Norwegian University of Science and Technology
Department of Mathematical Sciences

MA8109 Stochastic Processes and Differential Equations Autumn 2015

Exercise set #0

Please note: The notation used here may depart in some minor ways from the notation I generally use in the course. I hope it is clear enough, in any case.

$\boxed{1}$ Let X, Y, I be sets, and $f : X \to Y$ a function. Prove the following statements:

a) If $A, B \subset X$, then $(A \cap B)^c = A^c \cup B^c$.

b) If $A_i \subset X$ for $i \in I$, then $(\bigcup_{i\in I} A_i)^c = \bigcap_{i\in I} A_i^c$.

c) If $A \subset X$, then $f^{-1}(A^c) = f^{-1}(A)^c$.

d) If $A_i \subset X$ for $i \in I$, then $f^{-1}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f^{-1}(A_i)$.

Hint: $x \in \bigcup_{i\in I} A_i$ iff $\exists j \in I$ such that $x \in A_j$, and $x \in \bigcap_{i\in I} A_i$ iff $\forall i \in I$, $x \in A_i$.

$\boxed{2}$ Find the σ-algebra on a set X generated by:

a) $A, B \subset X$ where $A \cap B = \emptyset$.

b) $\mathcal{A} = \{\{i\}\}_{i\in\mathbb{N}}$ where $X = \mathbb{N}$.

$\boxed{3}$ (Øksendal, Problem 2.3)

Prove that for any collection $\{\mathcal{H}_i\}_{i\in I}$ of σ-algebras,

$$\mathcal{H} = \cap_{i\in I}\mathcal{H}_i$$

is a σ-algebra as well.

Hint: Verify all properties a σ-algebra should fulfill.

$\boxed{4}$ Let $(X, \mathcal{F}, m)$ be a measure space and $A_i \in \mathcal{F}$ for $i \in \mathbb{N}$. Prove that

a) If $A_1 \subset A_2$, then $m(A_1) \leq m(A_2)$.

b) $m(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} m(A_i)$.

c) If $A_1 \subset A_2 \subset A_3 \subset \dots$ and $A = \bigcup_{i=1}^{\infty} A_i$, then $m(A) = \lim_{i\to\infty} m(A_i)$.

Hint for b) and c): Make disjoint unions, use σ-additivity of m.

$\boxed{5}$ Let m_L be the Lebesgue measure on $\mathbb{R}$. Prove that $m_L(\mathbb{Q}) = 0$.

Hint: Let $\mathbb{Q} = \bigcup_{i=1}^{\infty}$ and consider the intervals $(q_i - 2^{-i}\epsilon, q_i + 2^{-i}\epsilon)$.

$\boxed{6}$ Let $f_1, f_2 : X \to \mathbb{R}$ be $\mathcal{F}$-measurable. Prove that $f(x) = \max(f_1(x), f_2(x))$ is $\mathcal{F}$-measurable.

Hint: Enough to prove that $f^{-1}((a, \infty)) \in \mathcal{F}$ for all $a \in \mathbb{R}$.

$\boxed{7}$ Let $(X, \mathcal{F}, m)$ be a measure space and $f_1, f_2 : A \to [0, \infty)$ $\mathcal{F}$-measurable. Prove that

$$\int (f_1 + f_2) dm = \int f_1 dm + \int f_2 dm.$$

Hint: Prove it for simple functions, approximate general functions, and go to the limit using MCT.

$\boxed{8}$ Let $(X, \mathcal{F}, m)$ be a measure space and $\phi : X \to [0, \infty)$ be $\mathcal{F}$-measurable. Prove that

$$\mu(\emptyset) = 0, \qquad \mu(A) = \int_A \phi \, dm \qquad \text{for any} \qquad A \in \mathcal{F},$$

defines a measure on $(X, \mathcal{F})$.

$\boxed{9}$ Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ a random variable on it. Define

$$\mu_X(A) = P(X^{-1}(A)) = P(\{X \in A\}) \qquad \text{for any} \qquad A \in \mathcal{F}.$$

a) Show that μ_X is a probability measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

Hint: $X^{-1}(\cap A_i) = \cap X^{-1}(A_i)$, $X^{-1}(\cup A_i) = \cup X^{-1}(A_i)$.

b) Let $f : \mathbb{R} \to [0, \infty)$ be $\mathcal{B}_{\mathbb{R}}$-measurable. Show that

$$E(f(X) = \int_X f(X)) \, dP(\omega) = \int_{\mathbb{R}} f(x) \, d\mu_X(x).$$

Hint: Prove it for simple functions, then approximate f, and go to the limit. You may use that $s_n \nearrow f$ then also $s_n \circ X \nearrow f \circ X$.