Examination paper for
MA8109 Stochastic Processes and Differential Equations

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**Examination date:** December 7, 2015
**Examination time (from–to):** 09:00–13:00
**Permitted examination support material:** C: Approved simple calculator. English dictionaries or bilingual dictionaries to/from English are allowed. No further written or printed support materials are allowed.

**Other information:**
**Please note:** This exam is given in English only.
Answers are accepted in English or a Scandinavian language.
*There is a list of useful formulas at the end of this exam.*
It is OK to use a result stated earlier in a problem, even if you have not managed to prove it.
The problems are not necessarily given in increasing order of difficulty.

**Language:** English
**Number of pages:** 2
**Number of pages enclosed:** 1

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Date Signature
Problem 1

a. Let $(B_t)_{t \geq 0}$ be a standard Brownian motion, and define $A_t = B_t - tB_1$ for $0 \leq t \leq 1$. Compute the expectations $E(A_t)$ and $E(A_sA_t)$, and verify in particular that $E(A_t^2) = t - t^2$.

b. A stochastic process defined for $t \in [0, 1]$ with continuous paths and the same probability law as the process $(A_t)$ above, is called a Brownian bridge. Verify that one can also construct a Brownian bridge by setting $A_1 = 0$ and

$$A_t = (1 - t)B_t/(1-t), \quad 0 \leq t < 1.$$ 

Conversely, use this to show how Brownian motion can be constructed from a Brownian bridge. (Hint: $t/(1-t)$ is a strictly increasing function of $t$ for $0 \leq t < 1$.)

Problem 2

a. What is a martingale? State the martingale representation theorem.

b. What is a stopping time?

Let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration associated with $n$-dimensional Brownian motion. Let $(M_t)_{t \geq 0}$ be a martingale and $\tau$ a stopping time, both with respect to $(\mathcal{F}_t)$. Assuming that $E(M_t^2) < \infty$ for all $t \geq 0$, show that the stopped process $(M_{\tau \wedge t})_{t \geq 0}$ is a martingale.

(By definition, $\tau \wedge t = \min(\tau, t)$. The martingale representation theorem will be useful.)

c. With the same assumptions as in b, assume further that $M_{\tau \wedge t}$ is bounded, i.e., there exists some constant $C$ so that $|M_{\tau \wedge t}| \leq C$ a.s. (almost surely) for all $t$. Assume also that $\tau < \infty$ a.s. Show that then $E(M_\tau) = E(M_0)$.

(Hint: Consider $M_{\tau \wedge n}$ and let $n \to \infty$.)

d. Now let $(B_t)_{t \geq 0}$ be one-dimensional Brownian motion, and let $\tau$ be the first exit time from $(-\infty, 1)$. It follows from the law of iterated logarithms that $\tau < \infty$ a.s. Put $M_t^\alpha = e^{\alpha B_t - \alpha^2 t/2}$, where $\alpha > 0$. Then $(M_t^\alpha)_{t \geq 0}$ is a martingale. You do not need to prove the above assertions.

Show that $E(\tau) = \infty$. (Hint: What is $M_\tau^\alpha$? Assume that $E(\tau) < \infty$. Use the inequality $e^x \geq 1 + x$ and get a contradiction when $\alpha$ is sufficiently small.)
Problem 3

a. The Itô diffusion

\[ \begin{align*}
    dX_t &= Y_t \, dt \\
    dY_t &= -X_t \, dt + dB_t
\end{align*} \] (1)

represents the motion of a harmonic oscillator, \( X_t \) being the position of a mass on the end of a spring, and \( Y_t \) being the velocity of the mass. (Clearly, some rescaling of \( X, Y, \) and \( t \) has been done, whereby the mass and the spring stiffness have been normalized to 1.) What is the physical interpretation of the noise term \( dB_t? \)

The quantity \( U_t = \frac{1}{2}(X_t^2 + Y_t^2) \) represents the energy of the mass–spring system. Find an expression for \( dU_t \), and use it to compute \( E(U_t) \) given initial data \( X_0 = x, Y_0 = y. \)

b. Writing \( Z_t = (X_t, Y_t)^T \), we can write (1) as

\[ dZ_t = HZ_t \, dt + \sigma dB_t \]

for a \( 2 \times 2 \) matrix \( H \) and column vector \( \sigma \). Solve this given the initial condition \( Z_0 = (x, y)^T \). (Hint: Differentiate \( e^{-tH}Z_t \). Note that \( H^2 = -I \). To compute \( e^{tH} \), split the power series into even and odd powers.)

Compute \( E(U_t) \) explicitly using this solution, thus verifying your result from point a.

c. What is the infinitesimal generator for the given Itô diffusion? Use it to determine a partial differential equation satisfied by \( u(t, x, y) = E^{(x,y)}(U_t) \), where \( E^{(x,y)} \) means expectation assuming \( (X_0, Y_0) = (x, y). \)

Verify that the explicit answer from point b does indeed satisfy the equation.
List of useful formulas

Note: The list does not state the requirements for the formulas to be valid.

1D Gaussian variable: \( X \in \mathcal{N}(\mu, \sigma^2) \);

(i) \( \mathbb{E}\left((X - \mu)^4\right) = 3\sigma^4 \),

(ii) \( \Phi_X(u) = e^{in-\sigma^2u^2/2} \),

(iii) \( f_X(x) = \frac{1}{\sqrt{2\pi\sigma}}e^{(x-\mu)^2/(2\sigma)} \).

Conditional Expectations:

(i) If \( Y \) is \( \mathcal{H} \)-measurable, then \( \mathbb{E}(Y X | \mathcal{H}) = Y \mathbb{E}(X | \mathcal{H}) \).

(ii) If \( X \) is independent of \( \mathcal{H} \), then \( \mathbb{E}(X | \mathcal{H}) = \mathbb{E}(X) \).

(iii) If \( \mathcal{G} \subset \mathcal{H} \), then \( \mathbb{E}(\mathbb{E}(X | \mathcal{H}) | \mathcal{G}) = \mathbb{E}(X | \mathcal{G}) \).

Itô Isometry: \( \mathbb{E}\left|\int_0^T f(t, \omega) dB_t(\omega)\right|^2 = \int_0^T \mathbb{E}|f(t, \omega)|^2 dt = \|f\|_{L^2(\Omega \times [0,T])}^2 \).

2D Itô Formula: The “Rules” and
\[
dg(t, X_t, Y_t) = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} dX_t + \frac{\partial g}{\partial y} dY_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} (dX_t)^2 + \frac{\partial^2 g}{\partial x \partial y} dX_t dY_t + \frac{1}{2} \frac{\partial^2 g}{\partial y^2} (dY_t)^2.
\]

The Generator for \( dX_t = \beta_t(X_t) dt + \sigma(X_t) dB_t \):

\[
A(f)(x) = \sum_{i=1}^n \beta_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^n \text{tr}\left(\sigma(x)\sigma(x)^T D^2 f(x)\right), \quad \left(D^2 f(x)\right)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(x).
\]

Dynkin’s formula: \( \mathbb{E}^x\left(f(X_T)\right) = f(x) + \mathbb{E}^x\left(\int_0^T A f(X_s) ds\right) \).

Potential Solutions: \( \nabla^2 f = 0 \) for all \( x \in \mathbb{R}^n, \ |x| \neq 0 \):

- \( n = 2 \): \( f(x) = \log|x| \),
- \( n > 2 \): \( f(x) = |x|^{2-n} \).

Grönwall’s inequality: If \( v(t) \leq C + A \int_0^t v(s) ds \ldots \), then \( v(t) \leq Ce^{At} \).