Norwegian University of Science and Technology
Department of Mathematical Sciences

MA8109 Stochastic Processes and Differential Equations Autumn 2015

Exercise set #1

Please note: The notation used here may depart in some minor ways from the notation I generally use in the course. I hope it is clear enough, in any case.

$\boxed{1}$ (Øksendal 2:2.8) Let B_t be a Brownian motion in $\mathbb{R}$.

a) Show that $E(e^{iuB_t}) = e^{-\frac{1}{2}u^2 t}$ and use this to prove that $E(B_t^4) = 3t^2$.

b) Use $E(f(B_t)) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(x) e^{-\frac{x^2}{2t}} dx$ and integration by parts to show that $E(B_t^4) = 3t^2$.

$\boxed{2}$ Consider the standard Brownian motion in $\mathbb{R}^n$, B_t. Prove that

$$\mathsf{E}\left(|B_t - B_s|^4\right) = n\,(n+2)\,(t-s)^2\,. \tag{1}$$

Hint: For this problem you could use that

$$\mathsf{E}\left(X_1^2 + \cdots + X_n^2\right)^2 = \mathsf{E}\left(\sum_{i,j=1}^{N} X_i^2 X_j^2\right), \tag{2}$$

and use the independence and distribution of the increments and the fact that $E(B_i^4(t)) = 3t$ for $i = 1, \ldots n$ (previous exercise).

$\boxed{3}$ Let B_t be a Brownian motion and $\mathcal{F}_t$ the filtration generated by B_t. Show that $B_t^2 - t$ is a martingale w.r.t. $\mathcal{F}_t$.

Hint: $B_t^p = B_s^p + (B_t^p - B_s^p)$ for $p = 1, 2$, use different properties of the conditional expectation.

$\boxed{4}$ Prove from the definition of the Itô integral, that

$$\int_0^t s dB_s = tB_t - \int_0^t B_s ds.$$

Hint: Note that $B_0 = 0$ a.s. Prove and apply Abel's summation by part formula:

$$\sum_{j=0}^{n-1} \Delta\left(a_j b_j\right) = \sum_{j=0}^{n-1} a_j \Delta b_j + \sum_{j=0}^{n-1} b_{j+1} \Delta a_j\ , \quad \Delta x_j = x_{j+1} - x_j.$$

Also note the alternate form, similar to the integration by parts formula:

$$\sum_{j=0}^{n-1} a_j \Delta b_j = a_n b_n - a_0 b_0 - \sum_{j=0}^{n-1} b_{j+1} \Delta a_j$$

$\boxed{5}$ (Øksendal 3:3.4) Check whether the following processes X_t are martingales w.r.t. $\{\mathcal{F}_t\}_t$.

(i) $X_t = B_t + 4t$

(ii) $X_t = B_t^2$

(iii) $X_t = t^2 B_t - 2\int_0^t sB_s ds$

(iv) $X_t = B_1(t)B_2(t)$ where (B_1, B_2) is 2-dimensional Brownian Motion.

$\boxed{6}$ Øksendal Exercise 4.4 (Ch 4)

$\boxed{7}$ Øksendal Exercise 4.7 (Ch 4)

$\boxed{8}$ Øksendal Exercise 4.8 a) (Ch 4)

$\boxed{9}$ Øksendal Exercise 4.11 (Ch 4)

$\boxed{10}$ Øksendal Exercise 4.13 (Ch 4)