Lebesgue decomposition and Radon–Nikodym

Abstract: This little note presents a proof of the Lebesgue–Radon–Nikodym theorem. Historically, Lebesgue decomposition and Radon–Nikodym are considered separate results; but there is a natural connection between the two. The proof presented here is a variant of a proof found in Rudin's *Real and Complex Analysis*. Rudin credits John von Neumann with the proof, but does not provide a reference. This is the quickest proof I know of, but it does depend on a small amount of Hilbert space theory (namely, the Riesz representation theorem) and the completeness of L^2.

Let μ and ν be two measures on a measurable space $(\Omega, \mathcal{F})$.

We say that μ and ν are *mutually singular*, and write $\mu \perp \nu$, if there is a set $A \subseteq \Omega$ with $\mu(A) = 0$ and $\nu(A^c) = 0$.

We say that ν is *absolutely continuous* with respect to μ, and write $\nu \ll \mu$, if for all $A \in \mathcal{F}$, $\mu(A) = 0$ implies $\nu(A) = 0$.

Theorem *Let μ and ν be two finite measures on a measurable space $(\Omega, \mathcal{F})$. Then there is a set $N \in \mathcal{F}$ with $\mu(N) = 0$ and a μ-integrable function $f \geq 0$ so that, for every $A \in \mathcal{F}$,*

$$\nu(A) = \nu(A \cap N) + \int_A f \, d\mu.$$

As a consequence, we can write

$$\begin{aligned} \nu &= \nu_\perp + \nu_\ll \quad \text{where} \\ \nu_\perp(A) &= \nu(A \cap N) \quad \text{and} \\ \nu_\ll(A) &= \int_A f \, d\mu. \end{aligned}$$

Clearly, $\nu_\perp \perp \mu$ and $\nu_\ll \ll \mu$. This is the *Lebesgue decomposition* of ν with respect to μ. The function f is called the *Radon–Nikodym derivative* of $\nu_\ll$ with respect to μ.

The theorem can be extended to σ-finite measures without much effort. We leave this to the reader. Note that without a σ-finiteness assumption, the theorem does not hold. For example, consider counting measure and Lebesgue measure on the real line.

The proof below employs the *Iverson* or *indicator bracket*: where S denotes a statement, $[S] = 1$ if S is true, $[S] = 0$ if S is false.

See the next page for the proof.

Proof: Note that $\mu+\nu$ is a measure. Working in the Hilbert space $L^2(\mu+\nu)$, it is a simple application of Riesz' representation theorem to see that there exists a function $h \in L^2(\mu+\nu)$ so that

$$\int_\Omega g\,d\nu = \int_\Omega gh\,d(\mu+\nu) \qquad \text{for } g \in L^2(\mu+\nu). \tag{$*$}$$

Now any bounded measurable function belongs to $L^2(\mu+\nu)$, so we can apply this to $g(\omega) = [h(\omega) < 0]$ and $g(\omega) = [h(\omega) > 1]$, respectively, and immediately conclude that $0 \le h \le 1$ a.e. with respect to $\mu+\nu$.

Let

$$N = \{\omega \in \Omega\colon\ h(\omega) = 1\}.$$

With $g(\omega) = [h(\omega) = 1]$ we immediately get $\mu(N) = 0$.

Now rewrite $(*)$ in the form

$$\int_\Omega (1-h)g\,d\nu = \int_\Omega gh\,d\mu \qquad \text{for } g \in L^2(\mu+\nu)$$

and set

$$g(\omega) = \big[\omega \in A\big]\frac{\big[h(\omega) < 1 - n^{-1}\big]}{1-h(\omega)}$$

(which is bounded, and hence in L^2, so the formula applies) to get

$$\int_A \Big[h < 1 - \frac{1}{n}\Big]\,d\nu = \int_A \frac{h}{1-h}\Big[h < 1 - \frac{1}{n}\Big]\,d\mu.$$

Now let $n \to \infty$ and apply the monotone convergence theorem to conclude

$$\int_A [h < 1]\,d\nu = \int_A \frac{h}{1-h}[h < 1]\,d\mu,$$

where it should be understood that $f(\omega) = 0$ when $h(\omega) = 1$. Putting

$$f = \frac{h}{1-h}[h < 1]$$

and recalling the definition of N, we see that this becomes

$$\nu(A \setminus N) = \int_A f\,d\mu,$$

and the proof is easily completed. ∎

Remark: The main difference between the present proof and Rudin's is that Rudin expands $(1-h)^{-1}$ in a power series instead of cutting off the part where its value is near 1.