

Lebesgue decomposition and Radon–Nikodym

Abstract: This little note presents a proof of the Lebesgue–Radon–Nikodym theorem. Historically, Lebesgue decomposition and Radon–Nikodym are considered separate results; but there is a natural connection between the two. The proof presented here is a variant of a proof found in Rudin's *Real and Complex Analysis*. Rudin credits John von Neumann with the proof, but does not provide a reference. This is the quickest proof I know of, but it does depend on a small amount of Hilbert space theory (namely, the Riesz representation theorem) and the completeness of L^2 .

Let μ and ν be two measures on a measurable space (Ω, \mathcal{F}) .

We say that μ and ν are *mutually singular*, and write $\mu \perp \nu$, if there is a set $A \subseteq \Omega$ with $\mu(A) = 0$ and $\nu(A^c) = 0$.

We say that ν is *absolutely continuous* with respect to μ , and write $\nu \ll \mu$, if for all $A \in \mathcal{F}$, $\mu(A) = 0$ implies $\nu(A) = 0$.

Theorem *Let μ and ν be two finite measures on a measurable space (Ω, \mathcal{F}) . Then there is a set $N \in \mathcal{F}$ with $\mu(N) = 0$ and a μ -integrable function $f \geq 0$ so that, for every $A \in \mathcal{F}$,*

$$\nu(A) = \nu(A \cap N) + \int_A f d\mu.$$

As a consequence, we can write

$$\begin{aligned} \nu &= \nu_{\perp} + \nu_{\ll} \quad \text{where} \\ \nu_{\perp}(A) &= \nu(A \cap N) \quad \text{and} \\ \nu_{\ll}(A) &= \int_A f d\mu. \end{aligned}$$

Clearly, $\nu_{\perp} \perp \mu$ and $\nu_{\ll} \ll \mu$. This is the *Lebesgue decomposition* of ν with respect to μ . The function f is called the *Radon–Nikodym derivative* of ν_{\ll} with respect to μ .

The theorem can be extended to σ -finite measures without much effort. We leave this to the reader. Note that without a σ -finiteness assumption, the theorem does not hold. For example, consider counting measure and Lebesgue measure on the real line.

The proof below employs the *Iverson* or *indicator bracket*: where S denotes a statement, $[S] = 1$ if S is true, $[S] = 0$ if S is false.

Proof: Note that $\mu + \nu$ is a measure. Working in the Hilbert space $L^2(\mu + \nu)$, it is a simple application of Riesz' representation theorem to see that there exists a function $h \in L^2(\mu + \nu)$ so that

$$\int_{\Omega} g \, d\nu = \int_{\Omega} gh \, d(\mu + \nu) \quad \text{for } g \in L^2(\mu + \nu). \quad (*)$$

Now any bounded measurable function belongs to $L^2(\mu + \nu)$, so we can apply this to $g(\omega) = [h(\omega) < 0]$ and $g(\omega) = [h(\omega) > 1]$, respectively, and immediately conclude that $0 \leq h \leq 1$ a.e. with respect to $\mu + \nu$.

Let

$$N = \{\omega \in \Omega : h(\omega) = 1\}.$$

With $g(\omega) = [h(\omega) = 1]$ we immediately get $\mu(N) = 0$.

Now rewrite (*) in the form

$$\int_{\Omega} (1 - h)g \, d\nu = \int_{\Omega} gh \, d\mu \quad \text{for } g \in L^2(\mu + \nu)$$

and set

$$g(\omega) = [\omega \in A] \frac{[h(\omega) < 1 - n^{-1}]}{1 - h(\omega)}$$

(which is bounded, and hence in L^2 , so the formula applies) to get

$$\int_A \left[h < 1 - \frac{1}{n} \right] d\nu = \int_A \frac{h}{1 - h} \left[h < 1 - \frac{1}{n} \right] d\mu.$$

Now let $n \rightarrow \infty$ and apply the monotone convergence theorem to conclude

$$\int_A [h < 1] d\nu = \int_A \frac{h}{1 - h} [h < 1] d\mu,$$

where it should be understood that $f(\omega) = 0$ when $h(\omega) = 1$. Putting

$$f = \frac{h}{1 - h} [h < 1]$$

and recalling the definition of N , we see that this becomes

$$\nu(A \setminus N) = \int_A f \, d\mu,$$

and the proof is easily completed. ■

Remark: The main difference between the present proof and Rudin's is that Rudin expands $(1 - h)^{-1}$ in a power series instead of cutting off the part where its value is near 1.