NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for
MA8109 Stochastic Prosesses and Differential Equations

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Examination time (from–to): 09:00–13:00

Permitted examination support material: C: Approved simple calculator. English dictionaries or bilingual dictionaries to/from English are allowed. No further written or printed support materials are allowed.

Other information:
Please note: This exam is given in English only.
Answers are accepted in English or a Scandinavian language.
There is a list of useful formulas at the end of this exam.
It is OK to use a result stated earlier in a problem, even if you have not managed to prove it.
The problems are not necessarily given in increasing order of difficulty.

Language: English

Number of pages: 2

Number of pages enclosed: 1

Checked by:

Date Signature

Problem 1

a. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion, and define $A_t = B_t - tB_1$ for $0 \leq t \leq 1$. Compute the expectations $\mathsf{E}(A_t)$ and $\mathsf{E}(A_s A_t)$, and verify in particular that $\mathsf{E}(A_t^2) = t - t^2$.

b. A stochastic process defined for $t \in [0,1]$ with continuous paths and the same probability law as the process (A_t) above, is called a *Brownian bridge*. Verify that one can also construct a Brownian bridge by setting $A_1 = 0$ and

$$A_t = (1-t)B_{t/(1-t)}, \qquad 0 \leq t < 1.$$

Conversely, use this to show how Brownian motion can be constructed from a Brownian bridge. (*Hint*: $t/(1-t)$ is a strictly increasing function of t for $0 \leq t < 1$.)

Problem 2

a. What is a martingale? State the martingale representation theorem.

b. What is a stopping time?

Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration associated with n-dimensional Brownian motion. Let $(M_t)_{t\geq 0}$ be a martingale and τ a stopping time, both with respect to $(\mathcal{F}_t)$. Assuming that $\mathsf{E}(M_t^2) < \infty$ for all $t \geq 0$, show that the *stopped process* $(M_{\tau\wedge t})_{t\geq 0}$ is a martingale.

(By definition, $\tau \wedge t = \min(\tau, t)$. The martingale representation theorem will be useful.)

c. With the same assumptions as in **b**, assume further that $M_{\tau\wedge t}$ is bounded, i.e., there exists some constant C so that $|M_{\tau\wedge t}| \leq C$ a.s. (almost surely) for all t. Assume also that $\tau < \infty$ a.s. Show that then $\mathsf{E}(M_\tau) = E(M_0)$.

(*Hint*: Consider $M_{\tau\wedge n}$ and let $n \to \infty$.)

d. Now let $(B_t)_{t\geq 0}$ be one-dimensional Brownian motion, and let τ be the first exit time from $(-\infty, 1)$. It follows from the law of iterated logarithms that $\tau < \infty$ a.s. Put $M_t^\alpha = e^{\alpha B_t - \alpha^2 t/2}$, where $\alpha > 0$. Then $(M_t^\alpha)_{t\geq 0}$ is a martingale. You do *not* need to prove the above assertions.

Show that $\mathsf{E}(\tau) = \infty$. (*Hint*: What is M_τ^α? Assume that $\mathsf{E}(\tau) < \infty$. Use the inequality $e^x \geq 1 + x$ and get a contradiction when α is sufficiently small.)

Problem 3

a. The Itô diffusion

$$\begin{aligned} dX_t &= Y_t\,dt \\ dY_t &= -X_t\,dt + dB_t \end{aligned} \tag{1}$$

represents the motion of a harmonic oscillator, X_t being the position of a mass on the end of a spring, and Y_t being the velocity of the mass. (Clearly, some rescaling of X, Y, and t has been done, whereby the mass and the spring stiffness have been normalized to 1.) What is the physical interpretation of the noise term dB_t?

The quantity $U_t = \frac{1}{2}(X_t^2 + Y_t^2)$ represents the energy of the mass–spring system. Find an expression for dU_t, and use it to compute $\mathsf{E}(U_t)$ given initial data $X_0 = x$, $Y_0 = y$.

b. Writing $Z_t = (X_t, Y_t)^\mathsf{T}$, we can write (1) as

$$dZ_t = HZ_t\,dt + \sigma\,dB_t$$

for a 2×2 matrix H and column vector σ. Solve this given the initial condition $Z_0 = (x, y)^\mathsf{T}$. (*Hint*: Differentiate $e^{-tH}Z_t$. Note that $H^2 = -I$. To compute e^{tH}, split the power series into even and odd powers.)

Compute $\mathsf{E}(U_t)$ explicitly using this solution, thus verifying your result from point **a**.

c. What is the infinitesimal generator for the given Itô diffusion? Use it to determine a partial differential equation satisfied by $u(t, x, y) = \mathsf{E}^{(x,y)}(U_t)$, where $\mathsf{E}^{(x,y)}$ means expectation assuming $(X_0, Y_0) = (x, y)$.

Verify that the explicit answer from point **b** does indeed satisfy the equation.

List of useful formulas

Note: The list does not state the requirements for the formulas to be valid.

1D Gaussian variable: $X \in \mathcal{N}(\mu, \sigma^2)$;

(i) $\mathsf{E}\Big((X-\mu)^4\Big) = 3\sigma^4$,

(ii) $\Phi_X(u) = e^{i\mu - \sigma^2 u^2/2}$,

(iii) $f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma}} e^{(x-\mu)^2/(2\sigma)}$.

Conditional Expectations:

(i) If Y is $\mathcal{H}$-measurable, then $\mathsf{E}(YX \mid \mathcal{H}) = Y\,\mathsf{E}(X \mid \mathcal{H})$.

(ii) If X is independent of $\mathcal{H}$, then $\mathsf{E}(X \mid \mathcal{H}) = \mathsf{E}(X)$.

(iii) If $\mathcal{G} \subset \mathcal{H}$, then $\mathsf{E}\Big(\mathsf{E}(X \mid \mathcal{H}) \mid \mathcal{G}\Big) = \mathsf{E}(X \mid \mathcal{G})$.

Itô Isometry: $\mathsf{E}\left|\int_0^T f(t,\omega)\,dB_t(\omega)\right|^2 = \int_0^T \mathsf{E}|f(t,\omega)|^2 dt = \|f\|^2_{L^2(\Omega\times[0,T])}$

2D Itô Formula: The “Rules” and

$$dg(t, X_t, Y_t) = \frac{\partial g}{\partial t}\,dt + \frac{\partial g}{\partial x}\,dX_t + \frac{\partial g}{\partial y}\,dY_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dX_t)^2 + \frac{\partial^2 g}{\partial x \partial y} dX_t dY_t + \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(dY_t)^2.$$

The Generator for $dX_t = \beta_i(X_t)\,dt + \sigma(X_t)\,dB_t$:

$$A(f)(x) = \sum_{i=1}^n \beta_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^n \operatorname{tr}\Big(\sigma(x)\sigma(x)^{\mathsf{T}} D^2 f(x)\Big), \qquad \Big(D^2 f(x)\Big)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(x).$$

Dynkin's formula: $\mathsf{E}^x\Big(f(X_\tau)\Big) = f(x) + \mathsf{E}^x\Big(\int_o^\tau Af(X_s)ds\Big)$.

Potential Solutions: $\nabla^2 f = 0$ for all $x \in \mathbb{R}^n$, $|x| \neq 0$:

$$n = 2: \quad f(x) = \log|x|,$$
$$n > 2: \quad f(x) = |x|^{2-n}.$$

Grönwall's inequality: If $v(t) \le C + A\int_0^t v(s)ds \ldots$, then $v(t) \le Ce^{At}$.