The abstract Lebesgue integral

The point of this note is to define the abstract Lebesgue integral and to derive its basic properties with as little fuzz as possible.

The reader is assumed to already know what measurable spaces and measurable functions are. The note should be readable *in principle* without any previous knowledge of the abstract Lebesgue integral (nor indeed the one on the real line), but as no motivation is provided, it may prove tough going.

Throughout this note, a fixed measure space $(\Omega,\mathcal{F},\mu)$ is given. All functions are supposed to take their values in the extended real line $[-\infty,\infty]$. We do not bother with complex functions, as this represent a rather trivial extension of the integral.[1]

First, a bit of handy notation:

The *Iverson bracket* (or *indicator bracket*) $[\cdots]$ has the value 1 if the statement inside the brackets is true, and 0 otherwise. For example, if $f\colon\Omega\to\mathbb{R}$ is given, then

$$[f(x)>a]=\begin{cases}1 & \text{if } f(x)>a,\\ 0 & \text{otherwise.}\end{cases}$$

The *characteristic function* (or *indicator function*) of a set A is commonly written as χ_A, but I shall use the symbol $[A]$ instead.

It can be defined in terms of the Iverson bracket by the mysterious looking formula

$$[A](x)=[x\in A].$$

Note carefully that the square brackets on the left denote a *function*, while the one on the right denote either 0 or 1 (depending on x).

Be warned that this notation highly non-standard, and should be explained carefully whenever you use it.

As an example, convince yourself that $\left[f^{-1}((a,\infty))\right](x)=[f(x)>a]$.

[1] This note started life as a supplementary note for the course TMA4225 Foundations of analysis.

Integral of simple functions

A *simple function* is a function that can be written

$$\sum_{i=1}^{n} a_i\,[A_i]$$

where $a_i \in \mathbb{R}$ and $A_i \in \mathcal{F}$ for each $i \in \{1,\dots,n\}$. Equivalently, it is just a measurable function which takes only a finite number of different values.

We want to define the integral of a simple function:

$$\int_\Omega \sum_{i=1}^{n} a_i\,[A_i]\,d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$

We need to show that this is *well defined,* since the same simple function can be written in many different ways. (For example, the sets A_i need not be mutually disjoint.)

For this, the following lemma suffices (exercise: convince yourself of this):

1 Lemma *If* $\sum_{i=1}^{n} a_i\,[A_i] = 0$ *then* $\sum_{i=1}^{n} \mu(A_i) = 0$.

Proof: Define the function $\alpha\colon \Omega \to \{0,1\}^n$ by

$$\alpha(\omega) = \big([\omega \in A_1],\dots,[\omega \in A_n]\big),$$

and note that the kth component function of α is $[A_k]$. For any $\beta \in \{0,1\}^n$, put

$$B_\beta = \{\omega \in \Omega\colon\ \alpha(\omega) = \beta\}.$$

Now A_i is a disjoint union of those B_β for which $\beta_i = 1$, so we find

$$\begin{aligned}\sum_{i=1}^{n} a_i \mu(A_i) &= \sum_{i=1}^{n} a_i \sum_{\beta \in \{0,1\}^n} \mu(B_\beta)\,[\beta_i = 1]\\ &= \sum_{\beta \in \{0,1\}^n} \mu(B_\beta) \underbrace{\sum_{i=1}^{n} a_i\,[\beta_i = 1]}_{\nu(\beta)}.\end{aligned}$$

Now we note that if $\omega \in B_\beta$ then $[\beta_i = 1] = [\omega \in A_i] = [A_i](\omega)$, so the sum denoted $\nu(\beta)$ at the end of the calculation above is zero by the assumption of the lemma. Thus for every β, either $\nu(\beta) = 0$ or $B_\beta = \emptyset$, so in any case $\mu(B_\beta)\nu(\beta) = 0$, and summing this over $\beta \in \{0,1\}^n$, we arrive at the conclusion of the lemma. ∎

Thus we have shown that the integral is well defined for simple functions.

It is now obvious that the integral $\int_\Omega \varphi\,d\mu$ is linear function of the simple function φ, and that $\int_\Omega \varphi\,d\mu \ge 0$ if $\varphi \ge 0$ (because φ can be written as a sum $\sum a_k\,[A_i]$ with all $a_i \ge 0$).

Integral of nonnegative functions

Whenever $f\colon \Omega \to [0,\infty]$ is a measurable function, we define its integral to be

$$\int_\Omega f\,d\mu = \sup\left\{\int_\Omega \varphi\,d\mu\colon 0 \le \varphi \le f \text{ and } \varphi \text{ is a simple function}\right\}$$

In the proof of the following theorem, we need the obvious fact that $f \le g$ implies $\int_\Omega f\,d\mu \le \int_\Omega g\,d\mu$.

2 Theorem (Monotone convergence theorem, MCT) *Assume that* $0 \le f_1 \le f_2 \le f_3 \le \cdots$ *are Lebesgue measurable functions, and let*

$$f(x) = \lim_{k\to\infty} f_k(x).$$

Then

$$\int_\Omega f\,d\mu = \lim_{k\to\infty} \int_\Omega f_k\,d\mu.$$

Proof: First, note that $\left(\int_\Omega f_k\,d\mu\right)$ is a non-decreasing sequence, so it does have a limit. Also $f_k \le f$, so $\int_\Omega f_k\,d\mu \le \int_\Omega f\,d\mu$. Taking the limit, we conclude

$$\lim_{k\to\infty} \int_\Omega f_k\,d\mu \le \int_\Omega f\,d\mu.$$

It remains to prove the opposite inequality.

Take any simple function φ with $0 \le \varphi \le f$, and write

$$\varphi = \sum_{i=1}^{n} a_i\,[A_i]$$

where the sets A_i are mutually disjoint, and $a_i > 0$. Thus $\varphi(\omega) = a_i$ when $\omega \in A_i$. Further, let α be any real number with $0 < \alpha < 1$.

Now fix an index i, and note that the sets

$$\{\omega \in A_i\colon f_k(\omega) > \alpha a_i\}, \qquad k = 1, 2, \ldots$$

form a non-decreasing sequence of subsets of A_i (since the f_k form a non-decreasing sequence of functions) whose union is all of A_i (since $f_k \to f \ge a_i > \alpha a_i$ on A_i), and that these sets are all measurable (since each f_k is measurable). Thus

$$\lim_{k\to\infty} \mu\bigl(\{\omega \in A_i\colon f_k(\omega) > \alpha a_i\}\bigr) = \mu(A_i),$$

and we get (using that pairwise disjointness means the sum in the first line is ≤ 1)

$$\begin{aligned}
\lim_{k\to\infty}\int_\Omega f_k\,d\mu &\ge \lim_{k\to\infty}\int_\Omega f_k \sum_{i=1}^n [A_i]\,d\mu \\
&= \sum_{i=1}^n \lim_{k\to\infty}\int_\Omega f_k\,[A_i]\,d\mu \\
&\ge \sum_{i=1}^n \lim_{k\to\infty}\int_\Omega f_k\,[A_i]\,[f_k > \alpha a_i]\,d\mu \\
&\ge \sum_{i=1}^n \lim_{k\to\infty}\int_\Omega \alpha a_i\,[A_i]\,[f_k > \alpha a_i]\,d\mu \\
&= \sum_{i=1}^n \lim_{k\to\infty} \alpha a_i \mu\bigl(\{\omega\in A_i : f_k(\omega) > \alpha a_i\}\bigr) \\
&= \sum_{i=1}^n \alpha a_i \mu(A_i) = \alpha\int_\Omega \varphi\,d\mu.
\end{aligned}$$

Since $\alpha\in(0,1)$ was arbitrary, we conclude

$$\lim_{k\to\infty}\int_\Omega f_k\,d\mu \ge \int_\Omega \varphi\,d\mu,$$

and using the definition of the integral we arrive at the desired conclusion. ∎

Fatou's lemma

3 Lemma (Fatou's lemma) *Let $f_k \le 0$ be measurable for each $k\in\mathbb{N}$. Then*

$$\int_\Omega \varliminf_{k\to\infty} f_k\,d\mu \le \varliminf_{k\to\infty}\int_\Omega f_k\,d\mu.$$

The limit inferior on the left is to be taken pointwise: The value of the left integrand at a point ω is taken to be $\varliminf_{k\to\infty} f_k(\omega)$.

Proof: Let

$$g_k(\omega) = \inf_{i\ge k} f_k(\omega).$$

Then $g_k \nearrow \varliminf_{k\to\infty} f_k$, so the monotone convergence theorem gives

$$\int_\Omega \varliminf_{k\to\infty} f_k\,d\mu = \lim_{k\to\infty}\int_\Omega g_k\,d\mu.$$

Now just note that $g_k \le f_k$ in order to complete the proof. ∎

Additivity of the integral

The integral of a nonnegative function f clearly satisfies $\int_\Omega af\,d\mu = a\int_\Omega f\,d\mu$ for any constant $a\ge 0$. However, *additivity* is far less obvious. We need the following:

4 Lemma *If $f \ge 0$ is a measurable function then there exists a sequence of simple functions $\varphi_k \ge 0$ so that $\varphi_k \nearrow f$ pointwise.*

Proof: For any $k \in \mathbb{N}$ and $i \in \{1, 2, \ldots, 2^k k\}$, let $a_{ki} = 2^{-k} i$, and set

$$\varphi_k = \sum_{i=1}^{2^k k-1} a_{ki}[A_{ki}], \qquad A_{ki} = \{\omega \in \Omega \colon a_{ki} \le f(\omega) < a_{k(i+1)}\}.$$

The important things to note are:
– for any k, the sets A_{ki} are disjoint
– for any k, $0 \le \varphi_k \le f$
– for any k and ω, if $f(\omega) < k$ then $\varphi(\omega) \ge f(\omega) - 2^k$
– $[a_{ki}, a_{k(i+1)}) = [a_{(k+1)(2i)}, a_{(k+1)(2i+1)}) \sqcup [a_{(k+1)(2i+1)}, a_{(k+1)(2i+2)})$ (here $\sqcup$ indicates disjoint union), and so $A_{ki} = A_{(k+1)(2i)} \sqcup A_{(k+1)(2i+1)}$, from which we conclude $\varphi_k \le \varphi_{k+1}$

These are sufficient to complete the proof. ∎

Now we can prove:

5 Proposition *If f and g are nonnegative measurable functions then*

$$\int_\Omega (f+g)\,d\mu = \int_\Omega f\,d\mu + \int_\Omega g\,d\mu.$$

Proof: Pick nonnegative simple φ_k and γ_k with $\varphi_k \nearrow f$ and $\gamma_k \nearrow g$; then $\varphi_k + \gamma_k \nearrow f + g$, and

$$\int_\Omega (\varphi_k + \gamma_k)\,d\mu = \int_\Omega \varphi_k\,d\mu + \int_\Omega \gamma_k\,d\mu.$$

Now let $k \to \infty$ and apply the monotone convergence theorem to each integral. ∎

Integrating general functions

Any measurable function f can be written as a difference of nonnegative measurable functions: $f = f^+ - f^-$ where

$$f^+(\omega) = \begin{cases} f(\omega) & f(\omega) > 0, \\ 0 & \text{otherwise,} \end{cases} \qquad f^-(\omega) = \begin{cases} -f(\omega) & f(\omega) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

(note that $f^- = (-f)^+$).

We would like to define

$$\int_\Omega f\,d\mu = \int_\Omega g\,d\mu - \int_\Omega h\,d\mu$$

if $g \ge 0$ and $h \ge 0$ are *any* measurable functions with $f = g - h$, but then we need to show that is well defined. This turns out to be easy: If $f = g_1 - h_1 = g_2 - h_2$ with

$g_1 \geq 0$, $h_1 \geq 0$, $g_2 \geq 0$, and $h_2 \geq 0$, then $g_1 + h_2 = g_2 + h_1$, so the additivity results for nonnegative integrands yields

$$\int_\Omega g_1\,d\mu + \int_\Omega h_2\,d\mu = \int_\Omega g_2\,d\mu + \int_\Omega h_1\,d\mu,$$

which we rearrange into

$$\int_\Omega g_1\,d\mu - \int_\Omega h_1\,d\mu = \int_\Omega g_2\,d\mu - \int_\Omega h_2\,d\mu,$$

showing that this is indeed well defined.

However, this argument may break down if some integrals are infinite, since we cannot meaningfully perform the subtraction $\infty - \infty$.

We define f to be *integrable* if it is measurable with $\int_\Omega |f|\,d\mu < \infty$. Then the integrals of $f^\pm$ are also finite, and the above procedure works.

We can also define the integral of f if just *one* of the two integrals $\int_\Omega f^\pm\,d\mu$ is finite. This generality is rarely needed, but easily handled.

It is now not difficult to show that $\int_\Omega f\,d\mu$ is a linear function of f for *integrable* f.

Dominated convergence

6 Theorem (Dominated convergence theorem, DCT) *Assume that $f_k \to f$ pointwise, where all functions are measurable. Also assume that $|f_k| \leq g$ for all k, where g is an integrable function. Then*

$$\int_\Omega f\,d\mu = \lim_{k\to\infty} \int_\Omega f_k\,d\mu.$$

Proof: We apply Fatou's lemma to both sequences $g \pm f_k$, noting that these are nonnegative, and also that $g \pm f_k \to g \pm f$:

$$\int_\Omega (g \pm f)\,d\mu \leq \varliminf_{k\to\infty} \int_\Omega (g \pm f_k)\,d\mu$$

We subtract the integral of g from both sides. With the plus sign, we conclude

$$\int_\Omega f\,d\mu \leq \varliminf_{k\to\infty} \int_\Omega f_k\,d\mu,$$

and with the minus sign, we multiply by -1 and conclude

$$\int_\Omega f\,d\mu \geq \varlimsup_{k\to\infty} \int_\Omega f_k\,d\mu.$$

Together, these two inequalities complete the proof. ∎