{2k}}{(2k)!} I + \sum_{k=0}^\infty \frac{(-1)^k t^{2k+1}}{(2k+1)!} H \\ &= (\cos t)I + (\sin t)H = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \end{aligned}$$

On component form we can then write the solution as

$$X_t = x\cos t + y\sin t + \int_0^t \sin(t-s)\,dB_s,$$
$$Y_t = -x\sin t + y\cos t + \int_0^t \cos(t-s)\,dB_s.$$

To compute $\mathsf{E}(U_t)$, it may be better to stay with the vector formulation, noting that $U_t = \frac{1}{2}|Z_t|^2 = \frac{1}{2}Z_t^{\mathsf{T}}Z_t$. Plugging in (2), we note that when multiplying out the square, the cross term is another Itô integral, so it has expectation zero. Thus

$$\mathsf{E}(U_t) = \tfrac{1}{2}|e^{tH}(x,y)^{\mathsf{T}}|^2 + \tfrac{1}{2}\mathsf{E}\Bigl(\Bigl(\int_0^t e^{(t-s)H}\sigma\,dB_s\Bigr)^2\Bigr)$$

The first term is just $\frac{1}{2}|(x,y)^{\mathsf{T}}| = \frac{1}{2}(x^2+y^2)$, while the other term can be evaluated using the Itô isometry. The absolute value of the integrand turns out to be 1, so in the end we have

$$\mathsf{E}(U_t) = \frac{1}{2}(x^2+y^2+t).$$

c. The infinitesimal generator is

$$A = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \frac{1}{2}\frac{\partial^2}{\partial y^2}.$$

Since $u(t,x,y) = \mathsf{E}^{(x,y)}(g(X_t,Y_t))$ where $g(x,y) = \frac{1}{2}(x^2+u^2)$, this function should satify $\partial u/\partial t = Au$. With $u = \frac{1}{2}(x^2+y^2+t)$, we find $\partial u/\partial t = \frac{1}{2}$, while $Au = yx - xy + \frac{1}{2} = \frac{1}{2}$.