

## The abstract Lebesgue integral

The point of this note is to define the abstract Lebesgue integral and to derive its basic properties with as little fuzz as possible.

*The reader* is assumed to already know what measurable spaces and measurable functions are. The note should be readable *in principle* without any previous knowledge of the abstract Lebesgue integral (nor indeed the one on the real line), but as no motivation is provided, it may prove tough going.

Throughout this note, a fixed measure space  $(\Omega, \mathcal{F}, \mu)$  is given. All functions are supposed to take their values in the extended real line  $[-\infty, \infty]$ . We do not bother with complex functions, as this represents a rather trivial extension of the integral.<sup>1</sup>

First, a bit of handy notation:

The *Iverson bracket* (or *indicator bracket*)  $[\dots]$  has the value 1 if the statement inside the brackets is true, and 0 otherwise. For example, if  $f: \Omega \rightarrow \mathbb{R}$  is given, then

$$[f(x) > a] = \begin{cases} 1 & \text{if } f(x) > a, \\ 0 & \text{otherwise.} \end{cases}$$

The *characteristic function* (or *indicator function*) of a set  $A$  is commonly written as  $\chi_A$ , but I shall use the symbol  $[A]$  instead.

It can be defined in terms of the Iverson bracket by the mysterious looking formula

$$[A](x) = [x \in A].$$

Note carefully that the square brackets on the left denote a *function*, while the one on the right denote either 0 or 1 (depending on  $x$ ).

Be warned that this notation is highly non-standard, and should be explained carefully whenever you use it.

As an example, convince yourself that  $[f^{-1}((a, \infty))](x) = [f(x) > a]$ .

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<sup>1</sup>This note started life as a supplementary note for the course TMA4225 Foundations of analysis.

## Integral of simple functions

A *simple function* is a function that can be written

$$\sum_{i=1}^n a_i [A_i]$$

where  $a_i \in \mathbb{R}$  and  $A_i \in \mathcal{F}$  for each  $i \in \{1, \dots, n\}$ . Equivalently, it is just a measurable function which takes only a finite number of different values.

We want to define the integral of a simple function:

$$\int_{\Omega} \sum_{i=1}^n a_i [A_i] d\mu = \sum_{i=1}^n a_i \mu(A_i).$$

We need to show that this is *well defined*, since the same simple function can be written in many different ways. (For example, the sets  $A_i$  need not be mutually disjoint.)

For this, the following lemma suffices (exercise: convince yourself of this):

**1 Lemma** If  $\sum_{i=1}^n a_i [A_i] = 0$  then  $\sum_{i=1}^n \mu(A_i) = 0$ .

**Proof:** Define the function  $\alpha: \Omega \rightarrow \{0, 1\}^n$  by

$$\alpha(\omega) = ([\omega \in A_1], \dots, [\omega \in A_n]),$$

and note that the  $k$ th component function of  $\alpha$  is  $[A_k]$ . For any  $\beta \in \{0, 1\}^n$ , put

$$B_{\beta} = \{\omega \in \Omega: \alpha(\omega) = \beta\}.$$

Now  $A_i$  is a disjoint union of those  $B_{\beta}$  for which  $\beta_i = 1$ , so we find

$$\begin{aligned} \sum_{i=1}^n a_i \mu(A_i) &= \sum_{i=1}^n a_i \sum_{\beta \in \{0,1\}^n} \mu(B_{\beta}) [\beta_i = 1] \\ &= \sum_{\beta \in \{0,1\}^n} \mu(B_{\beta}) \underbrace{\sum_{i=1}^n a_i [\beta_i = 1]}_{v(\beta)}. \end{aligned}$$

Now we note that if  $\omega \in B_{\beta}$  then  $[\beta_i = 1] = [\omega \in A_i] = [A_i](\omega)$ , so the sum denoted  $v(\beta)$  at the end of the calculation above is zero by the assumption of the lemma. Thus for every  $\beta$ , either  $v(\beta) = 0$  or  $B_{\beta} = \emptyset$ , so in any case  $\mu(B_{\beta}) v(\beta) = 0$ , and summing this over  $\beta \in \{0, 1\}^n$ , we arrive at the conclusion of the lemma. ■

Thus we have shown that the integral is well defined for simple functions.

It is now obvious that the integral  $\int_{\Omega} \varphi d\mu$  is linear function of the simple function  $\varphi$ , and that  $\int_{\Omega} \varphi d\mu \geq 0$  if  $\varphi \geq 0$  (because  $\varphi$  can be written as a sum  $\sum a_k [A_i]$  with all  $a_i \geq 0$ ).

### Integral of nonnegative functions

Whenever  $f: \Omega \rightarrow [0, \infty]$  is a measurable function, we define its integral to be

$$\int_{\Omega} f d\mu = \sup \left\{ \int_{\Omega} \varphi d\mu : 0 \leq \varphi \leq f \text{ and } \varphi \text{ is a simple function} \right\}$$

In the proof of the following theorem, we need the obvious fact that  $f \leq g$  implies  $\int_{\Omega} f d\mu \leq \int_{\Omega} g d\mu$ .

**2 Theorem (Monotone convergence theorem, MCT)** *Assume that  $0 \leq f_1 \leq f_2 \leq f_3 \leq \dots$  are Lebesgue measurable functions, and let*

$$f(x) = \lim_{k \rightarrow \infty} f_k(x).$$

Then

$$\int_{\Omega} f d\mu = \lim_{k \rightarrow \infty} \int_{\Omega} f_k d\mu.$$

**Proof:** First, note that  $(\int_{\Omega} f_k d\mu)$  is a non-decreasing sequence, so it does have a limit. Also  $f_k \leq f$ , so  $\int_{\Omega} f_k d\mu \leq \int_{\Omega} f d\mu$ . Taking the limit, we conclude

$$\lim_{k \rightarrow \infty} \int_{\Omega} f_k d\mu \leq \int_{\Omega} f d\mu.$$

It remains to prove the opposite inequality.

Take any simple function  $\varphi$  with  $0 \leq \varphi \leq f$ , and write

$$\varphi = \sum_{i=1}^n a_i [A_i]$$

where the sets  $A_i$  are mutually disjoint, and  $a_i > 0$ . Thus  $\varphi(\omega) = a_i$  when  $\omega \in A_i$ . Further, let  $\alpha$  be any real number with  $0 < \alpha < 1$ .

Now fix an index  $i$ , and note that the sets

$$\{\omega \in A_i : f_k(\omega) > \alpha a_i\}, \quad k = 1, 2, \dots$$

form a non-decreasing sequence of subsets of  $A_i$  (since the  $f_k$  form a non-decreasing sequence of functions) whose union is all of  $A_i$  (since  $f_k \rightarrow f \geq a_i > \alpha a_i$  on  $A_i$ ), and that these sets are all measurable (since each  $f_k$  is measurable). Thus

$$\lim_{k \rightarrow \infty} \mu(\{\omega \in A_i : f_k(\omega) > \alpha a_i\}) = \mu(A_i),$$

and we get (using that pairwise disjointness means the sum in the first line is  $\leq 1$ )

$$\begin{aligned}
\lim_{k \rightarrow \infty} \int_{\Omega} f_k d\mu &\geq \lim_{k \rightarrow \infty} \int_{\Omega} f_k \sum_{i=1}^n [A_i] d\mu \\
&\geq \lim_{k \rightarrow \infty} \int_{\Omega} \sum_{i=1}^n f_k [A_i] [f_k > \alpha a_i] d\mu \\
&\geq \lim_{k \rightarrow \infty} \int_{\Omega} \sum_{i=1}^n \alpha a_i [A_i] [f_k > \alpha a_i] d\mu \\
&= \sum_{i=1}^n \lim_{k \rightarrow \infty} \alpha a_i \mu(\{\omega \in A_i : f_k(\omega) > \alpha a_i\}) \\
&= \sum_{i=1}^n \alpha a_i \mu(A_i) = \alpha \int_{\Omega} \varphi d\mu.
\end{aligned}$$

Since  $\alpha \in (0, 1)$  was arbitrary, we conclude

$$\lim_{k \rightarrow \infty} \int_{\Omega} f_k d\mu \geq \int_{\Omega} \varphi d\mu,$$

and using the definition of the integral we arrive at the desired conclusion. ■

### Fatou's lemma

**3 Lemma (Fatou's lemma)** *Let  $f_k \leq 0$  be measurable for each  $k \in \mathbb{N}$ . Then*

$$\int_{\Omega} \underline{\lim}_{k \rightarrow \infty} f_k d\mu \leq \underline{\lim}_{k \rightarrow \infty} \int_{\Omega} f_k d\mu.$$

The limit inferior on the left is to be taken pointwise: The value of the left integrand at a point  $\omega$  is taken to be  $\underline{\lim}_{k \rightarrow \infty} f_k(\omega)$ .

**Proof:** Let

$$g_k(\omega) = \inf_{i \geq k} f_k(\omega).$$

Then  $g_k \nearrow \underline{\lim}_{k \rightarrow \infty} f_k$ , so the monotone convergence theorem gives

$$\int_{\Omega} \underline{\lim}_{k \rightarrow \infty} f_k d\mu = \lim_{k \rightarrow \infty} \int_{\Omega} g_k d\mu.$$

Now just note that  $g_k \leq f_k$  in order to complete the proof. ■

### Additivity of the integral

The integral of a nonnegative function  $f$  clearly satisfies  $\int_{\Omega} a f d\mu = a \int_{\Omega} f d\mu$  for any constant  $a \geq 0$ . However, *additivity* is far less obvious. We need the following:

**4 Lemma** If  $f \geq 0$  is a measurable function then there exists a sequence of simple functions  $\varphi_k \geq 0$  so that  $\varphi_k \nearrow f$  pointwise.

**Proof:** For any  $k \in \mathbb{N}$  and  $i \in \{1, 2, \dots, 2^k k\}$ , let  $a_{ki} = 2^{-k} i$ , and set

$$\varphi_k = \sum_{i=1}^{2^k k} a_{ki} [A_{ki}], \quad A_{ki} = \{\omega \in \Omega: a_{ki} \leq f(\omega) < a_{k(i+1)}\},$$

$$A_{k(2^k k)} = \{\omega \in \Omega: k \leq f(\omega)\}.$$

The important things to note are:

- for any  $k$ , the sets  $A_{ki}$  are disjoint
- for any  $k$ ,  $0 \leq \varphi_k \leq f$
- for any  $k$  and  $\omega$ , if  $f(\omega) < k$  then  $\varphi_k(\omega) \geq f(\omega) - 2^{-k}$
- for any  $k$  and  $\omega$ , if  $f(\omega) \geq k$  then  $\varphi_k(\omega) = k$
- $[a_{ki}, a_{k(i+1)}) = [a_{(k+1)(2i)}, a_{(k+1)(2i+1)}) \sqcup [a_{(k+1)(2i+1)}, a_{(k+1)(2i+2)})$  (here  $\sqcup$  indicates disjoint union), and so  $A_{ki} = A_{(k+1)(2i)} \sqcup A_{(k+1)(2i+1)}$ , from which we conclude  $\varphi_k \leq \varphi_{k+1}$

These are sufficient to complete the proof. ■

Now we can prove:

**5 Proposition** If  $f$  and  $g$  are nonnegative measurable functions then

$$\int_{\Omega} (f + g) d\mu = \int_{\Omega} f d\mu + \int_{\Omega} g d\mu.$$

**Proof:** Pick nonnegative simple  $\varphi_k$  and  $\gamma_k$  with  $\varphi_k \nearrow f$  and  $\gamma_k \nearrow g$ ; then  $\varphi_k + \gamma_k \nearrow f + g$ , and

$$\int_{\Omega} (\varphi_k + \gamma_k) d\mu = \int_{\Omega} \varphi_k d\mu + \int_{\Omega} \gamma_k d\mu.$$

Now let  $k \rightarrow \infty$  and apply the monotone convergence theorem to each integral. ■

### Integrating general functions

Any measurable function  $f$  can be written as a difference of nonnegative measurable functions:  $f = f^+ - f^-$  where

$$f^+(\omega) = \begin{cases} f(\omega) & f(\omega) > 0, \\ 0 & \text{otherwise,} \end{cases} \quad f^-(\omega) = \begin{cases} -f(\omega) & f(\omega) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

(note that  $f^- = (-f)^+$ ).

We would like to define

$$\int_{\Omega} f d\mu = \int_{\Omega} g d\mu - \int_{\Omega} h d\mu$$

if  $g \geq 0$  and  $h \geq 0$  are *any* measurable functions with  $f = g - h$ , but then we need to show that is well defined. This turns out to be easy: If  $f = g_1 - h_1 = g_2 - h_2$  with  $g_1 \geq 0, h_1 \geq 0, g_2 \geq 0,$  and  $h_2 \geq 0,$  then  $g_1 + h_2 = g_2 + h_1,$  so the additivity results for nonnegative integrands yields

$$\int_{\Omega} g_1 d\mu + \int_{\Omega} h_2 d\mu = \int_{\Omega} g_2 d\mu + \int_{\Omega} h_1 d\mu,$$

which we rearrange into

$$\int_{\Omega} g_1 d\mu - \int_{\Omega} h_1 d\mu = \int_{\Omega} g_2 d\mu - \int_{\Omega} h_2 d\mu,$$

showing that this is indeed well defined.

*However,* this argument may break down if some integrals are infinite, since we cannot meaningfully perform the subtraction  $\infty - \infty$ .

We define  $f$  to be *integrable* if it is measurable with  $\int_{\Omega} |f| d\mu < \infty$ . Then the integrals of  $f^{\pm}$  are also finite, and the above procedure works.

We can also define the integral of  $f$  if just *one* of the two integrals  $\int_{\Omega} f^{\pm} d\mu$  is finite. This generality is rarely needed, but easily handled.

It is now not difficult to show that  $\int_{\Omega} f d\mu$  is a linear function of  $f$  for *integrable*  $f$ .

### Dominated convergence

**6 Theorem (Dominated convergence theorem, DCT)** *Assume that  $f_k \rightarrow f$  point-wise, where all functions are measurable. Also assume that  $|f_k| \leq g$  for all  $k$ , where  $g$  is an integrable function. Then*

$$\int_{\Omega} f d\mu = \lim_{k \rightarrow \infty} \int_{\Omega} f_k d\mu.$$

**Proof:** We apply Fatou's lemma to both sequences  $g \pm f_k$ , noting that these are nonnegative, and also that  $g \pm f_k \rightarrow g \pm f$ :

$$\int_{\Omega} (g \pm f) d\mu \leq \liminf_{k \rightarrow \infty} \int_{\Omega} (g \pm f_k) d\mu$$

We subtract the integral of  $g$  from both sides. With the plus sign, we conclude

$$\int_{\Omega} f d\mu \leq \liminf_{k \rightarrow \infty} \int_{\Omega} f_k d\mu,$$

and with the minus sign, we multiply by  $-1$  and conclude

$$\int_{\Omega} f d\mu \geq \overline{\lim}_{k \rightarrow \infty} \int_{\Omega} f_k d\mu.$$

Together, these two inequalities complete the proof. ■