Convexity in Complex Geometry

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ABSTRACT. These notes were written for the "Advanced Complex Analysis" course at NTNU. We shall partially follow [H1] and mainly concentrate on the notion of convexity in complex geometry.

CONTENTS

The study of convexity in complex geometry has found many applications in other fields of mathematics. The followings are some selected applications of one crucial result (the Ohsawa-Takegoshi extension theorem) in the convexity theory.

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1: Suita conjecture. Bergman kernel and Green function are crucial notions in complex analysis. For the unit disk $\mathbb{D}$ in $\mathbb{C}$, we have

$$K_{\mathbb{D}}(z) = \frac{1}{\pi(1-|z|^2)^2}, \quad G_{\mathbb{D}}(z,w) = \log\left|\frac{z-w}{1-\bar{w}z}\right|.$$

Hence

$$\log(\pi K_{\mathbb{D}}(z)) = 2\lim_{w\to z}\{G_{\mathbb{D}}(z,w) - \log|z-w|\}.$$

The *Suita conjecture* is the following inequality:

$$\log(\pi K_{\Omega}(z)) \geq 2\lim_{w\to z}\{G_{\Omega}(z,w) - \log|z-w|\}, \tag{0.1}$$

where Ω is an arbitrary (smoothly bounded) domain in $\mathbb{C}$. This conjecture is solved and generalized by Blocki [Bl] and Guan-Zhou [GZ] using the following version of the Ohsawa-Takegoshi extension theorem [OT, BL]: for every given $z_0 \in \Omega$ there exists a holomorphic function f on Ω with $f(z_0) = 1$ and

$$\int_{\Omega} |f(x+iy)|^2 dxdy \leq \limsup_{t\to-\infty} e^{-t}\int_{G<t} dxdy$$

for every non-positive G on Ω with $G(z) - 2\log|z-z_0|$ subharmonic on Ω.

Exercise 1: show that (0.1) follows if we take $G(z) = 2G_{\Omega}(z,z_0)$.

2: Strong openness conjecture. Let ϕ be a plurisubharmonic function on a neighborhood of the origin in $\mathbb{C}^n$ and F be a holomorphic function near the origin. Assume that $|F|^2e^{-\phi}$ is integrable near the origin then Demailly and Kollar [DK] conjecture that $|F|^2e^{-p\phi}$ is integrable near the origin for some $p > 1$. This conjecture is a well known fact in case $n = 1$. For general n with $F = 1$, it is solved by Berndtsson [B3]. The most general case is proved by Guan-Zhou [GZ0] using the Ohsawa-Takegoshi extension theorem.

3: Corona problem. Another application of the Ohsawa-Takegoshi extension theorem is the *Skoda L^2-division theorem* [D, page 58]*: Let $g := (g_1, \cdots, g_r)$ be r holomorphic functions on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ with $|g|^2 := |g_1|^2 + \cdots |g_r|^2 \geq 1$ on $\mathbb{B}$. Set $m = \min\{n, r-1\}$. Then for every $\varepsilon > 0$ there exist holomorphic functions $(h_1, \cdots, h_r)$ on $\mathbb{B}$ such that*

$$g_1h_1 + \cdots + g_rh_r = 1 \tag{0.2}$$

and

$$\int_{\mathbb{B}} |h|^2|g|^{-2(m+\varepsilon)}d\lambda \leq \left(1+\frac{m}{\varepsilon}\right)\frac{\pi^n}{n!},$$

where $d\lambda$ denotes the Lebesgue measure. In case $n = 1$, Carleson [C] proved that there also exist *bounded* holomorphic functions $(h_1, \cdots, h_r)$ on $\mathbb{B}$ satisfying (0.2). Finding *bounded* holomorphic solution of (0.2) is known as the Corona problem. It is still an open problem for $n \geq 2$.

4: Bernstein-Kushnirenko theorem. The fourth application of the Ohsawa-Takegoshi extension theorem is the *Bergman kernel asymptotic formula*, which implies the following formula:

$$\lim_{m\to\infty} \frac{\dim H^0(X, mL)}{m^n/n!} = \#\{x \in X : f_1(x) = \cdots = f_n(x) = 0\}, \tag{0.3}$$

(see [B10, page 40])where L is an ample line bundle over an n-dimensional compact complex manifold X and f_j, $1 \leq j \leq n$, are generic holomorphic sections of L. In case X and L are defined by a Delzant polytope P, (0.3) implies the *Bernstein-Kushnirenko theorem (which holds true for a general convex polytope* P *in* $\mathbb{R}^n$ *with integral vertices, see* [Be, KK]*)*:

$$n!|P| = \#\{z \in (\mathbb{C}^*)^n : f_1(z) = \cdots = f_n(z) = 0\}, \tag{0.4}$$

for generic $f_1, \cdots, f_n \in \left\{\sum_{u \in P \cap \mathbb{Z}^n} c_u z^u : c_u \in \mathbb{C}\right\}$, where $|P|$ denotes the volume of P.

5: Bourgain-Milman theorem. The final application of the Ohsawa-Takegoshi extension theorem that I want to mention is the Berndtsson's subharmonicity of the Bergman kernel [B06, B09], which implies the *Bourgain-Milman theorem* [BM, B21]:

$$|K| \cdot |K^\circ| \geq (1.604)^{-n} \frac{\pi^n}{n!}, \tag{0.5}$$

where K denotes the unit ball of a norm $|| \cdot ||$ on $\mathbb{R}^n$, i.e.

$$K = \{x \in \mathbb{R}^n : ||x|| \leq 1\},$$

K° denotes the unit ball of the dual norm, i.e.

$$K^\circ = \{y \in \mathbb{R}^n : x \cdot y \leq 1, \ \forall\, x \in K\}.$$

The famous *Mahler conjecture* (still open in case $n \geq 4$) says that (0.5) still holds true if we change the right hand side to $4^n/n!$ (lecture on Tuesday, 22th August, week 34).

1. Convex analysis background

1.1. Convex set and convex function.

Definition 1.1. *Let*

$$\phi : A \to \mathbb{R}$$

be a function on a non-empty open set $A \subset \mathbb{R}^n$*. We say that* A *is convex if*

$$tx + (1-t)y \in A, \ \ \forall\, x, y \in A, \ 0 < t < 1.$$

Assume that A *is convex, we say that* ϕ *is convex if*

$$\phi\left(tx + (1-t)y\right) \leq t\phi(x) + (1-t)\phi(y) \ \ \forall\, x, y \in A, \ 0 < t < 1. \tag{1.1}$$

Lemma 1.1. $\phi : A \to \mathbb{R}$ *(*A *is convex) is convex if and only if*

$$\frac{\phi(ty + x) - \phi(x)}{t} \text{ is increasing in } t \text{ when } x, x + ty \in A \text{ and } t \neq 0. \tag{1.2}$$

Proof. Notice that (1.1) is equivalent to

$$\frac{\phi((1-t)(y-x) + x) - \phi(x)}{1-t} \leq \frac{\phi(y - x + x) - \phi(x)}{1}.$$

Write $(y - x) = sz$, the above inequality gives

$$\frac{\phi((1-t)sz + x) - \phi(x)}{(1-t)|s|} \leq \frac{\phi(sz + x) - \phi(x)}{|s|}.$$

Hence (1.1) is equivalent to (1.2) with $t > 0$ or $t < 0$. But since (1.1) also gives

$$\frac{\phi(ty+x)-\phi(x)}{t} \geq \frac{\phi(-ty+x)-\phi(x)}{-t}, \ t>0.$$

We know that (1.1) is equivalent to (1.2). □

Proposition 1.2. *Assume that ϕ is smooth. Then ϕ is convex if and only if the Hessian matrix (ϕ_{jk}) is positive semi-definite.*

Proof. Notice that (1.2) implies that the derivative of

$$\psi(t) := \phi(ty+x)$$

is increasing. Thus if ϕ is smooth and convex then

$$\psi_{tt}(0) = \sum \phi_{jk}(x) y^j y^k \geq 0.$$

On the other hand, if (ϕ_{jk}) is positive semi-definite then $\psi_{tt} \geq 0$ for all t such that $ty + x \in A$, which implies that

$$\frac{d}{dt}\left(\frac{\phi(ty+x)-\phi(x)}{t}\right) = \frac{d}{dt}\int_0^1 \psi'(ts)ds = \int_0^1 \psi''(ts)sds \geq 0,$$

hence (1.2) follows. □

Proposition 1.3. *Let ϕ be a convex function on a non-empty open set $A \subset \mathbb{R}^n$. Then ϕ is locally Lipschitz continuous.*

Proof. We shall follow the proof of Hörmander in [H2, Theorem 2.1.22, page 55]. For every finite set $X := \{x_1, \cdots, x_N\}$ in A, let us denote by $\mathrm{ch}(X)$ its convex hull, then using (1.1) inductively we have

$$\sup_X \phi = \sup_{\mathrm{ch}(X)} \phi.$$

Let K be a compact subset of A. Fix $\varepsilon > 0$ such that

$$K_\varepsilon := \{z \in \mathbb{R}^n : |z-x| \leq \varepsilon \text{ for some } x \in K\} \subset A.$$

Since A is convex, we can find X such that $K_\varepsilon \subset \mathrm{ch}(X)$. In particular

$$x+y, x-y \in \mathrm{ch}(X), \ \ \forall\, x \in K, \ |y| \leq \varepsilon,$$

which gives

$$\phi(x) - \sup_X \phi \leq \phi(x) - \phi(x-y); \ \ \phi(x+y) - \phi(x) \leq \sup_X \phi - \phi(x).$$

Notice that (1.2) implies

$$\frac{\phi(-y+x)-\phi(x)}{-1} \leq \frac{\phi(ty+x)-\phi(x)}{t} \leq \frac{\phi(y+x)-\phi(x)}{1}$$

for every $-1 < t < 1$, hence

$$\left|\frac{\phi(ty+x)-\phi(x)}{t}\right| \leq |\phi(x) - \sup_X \phi|,$$

from which we know that ϕ is locally Lipschitz continuous (in particular, ϕ is continuous). □

Remark. *The above proof in fact implies that*

$$|\phi(x) - \phi(y)| \leq L\,|x-y|, \ \ \forall\, x, y \in K.$$

where $L := \frac{\sup_{K_\varepsilon}\phi - \inf_K \phi}{\varepsilon} < \infty$ *since* ϕ *is proved to be continuous (lecture on 25th August).*

1.2. **Brunn-Minkowski inequality and isoperimetric inequality.**

Theorem 1.4 (Brunn-Minkowski inequality)**.** *Let* A_1, A_2 *be bounded non-empty convex open sets in* $\mathbb{R}^n$*. Then*

$$|A_1 + A_2|^{\frac{1}{n}} \geq |A_1|^{\frac{1}{n}} + |A_2|^{\frac{1}{n}},$$

where $A_1 + A_2 := \{x + y : x \in A_1,\ y \in A_2\}$ *denotes the Minkowski sum.*

Exercise 2: Show that the Brunn-Minkowski inequality is equivalent to that for every bounded non-empty convex open sets A_1, A_2 in $\mathbb{R}^n$: $-|tA_1 + (1-t)A_2|^{\frac{1}{n}}$ is convex in $t \in (0,1)$.

Remark: *In case* $A_1 = A$ *has smooth boundary and* $A_2 = s\mathbb{B}$*, where* $\mathbb{B}$ *is the unit ball and* s *is a small positive number, the Brunn-Minkowski inequality gives*

$$|A + s\mathbb{B}|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |s\mathbb{B}|^{\frac{1}{n}} = |A|^{\frac{1}{n}} + s|\mathbb{B}|^{\frac{1}{n}},$$

which implies

$$\lim_{s\to 0+} \frac{|A + s\mathbb{B}|^{\frac{1}{n}} - |A|^{\frac{1}{n}}}{s} \geq |\mathbb{B}|^{\frac{1}{n}}.$$

On the other hand, if we put $f(s) = |A + s\mathbb{B}|$ *then*

$$f'(0+) = |\partial A|,$$

where $|\partial A|$ *denotes the* $(n-1)$*-dimensional volume of the boundary* ∂A *of* A*. Hence*

$$\lim_{s\to 0+} \frac{|A + s\mathbb{B}|^{\frac{1}{n}} - |A|^{\frac{1}{n}}}{s} = (f^{1/n})'(0+) = \frac{1}{n} f^{(1-n)/n}(0) f'(0+) = \frac{|\partial A|}{n|A|^{(n-1)/n}},$$

and we have

$$\frac{|\partial A|}{n|A|^{(n-1)/n}} \geq |\mathbb{B}|^{\frac{1}{n}}.$$

Note that $|\partial\mathbb{B}| = n|\mathbb{B}|$*, the above inequality gives the following classical isoperimetric inequality for convex sets.*

Isoperimetric inequality. *Let* A *be a smoothly bounded convex open set in* $\mathbb{R}^n$*. Then*

$$\frac{|\partial A|}{|A|^{(n-1)/n}} \geq \frac{|\partial \mathbb{B}|}{|\mathbb{B}|^{(n-1)/n}}.$$

1.3. **Legendre transform, gradient map and convex exhaustion functions.**

Definition 1.2 (Legendre transform)**.** *Let ψ be a convex function on a bounded non-empty convex open set $A \subset \mathbb{R}^n$. We call*

$$\psi^*(y) := \sup_{x \in A} x \cdot y - \psi(x),\ x \cdot y := \sum_{j=1}^n x^j y^j,$$

the Legendre transform of ψ (with respect to A).

Proposition 1.5. *Let ψ be a smooth strictly convex exhaustion function on a bounded non-empty convex open set $A \subset \mathbb{R}^n$ (exhaustion means that ψ tends to infinity at the boundary of A, more precisely, it means that for every $c \in \mathbb{R}$, the closure of $\{\psi < c\}$ is a bounded subset of A; strictly convex means that the Hessian matrix is positive definite). Then its Legendre transform ψ^* is also smooth, strictly convex, moreover the gradient map of ψ^**

$$\nabla\psi^* : y \mapsto x = \nabla\psi^*(y) := (\partial\psi^*/\partial y^1, \cdots, \partial\psi^*/\partial y^n), \tag{1.3}$$

defines a diffeomorphism from $\mathbb{R}^n$ onto A.

Proof. It is enough to prove that the gradient map of ψ defines a diffeomorphism from A to $\mathbb{R}^n$, ψ^* is smooth and $\nabla\psi^*$ is the inverse of $\nabla\psi$.

Step 1: $\nabla\psi$ is a diffeomorphism from A to $\mathbb{R}^n$. Since ψ is smooth and strictly convex, we know that $\nabla\psi$ is a local diffeomorphism.

1. $\nabla\psi$ is injective: assume that $\nabla\psi(x_1) = \nabla\psi(x_2) = y_0$, consider

$$\psi^{y_0}(x) := \psi(x) - y_0 \cdot x, \tag{1.4}$$

we know that ψ^{y_0} is smooth, strictly convex and

$$\nabla\psi^{y_0}(x_1) = \nabla\psi^{y_0}(x_2) = 0. \tag{1.5}$$

Consider the restriction, say g, of ψ^{y_0} to the line determined by x_1 and x_2, then g is convex with critical points x_1 and x_2. Thus g is a constant on the line segment from x_1 to x_2, moreover, strict convexity of g implies $x_1 = x_2$. Thus $\nabla\psi$ is injective.

2. $\nabla\psi(A) = \mathbb{R}^n$: fix $y \in \mathbb{R}^n$, since ψ^y tends to infinity at the boundary of A, strict convexity of ψ implies that ψ^y has a unique minimum point, say $x \in A$. Thus

$$0 = \nabla\psi^y(x) = \nabla\psi(x) - y.$$

Step 2: ψ^* is smooth. Notice that

$$\psi^*(\nabla\psi(x)) = \nabla\psi(x) \cdot x - \psi(x). \tag{1.6}$$

Thus $\psi^* \circ \nabla\psi$ is a smooth, which implies that ψ^* is smooth on $\mathbb{R}^n$.

Step 3: $\nabla\psi^*$ is the inverse of $\nabla\psi$. Apply the differential to (1.6), we get that

$$(\nabla\psi^* \circ \nabla\psi(x)) \cdot (\psi_{jk}) = x \cdot (\psi_{jk}),\ \forall\, x \in A. \tag{1.7}$$

Since (ψ_{jk}) is an invertible matrix function, the above formula gives $\nabla\psi^* \circ \nabla\psi = Id$. □

Exercise 3: (1) Let ϕ be a smooth strictly convex function on $\mathbb{R}^n$. Show that $\nabla\phi$ defines a diffeomorphism from $\mathbb{R}^n$ to $\nabla\phi(\mathbb{R}^n)$, ϕ^* is smooth strictly convex on $\nabla\phi(\mathbb{R}^n)$ and $\nabla\phi^*$ defines a diffeomorphism from $\nabla\phi(\mathbb{R}^n)$ to $\mathbb{R}^n$.

(2) Let A be a non-empty open set in $\mathbb{R}^n$. Use the following proposition to show that A is convex if and only if there exists a smooth convex exhaustion function on A.

Proposition 1.6. *Let A be a non-empty convex open set in $\mathbb{R}^n$. Then there exists a real analytic strictly convex exhaustion function on A.*

Proof. Inspired by [B21, Proposition 3.2], we shall look at the following "Bergman kernel" type function (we omit the Lebesgue measure in the integral)

$$B(x) := \int_{\mathbb{R}^n_t} \frac{e^{2t\cdot x}}{\int_{y\in A} e^{2t\cdot y-|y|^2}}, \tag{1.8}$$

which is always strictly convex and real analytic in A. Since

$$B(x) \geq \int_{\mathbb{R}^n_t} \frac{e^{2t\cdot x}}{\int_{\mathbb{R}^n_y} e^{2t\cdot y-|y|^2}} = \pi^n e^{|x|^2}, \tag{1.9}$$

we know that B is an exhaustion function when $A=\mathbb{R}^n$. In case $A\neq\mathbb{R}^n$, then A must have a boundary point. Take $x\in A$ such that $d(x,\partial A)=\varepsilon$, by a rotation, one may assume that

$$d(x,\partial A)=|x-x_0|,\ x_0=(|x_0|,0,\cdots,0),\ x=(|x_0|-\varepsilon,0,\cdots,0),\ A\subset\{x_1<|x_0|\},$$

which implies

$$B(x)\geq \int_{\mathbb{R}^n_t} \frac{e^{2t_1(|x_0|-\varepsilon)}}{\int_{y_1<|x_0|} e^{2t\cdot y-|y|^2}} = \pi^{n-1}\int_{\mathbb{R}_t} \frac{e^{2t(|x_0|-\varepsilon)}}{\int_{y<|x_0|} e^{2ty-y^2}} \geq \pi^{n-1}\int_{t>0} \frac{e^{2t(|x_0|-\varepsilon)}}{\int_{-\infty}^{|x_0|} e^{2ty}} = \frac{\pi^{n-1}}{2\varepsilon^2}.$$

Hence

$$B(x)\geq \pi^n \max\{e^{|x|^2}, (2\pi)^{-1} d(x,\partial A)^{-2}\},$$

from which we know that B is strictly convex, real analytic and exhaustion in A. □

Exercise 4: Prove (1.9).

1.4. **Mixed volume and Alexandrov-Fenchel inequality.** Let A be a bounded non-empty convex open set in $\mathbb{R}^n$. By Proposition 1.6, there exists a real analytic strictly convex exhaustion function, say ψ, on A. Put $\phi=\psi^*$, then Proposition 1.5 implies that $\nabla\phi$ is a diffeomorphism from $\mathbb{R}^n$ onto A, thus we can write the volume $|A|$ of A as

$$|A| = \int_A dy = \int_{\mathbb{R}^n} MA(\phi)\, dx,\ dx := dx^1\wedge\cdots\wedge dx^n,\ dy := dy^1\wedge\cdots\wedge dy^n. \tag{1.10}$$

where $MA(\phi) := \det(\phi_{jk})$ denotes the determinant of the Hessian of ϕ.

Exercise 5: Use the change of variable $y=\nabla\phi(x)$ to prove (1.10).

The following proposition is a generalization of (1.10).

Proposition 1.7. *Let $\phi_1, \cdots, \phi_N$ be smooth strictly convex functions such that each $\nabla\phi_j$ is a diffeomorphism from $\mathbb{R}^n$ onto a bounded convex open set A_j. Then we have*

$$|t_1A_1 + \cdots + t_NA_N| = \int_{\mathbb{R}^n} MA(t_1\phi_1 + \cdots + t_N\phi_N)\, dx, \ t_j > 0, \ \forall\, 1 \le j \le N. \tag{1.11}$$

Proof. By induction on N, it suffices to show that

$$\nabla(\phi_1 + \phi_2)(\mathbb{R}^n) = A_1 + A_2. \tag{1.12}$$

Obviously we have $\nabla(\phi_1+\phi_2)(\mathbb{R}^n) \subset A_1+A_2$. Thus it is enough to show that for every $y_1 \in A_1$ and every $y_2 \in A_2$, there exists $x_0 \in \mathbb{R}^n$ such that $\nabla(\phi_1 + \phi_2)(x_0) = y_1 + y_2$. Consider $\phi_j^{y_j}$ instead of ϕ_j, one may assume that $y_1 = y_2 = 0$. Choose x_1 and x_2 such that

$$\nabla\phi_1(x_1) = \nabla\phi_2(x_2) = 0. \tag{1.13}$$

Since ϕ_j is convex, we know that each x_j is the minimum point of ϕ_j. Thus strict convexity of ϕ_j implies that

$$\phi_j(x) \to \infty, \ \text{as } |x| \to \infty, \tag{1.14}$$

i.e. each ϕ_j is proper. Thus $\phi_1 + \phi_2$ is also proper. Hence there exists a unique minimum point, say x_0, of $\phi_1 + \phi_2$. Thus $\nabla(\phi_1 + \phi_2)(x_0) = 0$. The proof is complete. □

Remark: *The above proposition implies that*

$$p(t) := |t_1A_1 + \cdots + t_nA_n|,$$

is a polynomial of degree n. We call the coefficient of $t_1 \cdots t_n$ in the polynomial $p(t)$, i.e.

$$V(A_1, \cdots, A_n) := \frac{\partial^n |t_1A_1 + \cdots + t_nA_n|}{\partial t_1 \cdots \partial t_n}, \tag{1.15}$$

the mixed volume *of $A_1, \cdots, A_n$.*

Exercise 6: Show that (1.11) implies that $|t_1A_1 + \cdots + t_nA_n|$ is a polynomial of degree n in t and $V(A, \cdots, A) = n!|A|$.

Reading task 1: Read page 12-13 of [B14] for the related mixed discriminant of matrices.

Theorem 1.8 (Alexandrov-Fenchel inequality)**.** *Let $A_1, \cdots, A_n$ be bounded non-empty convex open sets in $\mathbb{R}^n$. Assume that $n \ge 2$. Then*

$$V(A_1, \cdots, A_n)^2 \ge V(A_1, A_1, A_3, \cdots, A_n)V(A_2, A_2, A_3, \cdots, A_n).$$

(5th September, no lecture on 29th August).

2. An invitation to toric varieties

For references on toric varieties, see [F, O, CLS], in particular, the readers can try to use Chapter 1-2 in [CLS] to generalize the following discussion to non-smooth toric varieties. We will only look at smooth toric varieties, they are called toric manifolds, which are special compact complex manifolds that possess $(\mathbb{C}^*)^n$ actions. A nice reference on compact complex manifolds is the famous book of Kodaira (see Chapter 2, especially section 2.2 in [K]). Compact complex manifolds are higher dimensional generalizations of compact Riemann surfaces.

Definition 2.1. *A compact topological space X is called a compact Riemann surface if it possesses a finite open covering*

$$X := \cup_{1\leq j\leq N} U_j,$$

and homeomorphism σ_j from U_j onto a domain in $\mathbb{C}$ such that

$$\sigma_k \circ \sigma_j^{-1} : \sigma_j(U_j \cap U_k) \to \sigma_k(U_j \cap U_k)$$

is conformal as long as $U_j \cap U_k \neq \emptyset$.

Example. *The complex projective space $\mathbb{P}^1 := (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*$ and the elliptic curves $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, where $\tau \in \mathbb{C}$ and $\operatorname{Im}\tau > 0$.*

Definition 2.2. *A compact topological space X is called an n-dimensional compact toric manifold if it possesses a finite open covering*

$$X := \cup_{1\leq v\leq l} U_v,$$

and homeomorphism Φ_v from U_v onto $\mathbb{C}^n$ such that

$$\Phi_{v_1} \circ \Phi_{v_2}^{-1} : \Phi_{v_2}(U_{v_1} \cap U_{v_2}) \to \Phi_{v_1}(U_{v_1} \cap U_{v_2})$$

are monomial isomorphisms, i.e. each $\Phi_{v_1} \circ \Phi_{v_2}^{-1}$ is of the type

$$u \mapsto (u^{\lambda_1}, \cdots, u^{\lambda_n}), \ \ \lambda_j \in \mathbb{Z}^n, \ \ u^{\lambda_j} = u_1^{\lambda_{j1}} \cdots u_n^{\lambda_{jn}},$$

and $\Phi_{v_2}(U_{v_1} \cap U_{v_2}) \subset \mathbb{C}^n$ is the collection of all $u \in \mathbb{C}^n$ such that $u_k^{\lambda_{jk}}$ are holomorphic.

Remark. *It is clear that $(\mathbb{C}^*)^n \subset \Phi_{v_2}(U_{v_1} \cap U_{v_2}) = (\mathbb{C}^*)^k \times \mathbb{C}^{n-k}$. The standard example is*

$$\mathbb{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*.$$

In this section, we shall study how to construct compact toric manifolds using convex polytopes.

2.1. **Toric variety and toric line bundle associated to a Delzant polytope.**

Definition 2.3. *Fix $\alpha_j \in \mathbb{Z}^n$, $r_j \in \mathbb{Z}$, $1 \leq j \leq N$, we call the associated convex set*

$$P := \{x \in \mathbb{R}^n : \alpha_j \cdot x - r_j \leq 0, \ 1 \leq j \leq N\}$$

a Delzant polytope if

(1) *P is bounded with non-empty interior;*

(2) *for every vertex v of P, the associated index set*

$$I_v := \{1 \leq j \leq N : \alpha_j \cdot v - r_j = 0\}$$

has precisely n elements and $\{\alpha_j\}_{j\in I_v}$ generates $\mathbb{Z}^n$;

(3) $\{1, \cdots, N\} = \cup_{v \text{ is a vertex of } P} I_v$. *(8th September)*

Exercise 7: Show that every vertex v of a Delzant polytope P is integral, i.e. $v \in \mathbb{Z}^n$.

Exercise 8 (hard): Show that the gradient map

$$\nabla\phi : x \mapsto (\phi_{x_1}(x), \cdots, \phi_{x_n}(x)) \tag{2.1}$$

of the convex function

$$\phi(x) := \log\left(\sum_{u\in P\cap\mathbb{Z}^n} e^{u\cdot x}\right) \tag{2.2}$$

on $\mathbb{R}^n$ satisfies

$$\nabla\phi(\mathbb{R}^n) = \text{the interior of } P. \tag{2.3}$$

Remark: *If P is Delzant then one may recover $\{(\alpha_j, r_j)\}_{1\leq j\leq N}$ from P. For each vertex v of P, one may define a convex cone*

$$\sigma_v := \left\{\sum_{j\in I_v} t_j\alpha_j : t_j \geq 0,\ \forall\, j \in I_v\right\} \tag{2.4}$$

generated by $\{\alpha_j\}_{j\in I_v}$ and its polar

$$\sigma_v^\circ := \{x \in \mathbb{R}^n : \alpha\cdot x \leq 0,\ \forall\, \alpha \in \sigma_v\}. \tag{2.5}$$

Then one may prove the following

Proposition 2.1. *The polar cone σ_v° is generated by the corner of P around the vertex v.*

Exercise 9: Use Definition 2.3 (2) to show that the solution $\{\beta_k\}_{k\in I_v}$ of

$$\alpha_j\cdot\beta_k + \delta_{jk} = 0, \quad j, k \in I_v,$$

satisfies $\beta_k \in \mathbb{Z}^n$ for all $k \in I_v$ and defines a basis of $\mathbb{Z}^n$; moreover σ_v° is generated by $\{\beta_k\}_{k\in I_v}$.

Exercise 10: Write $I_v = \{k_1, \cdots k_n\}$, use Exercise 9 to prove that

$$\Phi_v : z \mapsto u = \Phi_v(z) := (z^{\beta_{k_1}}, \cdots, z^{\beta_{k_n}}), \quad z^t := z_1^{t_1}\cdots z_n^{t_n}, \tag{2.6}$$

defines a one to one mapping from $(\mathbb{C}^*)^n$ onto $(\mathbb{C}^*)^n$ (we call Φ_v a monomial isomorphism).

Note that each isomorphism Φ_v in (2.6) defines an embedding (called torus embedding)

$$\Phi_v : (\mathbb{C}^*)^n \to \mathbb{C}^n. \tag{2.7}$$

The Delzant toric variety X_P is defined by gluing those embeddings via (the maximal extension of) $\Phi_{v_1}\circ\Phi_{v_2}^{-1}$, more precisely, we have

$$X_P = \left(\cup_{v \text{ is a vertex of P}}\, \mathbb{C}^n \times \{v\}\right)/\sim, \tag{2.8}$$

where $(u_1, v_1) \sim (u_2, v_2)$ if and only if

$$\Phi_{v_1}\circ\Phi_{v_2}^{-1}(u_2) = u_1. \tag{2.9}$$

One may verify that X_P is a complex manifold covered by l copies (l denotes the number of vertex of P) of $\mathbb{C}^n$. From the definition, we also know that X_P is fully determined by α_j. (12th September — Exercise 8 will be discussed in 15th)

Exercise 11: Show that $X_P \simeq \mathbb{P}^1$ if $P = [0, m]$ for some positive integer m.

Exercise 12: Find P_1, P_2 such that $X_{P_1} \simeq \mathbb{P}^2,\ X_{P_2} \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

Remark: *Note that all X_P in the above exercises are compact. In fact, one may prove that all X_P are compact. This fact is not obvious. One proof is to use the gradient map* (2.1).

Exercise 13 (hard): Consider $\boldsymbol{\nabla}\phi : (\mathbb{C}^*)^n \to P$ defined by

$$\boldsymbol{\nabla}\phi(z) := \nabla\phi(\log(|z_1|^2), \cdots, \log(|z_n|^2)), \tag{2.10}$$

show that $\boldsymbol{\nabla}\phi$ has a unique proper, smooth and surjective extension to X_P

$$\boldsymbol{\nabla}\phi : X_P \to P, \tag{2.11}$$

where "proper" means that the preimage of every compact set is compact. (15th September)

Hint for Exercise 13: Fix a vertex v of P, one may write

$$\phi(x) = v \cdot x + \log\left(\sum_{u\in(P-v)\cap\mathbb{Z}^n} e^{u\cdot x}\right).$$

Note that every $u \in (P - v) \cap \mathbb{Z}^n$ can be uniquely written as

$$u = \sum_{j\in I_v} c_j^u \beta_j, \;\; c_j^u \in \mathbb{Z}_{\geq 0},$$

where β_j are generators of σ_v° defined in Exercise 9. Thus we have

$$\nabla\phi(x) = v + \frac{\sum_{j\in I_v} c_j\beta_j}{\sum_{u\in(P-v)\cap\mathbb{Z}^n} e^{u\cdot x}}, \quad c_j := \sum_{u\in(P-v)\cap\mathbb{Z}^n} c_j^u e^{u\cdot x}.$$

Since $\{\beta_j\}_{j\in I_v}$ defines a $\mathbb{Z}$-basis of $\mathbb{Z}^n$, one may assume that it is the canonical basis of $\mathbb{Z}^n$ (try!) so that $I_v = \{1, \cdots, n\}$,

$$x \cdot \beta_j = x_j, \;\; c_j^u = u_j.$$

Moreover, after a translation of P, we can assume that $v = 0$. Then we have

$$\boldsymbol{\nabla}\phi(z) = \frac{\sum_{u\in P\cap\mathbb{Z}^n} u|z^u|^2}{\sum_{u\in P\cap\mathbb{Z}^n} |z^u|^2}.$$

Write $z' = (z_2, \cdots, z_n)$, we know $\boldsymbol{\nabla}\phi|_{z_1=0}$ can be written as

$$\boldsymbol{\nabla}\phi(0, z') = \frac{\sum_{u'\in P_1\cap\mathbb{Z}^{n-1}} u'|(z')^{u'}|^2}{\sum_{u'\in P_1\cap\mathbb{Z}^{n-1}} |(z')^{u'}|^2},$$

where

$$P_1 := \{x' \in \mathbb{R}^{n-1} : (0, x') \in P\}. \tag{2.12}$$

We observe that $\boldsymbol{\nabla}\phi(0, z')$ is precisely the mapping associated to the face P_1 of P. Hence one may do induction on n (the $n = 1$ case is explained in the class, try!). In this way, we obtain that $\boldsymbol{\nabla}\phi(X_P) = P$. The fact that $0 \in P - v$ for every vertex v implies that $\boldsymbol{\nabla}\phi$ is smooth as a map from X_P to $\mathbb{R}^n$. Denote by $T = \mathbb{R}/\mathbb{Z}$ the one-dimensional torus. One may verify that

$\nabla\phi^{-1}(p) \simeq T^n$ for all p in the interior of P and $\nabla\phi^{-1}(v) \simeq T^0$ for all vertices v of P. In general, $\nabla\phi^{-1}(x) \simeq T^k$ if x lies in a k dimensional open face of P. Thus the inverse of $\nabla\phi$ induces a homeomorphism, say

$$\nabla\phi^* : (P \times T^n)/\sim \ \to X_P, \tag{2.13}$$

where $(x, [a]) \sim (x, [b])$ ($a, b \in \mathbb{R}^n$ so that $[a], [b] \in \mathbb{R}^n/\mathbb{Z}^n = T^n$) if and only if

$$a - b \in \sum_{j \in I_x} c_j \alpha_j, \text{ for some } c_j \in \mathbb{R},$$

where

$$I_x := \{1 \leq j \leq N : \alpha_j \cdot x - r_j = 0\}.$$

Since $P \times T^n$ is compact, we know X_P is compact and $\nabla\phi$ is proper. (19th September)

Another proof of the compactness of X_P is to use the following fact.

Lemma 2.2. *With the notation in (2.4), we have*

$$\cup_{v \text{ vertex of } P} \ \sigma_v = \mathbb{R}^n. \tag{2.14}$$

Proof. Consider the support function of P defined by

$$h_P(\alpha) := \sup_{x \in P} \alpha \cdot x, \ \ \alpha \in \mathbb{R}^n$$

For a vertex $v \in P$, we observe that $h_P(\alpha) = \alpha \cdot v$ if and only if

$$\alpha \cdot (x - v) \leq 0, \ \ \forall\, x \in P.$$

Since $P - v$ generates the polar cone σ_v°, we obtain

$$\{\alpha \in \mathbb{R}^n : h_P(\alpha) = \alpha \cdot v\} = \sigma_v.$$

Thus the lemma follows from $h_P(\alpha) := \sup_{v \text{ vertex of } P} \alpha \cdot v$. □

Theorem 2.3. *X_P is covered by l (l denotes the number of vertex of P) closed polydiscs*

$$X_P = \cup_{v \text{ vertex of } P} \ C_v, \tag{2.15}$$

where each C_v is a polydisc in $\mathbb{C}^n \times \{v\}$ defined by

$$C_v := \{(u, v) \in \mathbb{C}^n \times \{v\} : |u_j| \leq 1, \ 1 \leq j \leq n\}.$$

In particular X_P is compact.

Proof. By induction on n, it suffices to show that

$$(\mathbb{C}^*)^n = \cup_{v \text{ vertex of } P} \ (\mathbb{C}^*)^n \cap C_v,$$

i.e. (with respect to the notation in (2.6))

$$(\mathbb{C}^*)^n = \cup_{v \text{ vertex of } P} \ \{z \in (\mathbb{C}^*)^n : |z^{\beta_k}| \leq 1, \ k \in I_v\},$$

or equivalently (write $x_j = \log |z_j|^2$)

$$\mathbb{R}^n = \cup_{v \text{ vertex of } P} \ \{x \in \mathbb{R}^n : x \cdot \beta_k \leq 0, \ k \in I_v\}.$$

Since $\{\beta_k\}_{k\in I_v}$ generated σ_v°, we have

$$\{x \in \mathbb{R}^n : x \cdot \beta_k \leq 0, \ k \in I_v\} = \sigma_v.$$

Thus our theorem follows from Lemma 2.2. □

Remark: *In (2.12), $(\nabla\phi)^{-1}(P_1) = X_{P_1}$ is an $(n-1)$-dimensional compact toric manifold defined by the Delzant polytope P_1. It is also a subset of X_P, with respect to the z-coordinate , it is defined by $z_1 = 0$. We call it a divisor of X_P. In general, if F is an $(n-1)$-dimensional face (also called facet) of P then $(\nabla\phi)^{-1}(F)$ is an $(n-1)$-dimensional compact toric submanifold of X_P. In case F is given by $\alpha_j \cdot x = r_j$, we shall write*

$$Z_{\alpha_j} := (\nabla\phi)^{-1}(F) \tag{2.16}$$

and call it the α_j divisor of X_P.

One may similarly define the Delzant line bundle L_P over our Delzant toric variety X_P, the idea is to look at the embedding of $(\mathbb{C}^*)^n \times \mathbb{C}$

$$\Psi_v : (\mathbb{C}^*)^n \times \mathbb{C} \to \mathbb{C}^n \times \mathbb{C}, \tag{2.17}$$

where

$$\Psi_v(z, \xi) := (\Phi_v(z), z^{-v}\xi) \tag{2.18}$$

The Delzant line bundle L_P is defined by gluing those embeddings via (the maximal extension of) $\Psi_{v_1} \circ \Psi_{v_2}^{-1}$. (22th September)

Proposition 2.4. *Every $u \in P \cap \mathbb{Z}^n$ defines a holomorphic section, say s_u, of L_P over X_P.*

Proof. It suffices to show that the section

$$(\mathbb{C}^*)^n \to (\mathbb{C}^*)^n \times \mathbb{C}$$

defined by $z \mapsto (z, z^u)$ extends holomorphically over each Ψ_v embedding. Since

$$\Psi_v(z, z^u) = (\Phi_v(z), z^{u-v}),$$

we need to prove that z^{u-v} is holomorphic with respect to the $z^{\beta_{k_1}}, \cdots, z^{\beta_{k_n}}$ coordinates, i.e. $u - v$ is lies in the polar cone σ_v° generated by the corner of P around v, which follows from

$$P - v \in \sigma_v^\circ = \{x \in \mathbb{R}^n : \alpha_j \cdot x \leq 0, \ \forall\, j \in I_v\}.$$

□

Exercise 14: Show that $P - v \in \sigma_v^\circ$.

Exercise 15: Assume that $0 \in P$, show that the divisor given by $s_0 = 0$ can be written as

$$[s_0 = 0] = \sum_{1\leq j\leq N} r_j Z_{\alpha_j} \tag{2.19}$$

with respect to the notation (2.16).

2.2. **$(\mathbb{C}^*)^n$-action and holomorphic sections of Delzant line bundles.** In this section, we shall use the structure theorem for $(\mathbb{C}^*)^n$-action to prove the following converse of Proposition 2.4.

Theorem 2.5. *Let P be a Delzant polytope. Then $\{s_u\}_{u\in P\cap\mathbb{Z}^n}$ defines a basis of the space $H^0(X_P, L_P)$ of holomorphic sections of L_P over X_P, i.e.*

$$H^0(X_P, L_P) = \mathrm{Span}_{\mathbb{C}}\{s_u\}_{u\in P\cap\mathbb{Z}^n}. \tag{2.20}$$

Proof. The natural $(\mathbb{C}^*)^n$ action on $(\mathbb{C}^*)^n \times \mathbb{C}$ defined by

$$\rho(t)(z,\xi) = (t_1 z_1, \cdots, t_n z_n, \xi), \quad t \in (\mathbb{C}^*)^n, \tag{2.21}$$

induces a $(\mathbb{C}^*)^n$ action ρ on $H^0(X_P, L_P)$:

$$(\rho(t)s)(\rho(t)z) = \rho(t)(s(z)), \;\; u \in H^0(X_P, L_P). \tag{2.22}$$

Hence, if $s(z) = (z, \sum_{u\in\mathbb{Z}^n} c_u z^u)$ is the laurent series expansion of $s \in H^0(X_P, L_P)$ then

$$(\rho(t)s)(z) = \left(z, \sum_{u\in\mathbb{Z}^n} c_u t^{-u} z^u\right), \quad t \in (\mathbb{C}^*)^n. \tag{2.23}$$

The eigenvectors associated to this action are precisely those monomial sections s_u defined by z^u for $u \in P \cap \mathbb{Z}^n$. Hence our theorem follows from Theorem 2.7 below. □

Definition 2.4. *A $(\mathbb{C}^*)^n$ action on $\mathbb{C}^m$ is a holomorphic group homomorphism*

$$\rho : (\mathbb{C}^*)^n \to GL(m, \mathbb{C}),$$

where $GL(m, \mathbb{C})$ denote the space of $\mathbb{C}$-linear isomorphisms on $\mathbb{C}^m$.

Put $\mathbb{T} = \{t \in \mathbb{C}^* : |t| = 1\}$, then $\mathbb{T}^n$ is a subgroup of $(\mathbb{C}^*)^n$. Via the argument map

$$e^{2\pi i x} := (e^{2\pi i x_1}, \cdots, e^{2\pi i x_1}) \mapsto ([x_1], \cdots, [x_n]) \in (\mathbb{R}/\mathbb{Z})^n,$$

one may identify $\mathbb{T}^n$ with $(\mathbb{R}/\mathbb{Z})^n$. Thus

$$dx := dx_1 \cdots dx_n$$

defines a Haar probability measure on $\mathbb{T}^n$. Since ρ is holomorphic, we know that ρ is uniquely determined by its restriction to $\mathbb{T}^n$.

Definition 2.5. *A $\mathbb{T}^n$ action on $\mathbb{C}^m$ is a continuous group homomorphism*

$$\rho : \mathbb{T}^n \to GL(m, \mathbb{C}).$$

We shall also call $(\rho, \mathbb{C}^m)$ a finite-dimensional complex representation of $\mathbb{T}^n$ and denote by $\chi_\rho(t) := \mathrm{tr}\,\rho(t)$ the character of ρ. A subspace V of $\mathbb{C}^m$ is said to be ρ-invariant if $\rho(\mathbb{T}^n)V = V$. $(\rho, \mathbb{C}^m)$ is called irreducible if $\mathbb{C}^m$ has no proper non-zero ρ-invariant subspaces.

The main result that we want to prove is the following.

Theorem 2.6. *Let ρ be a $\mathbb{T}^n$ action on $\mathbb{C}^m$, then there exists a basis $\{e_j\}_{1\le j\le m}$ of $\mathbb{C}^m$ and $\lambda_j \in \mathbb{Z}^n$ such that*

$$\rho(t)e_j = t^{\lambda_j} e_j, \quad 1 \le j \le m, \quad t \in \mathbb{T}^n, \tag{2.24}$$

where $t^{\lambda_j} = t_1^{\lambda_{j1}} \cdots t_n^{\lambda_{jn}}$.

Proof. Step 1: $(\rho, \mathbb{C}^m)$ is a direct sum of irreducible representations. The idea is to use the orthogonal decomposition with respect to the following ρ-invariant inner product

$$(v, w) := \int_{\mathbb{T}^n} \rho(e^{2\pi i x})v \cdot \overline{\rho(e^{2\pi i x})w}\, dx, \ \ v, w \in \mathbb{C}^m,$$

on $\mathbb{C}^m$. Let V be a non-zero ρ-invariant subspace of minimal dimension. It is clearly irreducible, and its orthogonal complement $V^\perp$ with respect to the above ρ-invariant inner product is also ρ-invariant. Hence the result follows from the dimension induction.

Step 2: Check that each $\rho(t)a := t^\lambda a$, $\lambda \in \mathbb{Z}^n$, $a \in \mathbb{C}$ defines a one-dimensional irreducible representation of $\mathbb{T}^n$. This is not hard, we leave it to the readers. Now it suffices to show that this construction gives all irreducible representations.

Step 3: Show that there are no more irreducible representations. The main idea is to use

$$T_{v_1 v_2}(w) := \int_{\mathbb{T}^n} (\rho(e^{2\pi i x})w, v_1)\rho(e^{-2\pi i x})v_2\, dx, \ \ v_1,\, v_2 \in \mathbb{C}^m.$$

One may check that $T_{v_1 v_2} : \mathbb{C}^m \to \mathbb{C}^m$ is ρ-invariant, i.e.

$$T_{v_1 v_2}(\rho(t)w) = \rho(t)T_{v_1 v_2}(w) \tag{2.25}$$

and $T_{v_1 v_2}$ satisfies

$$(T_{v_1 v_2}(w_1), w_2) = \int_{\mathbb{T}^n} (\rho(e^{2\pi i x})w_1, v_1)\overline{(\rho(e^{2\pi i x})w_2, v_2)}\, dx. \tag{2.26}$$

Now, if (ρ, V_1), (ρ, V_2) are two irreducible sub-representations and $v_1 \in V_1, v_2 \in V_2$, then by (2.25), we know that the kernel and image of

$$T_{v_1 v_2} : V_1 \to V_2$$

are all ρ-invariant. Hence if $T_{v_1 v_2}$ is not zero, then the irreducibility of V_1, V_2 gives

$$\ker T_{v_1 v_2} = 0, \ \ \mathrm{Im}\, T_{v_1 v_2} = V_2,$$

i.e. $T_{v_1 v_2}$ is an isomorphism. Denote by χ_j the character of (ρ, V_j), one may write

$$\chi_j(e^{2\pi i x}) = \sum_k (\rho(e^{2\pi i x})e_j^k, e_j^k),$$

where each $\{e_j^k\}$ denotes an orthonormal basis of V_j. Thus if

$$(\chi_1, \chi_2) := \int_{\mathbb{T}^n} \chi_1(e^{2\pi i x})\overline{\chi_2(e^{2\pi i x})}\, dx \neq 0,$$

then by (2.26), we must have $T_{v_1 v_2} \neq 0$ for some v_1, v_2, hence $(\chi_1, \chi_2) \neq 0$ implies that (ρ, V_1) is isomorphic to (ρ, V_2). Since $\{e^{2\pi i \lambda \cdot x}\}_{\lambda \in \mathbb{Z}^n}$ already defines an complete orthonormal basis of $L^2(\mathbb{T}^n)$, we know that there are no characters $\chi \in C(\mathbb{T}^n)$ orthogonal to $\{e^{2\pi i \lambda \cdot x}\}_{\lambda \in \mathbb{Z}^n}$. □

Since holomorphic functions on $\mathbb{C}^*$ is uniquely determined by its restriction to $\mathbb{T}$, the above theorem implies the following result.

Theorem 2.7. *Let ρ be a $(\mathbb{C}^*)^n$ action on $\mathbb{C}^m$, then there exists a basis $\{e_j\}_{1\leq j\leq m}$ of $\mathbb{C}^m$ and $\lambda_j \in \mathbb{Z}^n$ such that*

$$\rho(t)e_j = t^{\lambda_j} e_j, \quad 1 \leq j \leq m, \quad t \in (\mathbb{C}^*)^n, \tag{2.27}$$

where $t^{\lambda_j} = t_1^{\lambda_{j1}} \cdots t_n^{\lambda_{jn}}$.

For each $\lambda \in \mathbb{Z}^n$, denote by

$$W_\lambda := \{w \in \mathbb{C}^m : \rho(t)w = t^\lambda w\},$$

then the above theorem gives

$$\mathbb{C}^m = \oplus_{\lambda \in \mathbb{Z}^n} W_\lambda.$$

(26th September, section 2.3 is optional)

2.3. **Volume of Delzant line bundles and Bernstein-Kushnirenko theorem.** For a positive integer k, kP defines the same toric variety X_P, moreover, the transition function of L_{kP} is the k-th power of the transition function of L_P, hence we write $L_{kP} = kL_P$ (or $L_{kP} = L_P^{\otimes k}$). We call

$$|L_P| := \lim_{k\to\infty} \frac{\dim H^0(X_P, kL_P)}{k^n/n!}$$

the volume of L_P. By Theorem 2.5, we have

$$|L_P| = \lim_{k\to\infty} \frac{\#\{kP \cap \mathbb{Z}^n\}}{k^n/n!} = n!|P|, \tag{2.28}$$

which explains the Bernstein-Kushnirenko theorem (0.4).

Exercise 16: Prove (2.28).

Another proof of the Bernstein-Kushnirenko theorem is to use the function ϕ defined in (2.2). By a change of variable $x_j = \log|z_j|^2$, we obtain

$$\phi_P(z) := \phi(\log|z_1|^2, \cdots, \log|z_n|^2) = \log\left(\sum_{u\in P\cap\mathbb{Z}^n} |z^u|^2\right), \quad z \in (\mathbb{C}^*)^n.$$

Put

$$h_P(z,\xi) := |\xi|^2 e^{-\phi_P(z)}, \tag{2.29}$$

one may check (try!) that $h_P \circ \Psi_v^{-1}$ is smooth on $\mathbb{C}_v^n \times \mathbb{C}$ for every vertex v of P. Thus h_P extends to a smooth function on L_P and defines a Hermitian metric on each fiber of the natural mapping $L_P \to X_P$ (we call such a function a metric on L_P). In particular, for every section $s \in H^0(X_P, L_P)$, $h_P \circ s$ defines a smooth function on X_P that satisfies

$$\log h_P \circ s(z) = \log|s(z)|^2 - \phi_P(z), \quad z \in (\mathbb{C}^*)^n. \tag{2.30}$$

Proof of the Bernstein-Kushnirenko theorem. The idea is to use the Poincaré-Lelong formula, which gives

$$dd^c \log h_P \circ s = [Z_s] - dd^c\phi_p, \quad dd^c := \frac{i\partial\bar\partial}{2\pi},$$

where $[Z_s]$ denotes the current of integration along the zero set $Z_s := \{s = 0\}$ of s. Hence the Stokes' theorem gives

$$0 = \int_X (dd^c\phi_p)^{n-1} \wedge dd^c \log h_P \circ s = \int_{Z_s} (dd^c\phi_p)^{n-1} - \int_X (dd^c\phi_p)^n.$$

By induction on n, we thus obtain

$$\int_X (dd^c\phi_p)^n = \#\{x \in X : s_1(x) = \cdots = s_n(x) = 0\},$$

for generic sections $s_j \in H^0(X_P, L_P)$. Hence

$$\int_X (dd^c\phi_p)^n = n! \int_{\mathbb{R}^n} MA(\phi)dx = n!|P|$$

gives the Bernstein-Kushnirenko theorem. □

3. Brascamp-Lieb proof of the Prekopa theorem

3.1. **Prekopa's theorem.** We shall follow Berndtsson's note [B10, section 1.3] to prove the following Prekopa's theorem (which is also known as the functional version of the Brunn-Minkowski inequality in Theorem 1.4).

Notation: We say that ϕ is a (generalized) convex function on a convex open set $U \subset \mathbb{R}^N$ if $\phi = -\infty$ identically on U or ϕ is finite everywhere with

$$\phi\,(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y) \ \ \forall\, x, y \in U, \ 0 < t < 1. \tag{3.1}$$

The Prekopa theorem. *Let $\phi(t, x)$ be a convex function on $\mathbb{R}^m_t \times \mathbb{R}^n_x$. Define $\tilde{\phi}(t)$ by*

$$e^{-\tilde{\phi}(t)} := \int_{\mathbb{R}^n_x} e^{-\phi(t,x)},$$

where we omit the Lebesgue measure on $\mathbb{R}^n_x$ in the integral. Then $\tilde{\phi}$ is convex on $\mathbb{R}^m_t$.

Proof. By Fubini's theorem, it suffices to prove the $n = 1$ case. Since convexity means convexity on any line, one may further assume that $m = 1$. Write ϕ as the decreasing limit of a family of smooth strictly convex functions and

$$\int_{\mathbb{R}^n_x} e^{-\phi(t,x)} = \lim_{R\to\infty} \int_{|x|<R} e^{-\phi(t,x)},$$

it suffices to prove the following theorem. □

Theorem 3.1. *Let ϕ be a smooth strictly convex function on $\mathbb{R}_t \times \mathbb{R}_x$. Fix $R > 0$. Define $\tilde{\phi}(t)$ by*

$$e^{-\tilde{\phi}(t)} := \int_{|x|<R} e^{-\phi(t,x)}.$$

Then $\tilde{\phi}$ is smooth strictly convex on $\mathbb{R}_t$.

Proof. By a change of variable, one may assume that $R = 1$. Apply the t-derivative, we get

$$-\tilde{\phi}_t e^{-\tilde{\phi}(t)} = \int_{|x|<1} -\phi_t e^{-\phi(t,x)},$$

apply the t-derivative again, we get

$$(-\tilde{\phi}_{tt} + (\tilde{\phi}_t)^2) e^{-\tilde{\phi}(t)} = \int_{|x|<1} (-\phi_{tt} + (\phi_t)^2) e^{-\phi(t,x)}.$$

Write the probability measure $e^{-\phi(t,x)} dx / \int_{|x|<1} e^{-\phi(t,x)}$ as $d\mu$, we have

$$\tilde{\phi}_t = \int_{|x|<1} \phi_t \, d\mu, \quad \tilde{\phi}_{tt} = \int_{|x|<1} \phi_{tt} - (\phi_t)^2 \, d\mu + (\tilde{\phi}_t)^2.$$

Note that $\tilde{\phi}_t$ is the μ-average of ϕ_t, we have

$$\int_{|x|<1} \tilde{\phi}_t (\phi_t - \tilde{\phi}_t) \, d\mu = (\tilde{\phi}_t)^2 - (\tilde{\phi}_t)^2 = 0,$$

which implies

$$\int_{|x|<1} (\phi_t - \tilde{\phi}_t)^2 \, d\mu = \int_{|x|<1} (\phi_t)^2 \, d\mu - (\tilde{\phi}_t)^2,$$

hence we get

$$\tilde{\phi}_{tt} = \int_{|x|<1} \phi_{tt} \, d\mu - \int_{|x|<1} (\phi_t - \tilde{\phi}_t)^2 \, d\mu.$$

By the lemma below, we then have

$$\tilde{\phi}_{tt} \geq \int_{|x|<1} \phi_{tt} \, d\mu - \int_{|x|<1} \frac{(\phi_{tx})^2}{\phi_{xx}} \, d\mu.$$

Since ϕ is strictly convex, we have

$$\phi_{tt} > \frac{(\phi_{tx})^2}{\phi_{xx}},$$

hence the theorem follows. □

Lemma 3.2. *Let ψ be a smooth strictly convex function on $\mathbb{R}$. Let u be a smooth function on $\mathbb{R}$ with $\int_{|x|<1} u e^{-\psi} = 0$. Then*

$$\int_{|x|<1} u^2 e^{-\psi} \leq \int_{|x|<1} \frac{(u_x)^2}{\psi_{xx}} e^{-\psi}.$$

Proof. One may follow the proof of Lemma 2.7 in [B14, page 4], here we shall introduce another proof based on the Bochner identity below. For every smooth function α with compact support in $(-1, 1)$ (i.e. $\alpha \in C_0^\infty(-1, 1)$), by the Cauchy-Schwarz inequality and (3.5), we have

$$\left(\int_{|x|<1} u_x \alpha e^{-\psi} \right)^2 \leq \int_{|x|<1} \alpha^2 \psi_{xx} e^{-\psi} \int_{|x|<1} \frac{(u_x)^2}{\psi_{xx}} e^{-\psi} \leq \int_{|x|<1} (\delta\alpha)^2 e^{-\psi} \int_{|x|<1} \frac{(u_x)^2}{\psi_{xx}} e^{-\psi}.$$

Consider the following inner product

$$(\alpha, \beta)_{\square} := \int_{|x|<1} (\delta\alpha)(\delta\beta)e^{-\psi}$$

on $C_0^\infty(-1,1)$, by the above inequality we know that

$$\alpha \mapsto \int_{|x|<1} u_x \alpha e^{-\psi}$$

defines a bounded $\mathbb{R}$-linear functional on $(C_0^\infty(-1,1), ||\cdot||_{\square})$, which extends to a functional on its Hilbert completion H_0^1. Thus the Riesz representation theorem gives $v \in H_0^1$ such that

$$\int_{|x|<1} (\delta v)(\delta\alpha)e^{-\psi} = (v, \alpha)_{\square} = \int_{|x|<1} u_x \alpha e^{-\psi}, \ \forall\, \alpha \in C_0^\infty(-1,1), \tag{3.2}$$

with

$$\int_{|x|<1} (\delta v)^2 e^{-\psi} = ||v||_{\square}^2 \leq \int_{|x|<1} \frac{(u_x)^2}{\psi_{xx}} e^{-\psi}. \tag{3.3}$$

Think of δv as a distribution, we have

$$\int_{|x|<1} (\delta v)(\delta\alpha)e^{-\psi} = -\int_{|x|<1} (\delta v)_x \alpha e^{-\psi}, \ \forall\, \alpha \in C_0^\infty(-1,1),$$

hence (3.2) gives $-(\delta v)_x = u_x$ in the sense of distribution. Since $v \in H_0^1$, we have

$$\int_{|x|<1} (\delta v)e^{-\psi} = 0,$$

hence $-(\delta v) \perp \ker\,()_x$ in $L^2_{(-1,1)}(e^{-\psi})$ which implies that $-(\delta v)$ is the L^2-minimal solution of

$$(\cdot)_x = u_x.$$

Hence we must have $-(\delta v) = u$, thus (3.3) gives our estimate. □

The Bochner identity. *For every smooth function α in $\mathbb{R}$, we have*

$$(\alpha^2 e^{-\psi})_{xx} = ((\delta\alpha)^2 + 2(\delta\alpha)_x \alpha + \alpha_x^2 + \psi_{xx}\alpha^2)e^{-\psi}, \tag{3.4}$$

where $\delta\alpha := \alpha_x - \psi_x \alpha$. If α has compact support we further have

$$\int_{\mathbb{R}} (\delta\alpha)^2 e^{-\psi} = \int_{\mathbb{R}} \alpha_x^2 e^{-\psi} + \int_{\mathbb{R}} \psi_{xx}\alpha^2 e^{-\psi}. \tag{3.5}$$

Proof. Take derivative of $(\alpha^2 e^{-\psi})_x = (\delta\alpha)\alpha e^{-\psi} + \alpha\alpha_x e^{-\psi}$, we get

$$(\alpha^2 e^{-\psi})_{xx} = ((\delta\alpha)^2 + (\delta\alpha)_x \alpha + \alpha_x^2 + \alpha\delta(\alpha_x))e^{-\psi},$$

hence (3.4) follows from

$$\delta(\alpha_x) = (\delta\alpha)_x + \psi_{xx}\alpha.$$

Integrating (3.4) over $\mathbb{R}$, we get (3.5). (30th September, section 3.2 is for home-reading.) □

Reading task 2: Read page 5-8 of [B10] for the complex version of the above theory.

3.2. **Prekopa proof of the Brunn-Minkowski inequality.** We shall show that the Prekopa theorem implies the following "variational" version of the Brunn-Minkowski inequality.

Theorem 3.3. *Let A be a convex open set in $\mathbb{R}^m_t \times \mathbb{R}^n_x$ and let A_t be the slices*

$$A_t := \{x \in \mathbb{R}^n_x : (t,x) \in A\}.$$

Let $|A_t|$ be the Lebesgue measure of A_t. Then $-\log|A_t|$ is convex on $U := \{t : A_t \neq \emptyset\}$.

Proof. Since A is the increasing limit of bounded convex open sets, one may assume that A is bounded, then $-\log|A_t|$ is finite everywhere on U. Put

$$\phi(t,x) = \begin{cases} 0 & (t,x) \in \bar{A} \\ \infty & (t,x) \notin \bar{A}, \end{cases}$$

we have

$$|A_t| = \int_{\mathbb{R}^n_x} e^{-\phi(t,x)}.$$

Notice that ϕ is the increasing limit of a family of smooth convex functions, the Prekopa theorem implies that $-\log|A_t|$ is the increasing limit of a family of convex functions, which implies that $-\log|A_t|$ satisfies (3.1). Since $-\log|A_t|$ is also finite everywhere on U, we know that it is convex (see the notation at the beginning of this section). □

Remark: *Let us apply the above theorem to*

$$A := \{(t,x) \in \mathbb{R}^2_t \times \mathbb{R}^n_x : t_1, t_2 > 0, \ \ x \in t_1A_1 + t_2A_2\},$$

where A_1, A_2 are bounded non-empty convex open sets in $\mathbb{R}^n$ and

$$t_1A_1 + t_2A_2 := \{t_1x_1 + t_2x_2 : x_1 \in A_1, \ x_2 \in A_2\}.$$

Then A is convex. Hence the above theorem implies that $-\log|t_1A_1 + t_2A_2|$ is convex in $\mathbb{R}^2_+$. Lemma 3.4 below implies the following "additive" version of the Brunn-Minkowski inequality.

Brunn-Minkowski inequality. *Let A_1, A_2 be bounded non-empty convex open sets in $\mathbb{R}^n$. Then*

$$|A_1 + A_2|^{\frac{1}{n}} \geq |A_1|^{\frac{1}{n}} + |A_2|^{\frac{1}{n}}.$$

Lemma 3.4. *Let f be a positive smooth function on an open convex cone, say $\mathcal{K}$, in $\mathbb{R}^N$. Assume that f is 1-homogeneous, i.e.*

$$f(tx) \equiv tf(x), \ \forall\, t > 0, \ x \in \mathcal{K}.$$

Then the following statements are equivalent:

A1: $f(x+y) \geq f(x) + f(y)$, $\forall\, x, y \in \mathcal{K}$;
A2: $-f$ is convex;
A3: $-\log f$ is convex;
A4: For every $x', y' \in \mathcal{K}$, $t \mapsto -\log f(tx' + (1-t)y')$ is convex on $(0,1)$.

Proof. Since f is 1-homogeneous, $A1$ implies

$$f(tx+(1-t)y) \geq tf(x)+(1-t)f(y). \tag{3.6}$$

Thus $A1 \Rightarrow A2$. Since

$$(-\log f)_{\xi\xi} = \frac{-f_{\xi\xi}}{f} + \frac{(f_\xi)^2}{f^2},\ f_\xi = \sum \xi^j f_{x_j}, \tag{3.7}$$

we know $A2 \Rightarrow A3$. Since $A3 \Rightarrow A4$ is trivial, it is enough to show $A4 \Rightarrow A1$: notice that $A4$ implies

$$f(tx'+(1-t)y') \geq f(x')^t f(y')^{1-t}. \tag{3.8}$$

Take

$$x' = \frac{x}{f(x)},\ y' = \frac{y}{f(y)},\ t = \frac{f(x)}{f(x)+f(y)}, \tag{3.9}$$

we get $A1$. The proof is complete. □

4. A SHORT SEVERAL COMPLEX VARIABLES COURSE

We will mainly follow the Hörmander book [H1]. The aim is to prepare for the next section.

4.1. **Holomorphic function of several variables.** We will follow section 2.1-2.2 of [H1]. Let u be a complex valued function in $C^1(\Omega)$, where Ω is an open set in $\mathbb{C}^n$, which we identify with $\mathbb{R}^{2n}$. We shall denote the real coordinates by x_j, $1 \leq j \leq n$, and the complex coordinates by $z_j = x_{2j-1} + i\, x_{2j}$, $j = 1, \cdots, n$. Using the notations

$$\frac{\partial u}{\partial z_j} := \frac{1}{2}\left(\frac{\partial u}{\partial x_{2j-1}} - i\frac{\partial u}{\partial x_{2j}}\right),\ \ \frac{\partial u}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial u}{\partial x_{2j-1}} + i\frac{\partial u}{\partial x_{2j}}\right)$$

and

$$dz_j := dx_{2j-1} + i\, dx_{2j},\ \ d\bar{z}_j := dx_{2j-1} - i\, dx_{2j},$$

we can express $du = \sum \frac{\partial u}{\partial x_j} dx_j$ as a linear combination of the differentials dz_j and $d\bar{z}_j$,

$$du = \sum_{j=1}^n \frac{\partial u}{\partial z_j} dz_j + \sum_{j=1}^n \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j. \tag{4.1}$$

With the notation

$$\partial u := \sum_{j=1}^n \frac{\partial u}{\partial z_j} dz_j,\ \ \overline{\partial} u := \sum_{j=1}^n \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j,$$

one may also write

$$du = \partial u + \overline{\partial} u. \tag{4.2}$$

Differential forms which are linear combination of the differentials dz_j are said to be of type $(1,0)$, and those which are linear combinations of $d\bar{z}_j$ are said to be of type $(0,1)$. Thus ∂u (resp. $\overline{\partial} u$) is the component of du of type $(1,0)$ (resp. $(0,1)$).

Definition 4.1. *A function $u \in C^1(\Omega)$ is said to be holomorphic in Ω if du is of type $(1,0)$, that is, if*

$$\overline{\partial} u = 0 \ \ (\textit{the Cauchy-Riemann equations}).$$

The set of all holomorphic functions in Ω is denoted by $\mathcal{O}(\Omega)$.

Reading task 3: Read page 23-29 of [H1] for classical results on holomorphic functions.

4.2. **Subharmonic functions.** We will follow section 1.6 of [H1]. We recall that a C^2 function h in an open set $\Omega \subset \mathbb{C}$ is called harmonic of $\Delta h = 4\partial^2 h/\partial z \partial \bar{z} = 0$ in Ω.

Definition 4.2. *A function u defined in an open set $\Omega \subset \mathbb{C}$ and with values in $[-\infty, \infty)$ is called subharmonic if*

(a) *u is is upper semi-continuous, that is, $\{z \in \Omega : u(z) < s\}$ is open for every $s \in \mathbb{R}$;*
(b) *for each compact set $K \subset \Omega$ and every continuous function h on K which is harmonic in the interior of K and is $\geq u$ on ∂K we have $u \leq h$ in K.*

By our definition the function which is $-\infty$ identically is subharmonic; sometimes this is excluded in the definition.

Theorem 4.1. *If u is subharmonic and $0 < c \in \mathbb{R}$, it follows that cu is subharmonic. If u_α, $\alpha \in A$, is a family of subharmonic functions, then $u = \sup_\alpha u_\alpha$ is subharmonic if $u < \infty$ and u is upper semi-continuous, which is always the case if A is finite. If $u_1, u_2, \cdots$ is a decreasing of subharmonic functions, then $u = \lim_{j\to\infty} u_j$ is also subharmonic.*

Remark. *An upper semi-continuous function u defined in an open set $\Omega \subset \mathbb{C}$ is subharmonic if and only if for every closed disc $D \subset \Omega$ and every holomorphic polynomial f with $u \leq \mathrm{Re} f$ on ∂D, we have $u \leq \mathrm{Re} f$ in D (see Theorem 1.6.3 (i) in* [H1]*). In particular, we know that $\log|f|$ is subharmonic in Ω for every $f \in \mathcal{O}(\Omega)$ (see Corollary 1.6.6 in* [H1]*).*

Reading task 4: Read page 17-21 of [H1] for classical results on subharmonic functions.

4.3. **Plurisubharmonic functions and pseudoconvexity.** We will follow section 2.6 of [H1].

Definition 4.3. *A function u defined in an open set $\Omega \subset \mathbb{C}^n$ and with values in $[-\infty, \infty)$ is called plurisubharmonic if*

(a) *u is is upper semi-continuous;*
(b) *for arbitrary z and $w \in \mathbb{C}^n$, the function $\tau \mapsto u(z + \tau w)$ is subharmonic in the part of $\mathbb{C}$ where it is defined.*

We shall denote the set of all such functions by $P(\Omega)$.

Remark. *We know that $\log|f| \in P(\Omega)$ for every $f \in \mathcal{O}(\Omega)$. A function $u \in C^2(\Omega)$ is plurisubharmonic if and only if (see Theorem 2.6.2 in* [H1]*)*

$$\sum_{j,k=1}^{n} \frac{\partial^2 u(z)}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geq 0, \ \forall\, z \in \Omega, \ w \in \mathbb{C}^n. \tag{4.3}$$

We say that u is strictly plurisubharmonic if strict inequality holds true in (4.3) *for every $z \in \Omega$ and no-zero w.*

Reading task 3: Read the proof of Theorem 2.6.2, 2.6.3, 2.6.4, 2.6.11 in [H1].

Definition 4.4. *An open set $\Omega \subset \mathbb{C}^n$ is called pseudoconvex if there exists a smooth strictly plurisubharmonic exhaustion function on Ω.*

Exercise 17: Let u be a smooth function on a convex open set A in $\mathbb{C}^n$, show that

$$\tilde{u} : z \mapsto u(\operatorname{Re} z)$$

is plurisubharmonic in $A \times i\mathbb{R}^n$ if and only if u is convex in A. Use this fact to prove that $A \times i\mathbb{R}^n$ is pseudoconvex (in fact, one may use Theorem 2.6.7 in [H1] to prove that $A \times i\mathbb{R}^n$ is pseudoconvex if and only if A is convex).

Definition 4.5. *An open set $\Omega \subset \mathbb{C}^n$ is called a smoothly bounded strongly pseudoconvex if there exists a smooth strictly plurisubharmonic function ρ in a neighborhood U of $\bar{\Omega}$ such that*

$$\Omega = \{z \in U : \rho(x) < 0\}$$

and $d\rho \neq 0$ on $\partial\Omega$.

Exercise 18: Show that smoothly bounded strongly pseudoconvex implies pseudoconvex. Hint: consider $u = -\log(-\rho)$. (3rd October)

5. Subharmonicity of Bergman kernels

This is a highly non-trivial complex generalization of the Prekopa theorem. Here we will introduce the Hörmander L^2-theory and prove results in section 4 of the Hörmander book.

5.1. L^2-estimates for the $\bar\partial$-equation on pseudoconvex domains in $\mathbb{C}^n$.

We shall use the Bochner methods to rewrite section 4.4 of [H1]. Similar to the real case, we have the following:

The complex Bochner identity. *For smooth functions α, β in a domain $\Omega \subset \mathbb{C}^n$, we have*

$$(\alpha\bar\beta e^{-\psi})_{j\bar k} = \left((\delta_j\alpha)(\overline{\delta_k\beta}) + (\delta_j\alpha)_{\bar k}\bar\beta + \alpha_{\bar k}\overline{\beta_{\bar j}} + \alpha\overline{(\delta_k\beta)_{\bar j}} + \psi_{j\bar k}\alpha\bar\beta\right) e^{-\psi}, \tag{5.1}$$

where ψ denotes a real smooth function in Ω, $\delta_j\alpha := \alpha_j - \psi_j\alpha$. Assume further that α has compact support we have

$$\int_\Omega (\delta_j\alpha)(\overline{\delta_k\beta})e^{-\psi} = \int_\Omega \alpha_{\bar k}\overline{\beta_{\bar j}}e^{-\psi} + \int_\Omega \psi_{j\bar k}\alpha\bar\beta e^{-\psi}. \tag{5.2}$$

Proof. We have

$$(\alpha\bar\beta e^{-\psi})_{j\bar k} = \left((\delta_j\alpha)\bar\beta e^{-\psi} + \alpha\overline{\beta_{\bar j}}e^{-\psi}\right)_{\bar k}$$

and

$$(\alpha\bar\beta e^{-\psi})_{j\bar k} = \left((\delta_j\alpha)(\overline{\delta_k\beta}) + (\delta_j\alpha)_{\bar k}\bar\beta + \alpha_{\bar k}\overline{\beta_{\bar j}} + \alpha\overline{\delta_k(\beta_{\bar j})}\right) e^{-\psi},$$

thus

$$\delta_k(\beta_{\bar j}) = (\delta_k\beta)_{\bar j} + \psi_{k\bar j}\beta$$

gives (5.1). Integration by parts gives (5.2). □

Definition 5.1. *We call $u := \sum_{p=1}^n u_{\bar p} d\bar z_p$ a smooth $(0,1)$-form on Ω if $u_{\bar p}$ (NOT the derivatives of u, just an n-tuple of functions!) are smooth functions on Ω. Fix a smooth strictly plurisubharmonic function ϕ on Ω, we define*

$$\delta u := \sum_{j=1}^n \delta_j u^j, \quad u^j := \sum_{p=1}^n u_{\bar p}\phi^{\bar p j}, \tag{5.3}$$

where $(\phi^{\bar p j})$ denotes the inverse matrix of the complex Hessian matrix $(\phi_{j\bar q})$, i.e. it is defined such that $(\sum_j \phi^{\bar p j}\phi_{j\bar q})_{1\le p,q\le n}$ is the identity matrix.

Apply (5.2) to $\alpha = u^j, \beta = u^k$, we obtain

$$\int_\Omega |\delta u|^2 e^{-\psi} = \sum_{j,k=1}^n \int_\Omega (u^j)_{\bar k}\overline{(u^k)_{\bar j}} e^{-\psi} + \sum_{j,k=1}^n \int_\Omega \psi_{j\bar k} u^j \overline{u^k} e^{-\psi}. \tag{5.4}$$

In order to solve the $\bar\partial$-equation, we need a lower bound for the left hand side of (5.4). The problem is that the sign of

$$J := \sum_{j,k=1}^n (u^j)_{\bar k}\overline{(u^k)_{\bar j}}$$

is not clear. We need the following lemma. (6th October)

Lemma 5.1. *$J = |\nabla_\phi u|^2_\phi - |\bar\partial u|^2_\phi$, where*

$$|\nabla_\phi u|^2_\phi := \sum (u^k)_{\bar q}\overline{(u^l)_{\bar p}}\phi^{\bar q p}\phi_{k\bar l}, \ |\bar\partial u|^2_\phi := \frac{1}{2}\sum ((u_{\bar l})_{\bar q} - (u_{\bar q})_{\bar l})\overline{((u_{\bar s})_{\bar p} - (u_{\bar p})_{\bar s})}\phi^{\bar q p}\phi^{\bar l s}.$$

Proof. Put

$$u^p_{\bar l} := \sum_{k,q=1}^n (u^k)_{\bar q}\phi^{\bar q p}\phi_{k\bar l}.$$

A direct computation gives

$$\sum u^p_{\bar l}\overline{u^l_{\bar p}} = J, \ \ \sum u^p_{\bar l}\overline{(u^l)_{\bar p}} = \sum (u^p)_{\bar l}\overline{u^l_{\bar p}} = |\nabla_\phi u|^2_\phi.$$

Hence

$$u^p_{\bar l}\overline{(u^l)_{\bar p}} + (u^p)_{\bar l}\overline{u^l_{\bar p}} - (u^p)_{\bar l}\overline{(u^l)_{\bar p}} - u^p_{\bar l}\overline{u^l_{\bar p}} = (u^p_{\bar l} - (u^p)_{\bar l})\overline{((u^l)_{\bar p} - u^l_{\bar p})}$$

gives

$$2|\nabla_\phi u|^2_\phi - 2J = \sum (u^p_{\bar l} - (u^p)_{\bar l})\overline{((u^l)_{\bar p} - u^l_{\bar p})}.$$

Thus our lemma follows from

$$u^p_{\bar l} - (u^p)_{\bar l} = \sum_{k,q=1}^n ((u^k)_{\bar q}\phi_{k\bar l} - (u^k)_{\bar l}\phi_{k\bar q})\phi^{\bar q p} = \sum_{q=1}^n ((u_{\bar l})_{\bar q} - (u_{\bar q})_{\bar l})\phi^{\bar q p}.$$

(The readers should try to add the details, see section 1.4 in [B95] for the $\phi(z) = |z|^2$ case). ☐

Remark. *One may check that*

$$\nabla_\phi u := \sum (u^k)_{\bar{q}} d\bar{z}_q \otimes \frac{\partial}{\partial z_k}, \quad \overline{\partial} u := \sum (u_{\bar{j}})_{\bar{k}} d\bar{z}_k \wedge d\bar{z}_j. \tag{5.5}$$

are independent of the choice of the coordinate z, thus the above computations can also be generalized to complex manifolds, see [B10] *for a nice coordinate free computation on manifolds. Note that $d\bar{z}_k \wedge d\bar{z}_j = -d\bar{z}_j \wedge d\bar{z}_k$, we know that*

$$\overline{\partial} u = \frac{1}{2} \sum ((u_{\bar{j}})_{\bar{k}} - (u_{\bar{k}})_{\bar{j}}) d\bar{z}_k \wedge d\bar{z}_j.$$

If you are not familiar with the wedge product, then you might think of $\overline{\partial} u$ as a tuple of functions

$$(\overline{\partial} u)_{\bar{j}\bar{k}} = (u_{\bar{j}})_{\bar{k}} - (u_{\bar{k}})_{\bar{j}}.$$

Since $(\overline{\partial} u)_{\bar{j}\bar{k}} = -(\overline{\partial} u)_{\bar{k}\bar{j}}$, we call $\bar{u}$ a $(0,2)$-form.

Definition 5.2. *A call a tuple of smooth function $v := (v_{\bar{j}_1 \cdots \bar{j}_q})$ a smooth $(0,q)$-form if*

$$v_{\bar{j}_1 \cdots \bar{j}_q} = \mathrm{sgn}\sigma \, v_{\overline{\sigma(j_1)} \cdots \overline{\sigma(j_q)}}.$$

where $\mathrm{sgn}\sigma$ denotes the sign of the permutation $(j_1, \cdots, j_q) \mapsto (\sigma(j_1), \cdots, \sigma(j_q))$ (it is defined to be zero if $j_l = j_k$ for some $l \neq k$). $\overline{\partial} v$ is defined as a $(0, q+1)$-form

$$(\overline{\partial} v)_{\bar{j}_1 \cdots \bar{j}_{q+1}} := (v_{\bar{j}_1 \cdots \bar{j}_q})_{\bar{j}_{q+1}} + (-1)(v_{\bar{j}_1 \cdots \bar{j}_{q-1} \bar{j}_{q+1}})_{\bar{j}_q} + \cdots + (-1)^q (v_{\bar{j}_2 \cdots \bar{j}_q})_{\bar{j}_1}. \tag{5.6}$$

We say that u is $\overline{\partial}$-exact if $u = \overline{\partial} v$ for some v. v is said to be $\overline{\partial}$-closed of $\overline{\partial} v = 0$. The inner product of two $(0,q)$-forms v, w is defined as

$$(v, w) = \frac{1}{q!} \sum \int_\Omega v_{\bar{j}_1 \cdots \bar{j}_q} \overline{w_{\bar{k}_1 \cdots \bar{k}_q}} \phi^{\bar{j}_1 k_1} \cdots \phi^{\bar{j}_q k_q} e^{-\psi}, \tag{5.7}$$

where we omit the Lebesgue measure in the integral. The adjoint operator $\overline{\partial}^*$ is defined such that

$$(v, \overline{\partial} u) = (\overline{\partial}^* v, u),$$

where we assume that u has compact support. The Laplacian operator is defined as

$$\Box := \overline{\partial}\overline{\partial}^* + \overline{\partial}^* \overline{\partial}.$$

Exercise 19: Show that $\overline{\partial}\overline{\partial} v = 0$, in particular, every $\overline{\partial}$-exact form is $\overline{\partial}$-closed.

Exercise 20: For smooth $(0,2)$-form v and $(0,1)$-form u. Show that

$$(\overline{\partial}^* v)_{\bar{m}} = -\sum \phi_{l\bar{m}} \delta_k (v_{\bar{p}\bar{q}} \phi^{\bar{p}l} \phi^{\bar{q}k}), \quad \overline{\partial}^* u = -\sum \delta_k (u_{\bar{j}} \phi^{\bar{j}k}) = -\delta u. \tag{5.8}$$

Exercise 21: Show that the leading order term of $\Box$ is given by

$$-\sum_{j,k=1}^n \phi^{\bar{k}j} \frac{\partial^2}{\partial z_j \partial \bar{z}_k}.$$

In particular, $\Box$ is always elliptic.

By Lemma 5.1, (5.4) and (5.8), we have

Theorem 5.2. *For every real smooth function ψ and smooth strictly plurisubharmonic function ϕ on Ω, we have*

$$(\square u, u) = ||\overline{\partial} u||^2 + ||\overline{\partial}^* u||^2 = \int_\Omega |\nabla_\phi u|^2_\phi e^{-\psi} + \sum_{j,k=1}^n \int_\Omega \psi_{j\bar{k}} u^j \overline{u^k} e^{-\psi}, \tag{5.9}$$

where u is an arbitrary smooth $(0,1)$-form with compact support in Ω.

Thanks to the Riesz representation theorem, the above theorem gives

Theorem 5.3. *With the notation above, assume further that ψ is strictly plurisubharmonic. Then for every smooth $(0,1)$-form u with*

$$||u||^2_\psi := \sum_{j,k=1}^n \int_\Omega \psi^{\bar{j}k} u_{\bar{j}} \overline{u_{\bar{k}}} e^{-\psi} < \infty, \tag{5.10}$$

there exist a smooth $(0,1)$-form v with $\square v = u$ and

$$(\square v, v) = ||\overline{\partial} v||^2 + ||\overline{\partial}^* v||^2 \leq ||u||^2_\psi. \tag{5.11}$$

Proof. The proof is simiar to that of Lemma 3.2. We leave to to the readers. Hint:

$$|(u,\alpha)|^2 \leq ||u||^2_\psi \sum_{j,k=1}^n \int_\Omega \psi_{j\bar{k}} \alpha^j \overline{\alpha^k} e^{-\psi} \leq ||u||^2_\psi (\square \alpha, \alpha),$$

for every smooth $(0,1)$-form α with compact support in Ω. Smoothness of v follows from the standard regularity theorem for elliptic operators (see [W, Theorem 6.5]) □

Exercise 22: Finish the proof of Theorem 5.3. (10th October)

Now we are ready to prove the following main theorem in Hörmander L^2-theory.

Theorem 5.4. *Let ψ be a smooth strictly plurisubharmonic function on a pseudoconvex domain $\Omega \subset \mathbb{C}^n$. Then for every smooth $\overline{\partial}$-closed $(0,1)$-form u with*

$$||u||^2_\psi := \sum_{j,k=1}^n \int_\Omega \psi^{\bar{j}k} u_{\bar{j}} \overline{u_{\bar{k}}} e^{-\psi} < \infty, \tag{5.12}$$

there exists a smooth function f with $\overline{\partial} f = u$ on Ω and

$$\int_\Omega |f|^2 e^{-\psi} \leq ||u||^2_\psi. \tag{5.13}$$

Proof. Let $\rho > 0$ be smooth strictly plurisubharmonic exhaustion function on Ω. Fix $\varepsilon > 0$, apply Theorem 5.3 to $\phi = \psi + \varepsilon\rho^2$, we obtain $\square_\varepsilon v_\varepsilon = u$. Let f be the weak limit of $\overline{\partial}^* v_\varepsilon$ as $\varepsilon \to 0$, we know that (5.11) implies (5.13). Hence it suffices to show that $\overline{\partial}^* \overline{\partial} v_\varepsilon \to 0$ in the sense of distribution. Note that $\overline{\partial} u = 0$ gives $\overline{\partial}\,\overline{\partial}^* \overline{\partial} v_\varepsilon = 0$, hence (here $(\overline{\partial} a \wedge u)_{\bar{j}\bar{k}} := (a)_{\bar{k}} u_{\bar{j}} - (a)_{\bar{j}} u_{\bar{k}}$)

$$0 = (\chi(\varepsilon\rho)^2 \overline{\partial}\,\overline{\partial}^* \overline{\partial} v_\varepsilon, \overline{\partial} v_\varepsilon) = ||\chi(\varepsilon\rho)(\overline{\partial}^* \overline{\partial} v_\varepsilon)||^2 - \varepsilon\,(2\chi(\varepsilon\rho)\chi'(\varepsilon\rho)\overline{\partial}\rho \wedge (\overline{\partial}^* \overline{\partial} v_\varepsilon), \overline{\partial} v_\varepsilon), \tag{5.14}$$

where $0 \leq \chi \leq 1$ is a smooth function on $\mathbb{R}$ with $|\chi'| \leq 1$ such that $\chi = 1$ on $(-\infty, 1)$ and $\chi = 0$ on $(3, \infty)$. Note that (5.14) gives

$$||\chi(\varepsilon\rho)(\overline{\partial}^*\overline{\partial} v_\varepsilon)||^4 \leq 2\varepsilon||\chi(\varepsilon\rho)(\overline{\partial}^*\overline{\partial} v_\varepsilon)||^2||u||^2_\psi, \tag{5.15}$$

which implies that $\overline{\partial}^*\overline{\partial} v_\varepsilon \to 0$ in the sense of distribution as $\varepsilon \to 0$. □

Exercise 23: Prove (5.15) and think why the solution f is always smooth. (13th October)

5.2. **Variation of Bergman projections and implicit function theorem for Banach spaces.**

5.2.1. *Variation of Bergman projections.*

Definition 5.3. *We shall denote by H the space of measurable functions on a domain $\Omega \subset \mathbb{C}^2$ with finite L^2-norm (ϕ is a given real smooth function on Ω)*

$$||u||^2_\phi := \int_\Omega |u|^2 \, e^{-\phi} < \infty, \ (\textit{we omit the Lebesgue measure in the integral}).$$

We call the collection, say

$$H_0 := \{u \textit{ is holomorphic on } \Omega : ||u||_\phi < \infty\},$$

of those holomorphic functions u in H the Bergman space.

Take a non-positive upper semi-continuous function

$$G : \Omega \to [-\infty, 0]$$

and a smooth function $\chi : \mathbb{R} \to [0, \infty)$ that vanishes on $(-\infty, 0]$. For each $t \in \mathbb{R}$, let us define

$$||u||^2_t := \int_\Omega |u(z)|^2 \, e^{-\phi(z) - \chi(G(z) - t)} \tag{5.16}$$

for $u \in H$ (see Definition 5.3, hence each $|| \cdot ||_t$ is a new Hilbert norm on H).

Definition 5.4. *We call the orthogonal projection*

$$P^t : H \to H_0 \ (\textit{for } H \textit{ and } H_0 \textit{ see Definition } 5.3)$$

with respect to the above $|| \cdot ||_t$ norm a Bergman projection.

By using the implicit function theorem, Berndtsson proved the following smoothness theorem.

Theorem 5.5 (Berndtsson's smoothness theorem)**.** *The Bergman projections P^t in Definition 5.4 depend smoothly on t, more precisely*

$$\begin{aligned} P : \mathbb{R} \times H &\to H_0 \\ (t, u) &\mapsto P^t u, \end{aligned}$$

is a smooth map from $\mathbb{R} \times (H, || \cdot ||_\phi)$ to $(H_0, || \cdot ||_\phi)$.

Proof. Fix $(t_0, u_0) \in \mathbb{R} \times H$. Notice that the following mapping

$$\begin{aligned} F : \mathbb{R} \times H \times H_0 &\to H_0 \\ (t, u, v) &\mapsto P^{t_0}(e^{\chi(G-t_0)-\chi(G-t)}(v-u)) \end{aligned}$$

is smooth, moreover $F(t, u, v) = 0$ iff $v = P^t u = P(t, u)$. Since $F_{H_0}(t_0, u_0, v) \equiv id_{H_0}$, we know smoothness of P follows directly Theorem 5.10 in the next subsection. □

Reading task 5: Read the next subsection for the proof of Theorem 5.10.

Remark: *For every bounded $\mathbb{C}$-linear mapping $f : H_0 \to \mathbb{C}$, one may define its functional norm with respect to $||\cdot||_t$ as*

$$||f||_t := \sup\{|f(u)| : u \in H_0,\ ||u||_t = 1\}.$$

Then Berndtsson's smoothness theorem gives the following result.

Lemma 5.6. *$||f||_t^2$ depends smoothly on t.*

Proof. Denote by u_f the Riesz representation of f in $(H_0, ||\cdot||_\phi)$, i.e.

$$f(u) = (u, u_f)_\phi.$$

Notice that $(u, u_f)_\phi = (u, e^{\chi(G-t)}u_f)_t$, hence

$$u_f^t := P^t(e^{\chi(G-t)}u_f)$$

is the Riesz representation of f in $(H_0, ||\cdot||_t)$. Thus

$$||f||_t^2 = ||P^t(e^{\chi(G-t)}u_f)||_t^2 = (P^t(e^{\chi(G-t)}u_f), e^{-\chi(G-t)}P^t(e^{\chi(G-t)}u_f))_\phi$$

depends smoothly on t by Theorem 5.5. (17th October) □

Similar to the proof of Theorem 3.1, we can continue to prove the following result.

Theorem 5.7. *With respect to the notation in the proof of Lemma 5.6, we have*

$$\frac{d^2}{dt^2}\left(\log ||f||_t^2\right) \geq \frac{(\chi''(G-t)u_f^t, u_f^t)_t - ||\chi'(G-t)u_f^t - P^t(\chi'(G-t)u_f^t)||_t^2}{||u_f^t||_t^2}. \tag{5.17}$$

Proof. Apply the t-derivative to $||f||_t^2 = ||u_f^t||_t^2 = \int_\Omega |u_f^t|^2 e^{-\phi-\chi(G-t)}$, we get

$$\frac{d}{dt}\left(||f||_t^2\right) = \int_\Omega (\partial_t u_f^t + \chi'(G-t)u_f^t)\,\overline{u_f^t}\,e^{-\phi^t} + \int_\Omega u_f^t\,\overline{\partial_t u_f^t}\,e^{-\phi^t}, \tag{5.18}$$

where

$$\phi^t(z) := \phi(t, z) := \phi(z) + \chi(G(z) - t). \tag{5.19}$$

Since for every $u \in H_0$, we have

$$\int_\Omega (\partial_t u_f^t + \chi'(G-t)u_f^t)\,\overline{u}\,e^{-\phi^t} = \frac{d}{dt}\int_\Omega u_f^t\,\overline{u}\,e^{-\phi^t} = \frac{d}{dt}\int_\Omega u_f \overline{u}\,e^{-\phi} = 0, \tag{5.20}$$

we know that (5.18) reduces to

$$\frac{d}{dt}\left(||f||_t^2\right) = \int_\Omega u_f^t\,\overline{\partial_t u_f^t}\,e^{-\phi^t}. \tag{5.21}$$

Take the derivative again, we get

$$\frac{d^2}{dt^2}\left(||f||_t^2\right) = ||\partial_t u_f^t||_t^2 + \int_\Omega u_f^t\, \overline{\partial_t(\partial_t u_f^t + \chi'(G-t)u_f^t)}\, e^{-\phi^t} + (\chi''(G-t)u_f^t, u_f^t)_t,$$

which implies (together with (5.21))

$$\frac{d^2}{dt^2}\left(\log ||f||_t^2\right) \geq \frac{\int_\Omega u_f^t\, \overline{\partial_t(\partial_t u_f^t + \chi'(G-t)u_f^t)}\, e^{-\phi^t} + (\chi''(G-t)u_f^t, u_f^t)_t}{||u_f^t||_t^2}. \tag{5.22}$$

Now by (5.20), we have $\partial_t u_f^t + \chi'(G-t)u_f^t \perp H_0$. Put

$$\partial_t u_f^t + \chi'(G-t)u_f^t := u_{H_0^\perp},$$

note that $\partial_t u_f^t \in H_0$, hence we must have

$$u_{H_0^\perp} = \chi'(G-t)u_f^t - P^t(\chi'(G-t)u_f^t).$$

Hence

$$0 = \frac{d}{dt}\int_\Omega u_f^t\, \overline{u_{H_0^\perp}}\, e^{-\phi^t} = ||u_{H_0^\perp}||_t^2 + \int_\Omega u_f^t\, \overline{\partial_t u_{H_0^\perp}}\, e^{-\phi^t}$$

gives

$$\int_\Omega u_f^t\, \overline{\partial_t(\partial_t u_f^t + \chi'(G-t)u_f^t)}\, e^{-\phi^t} = \int_\Omega u_f^t\, \overline{\partial_t u_{H_0^\perp}}\, e^{-\phi^t} = -||u_{H_0^\perp}||_t^2,$$

and (5.17) follows from (5.22). □

Now we can apply the complex version (see Theorem 5.4) of Lemma 3.2 to prove the following convexity theorem of Berndtsson (compare with Theorem 3.1).

Convexity theorem of Berndtsson. *Let $G \leq 0$ be a function on a pseudoconvex domain $\Omega \subset \mathbb{C}^n$ such that e^G is smooth and*

$$\phi + \lambda G \text{ is smooth and strictly plurisubharmonic} \tag{5.23}$$

on $\{G \neq -\infty\}$ for some constant $\lambda > 0$, where ϕ is a given smooth strictly plurisubharmonic function on Ω. Let $\chi : \mathbb{R} \to [0, \infty)$ be a smooth convex increasing function that vanishes on $(-\infty, 0]$ with $\chi' \leq \lambda$. Let f be a bounded $\mathbb{C}$-linear functional on

$$H_0 := \left\{ u \text{ is holomorphic on } \Omega : ||u||_\phi^2 := \int_\Omega |u|^2 e^{-\phi} < \infty \right\},$$

then

$$\log ||f||_t^2 := \sup\left\{ \log |f(u)|^2 : u \in H_0, \int_\Omega |u|^2 e^{-\phi-\chi(G-t)} = 1 \right\}$$

is convex in $t \in \mathbb{R}$.

Proof. By our assumptions and Theorem 5.4, we have

$$||\chi'(G-t)u_f^t - P^t(\chi'(G-t)u_f^t)||_t^2 \leq \int_\Omega \chi''(G-t)|u_f^t|^2\, e^{-\phi^t}, \tag{5.24}$$

hence our theorem follows from (5.17). (20th October) □

Exercise 24: Prove (5.24).

Remark: *Our assumptions imply that*

$$(\tau, z) \mapsto \phi(z) + \chi(G(z) - \mathrm{Re}\,\tau)$$

is plurisubharmonic in $(\tau, z) \in \mathbb{C} \times \Omega$*, which can be seen a complex version of the convexity assumption of* ϕ *in Theorem 3.1. In applications, the best* χ *would be*

$$\sup\{\chi(s) : \chi : \mathbb{R} \to [0, \infty) \text{ is smooth convex increasing, vanishes on } (-\infty, 0] \text{ and } \chi' \leq \lambda\},$$

which $= \lambda \max\{s, 0\}$*. Let us choose a family of* χ*, satisfying the above assumptions, with limit* $\lambda \max\{s, 0\}$*, then the above theorem gives the following result.*

Theorem 5.8. *Let* $G \leq 0$ *be a function on a pseudoconvex domain* $\Omega \subset \mathbb{C}^n$ *such that* e^G *is smooth and*

$$\phi + \lambda G \text{ is smooth and strictly plurisubarmonic} \tag{5.25}$$

on $\{G \neq -\infty\}$ *for some constant* $\lambda > 0$*, where* ϕ *is a given smooth strictly plurisubarmonic function on* Ω*. Let* f *be a bounded* $\mathbb{C}$*-linear functional on*

$$H_0 := \left\{ u \text{ is holomorphic on } \Omega : ||u||^2_\phi := \int_\Omega |u|^2 e^{-\phi} < \infty \right\},$$

then

$$\log ||f||^2_t := \sup \left\{ \log |f(u)|^2 : u \in H_0, \int_\Omega |u|^2 e^{-\phi - \lambda \max\{G-t, 0\}} = 1 \right\}$$

is convex in $t \in \mathbb{R}$*.*

5.2.2. *Implicit function theorem for Banach spaces.* In this section, we shall recall the implicit function theorem for Banach spaces by following Hörmander's book [H0].

Definition 5.5. *Let* H *be a real vector space. We call*

$$|| \cdot || : H \to [0, \infty), \tag{5.26}$$

a norm on H *if*

$$\begin{aligned} ||cx|| &= |c| \cdot ||x||, && \forall\, c \in \mathbb{R}, \\ ||x + y|| &\leq ||x|| + ||y||, && \forall\, x, y \in H, \\ ||x|| &> 0, && \text{if } x \neq 0. \end{aligned}$$

Let H be a real vector space with norm $|| \cdot ||$. Then H is a metric space with distance function:

$$d(x, y) := ||x - y||. \tag{5.27}$$

Recall that a metric space H is complete if each Cauchy sequence in H has a unique limit point in H. If H is not complete then one may consider its completion $\tilde{H}$, which is defined to be the space of equivalent Cauchy sequences in H.

Definition 5.6. *Let* H *be a real vector space with norm* $|| \cdot ||$*. We call* H *a real Banach space if* H *is complete as a metric space.*

Remark: *Let H_1 and H_2 be two real Banach spaces. Then $H_1 \times H_2$ is also a Banach space with norm*

$$||(x,y)||^2 := ||x||^2 + ||y||^2, \ \forall\, (x,y) \in H_1 \times H_2. \tag{5.28}$$

If we denote by $L(H_1, H_2)$ the space of bounded $\mathbb{R}$-linear maps from H_1 to H_2 then we know that $L(H_1, H_2)$ is also a Banach space with norm

$$||T|| := \sup\{||Tx|| : ||x|| \leq 1, \ x \in H_1\}, \ \forall\, T \in L(H_1, H_2). \tag{5.29}$$

Notice that if H_1 is $\mathbb{R}$ with the usual norm then $L(H_1, H_2)$ is isomorphic to H_2. Now we can define the notion of differentiability on Banach space.

Definition 5.7. *Let H_1 and H_2 be two real Banach spaces. Let f be a map from an open set U in H_1 to H_2. We say f is differentiable at $x \in U$ if there exists $T \in L(H_1, H_2)$ such that*

$$||f(x+h) - f(x) - Th|| = o||h||, \ i.e. \lim_{||h|| \to 0, \ h \neq 0} \frac{||f(x+h) - f(x) - Th||}{||h||} = 0. \tag{5.30}$$

If f is differentiable at x then we shall write $T = f'(x)$. We call f is C^1 on U if f is differentiable at all points in U and its derivative

$$f' : x \mapsto f'(x), \tag{5.31}$$

is a continuous map from U to $L(H_1, H_2)$.

Remark: *Let f be a C^1 map, we say f is C^2 if the derivative, say $f^{(2)}$, of f' is C^1. Then we can inductively define the notion of a C^k map, and define $f^{(k+1)}$ as the derivative of $f^{(k)}$ for every $k \geq 2$. And we say f is C^∞ or smooth if f is C^k for every k. Now we can state the inverse function theorem for Banach spaces:*

Theorem 5.9 (Inverse function theorem)**.** *Let f be a C^1 map from an open set U in a real Banach space H to H. Assume that f is C^1 on U, $0 \in U$, $f(0) = 0$ and $f'(0) = id_H$ is the identity map on H. Then there exists an open neighborhood $V \subset U$ of 0 such that $f|_V$ is a homeomorphism onto the open set $f(V)$ and its inverse*

$$f^{-1} : f(V) \to V \tag{5.32}$$

is also C^1. Assume further that f is C^k then f^{-1} is also C^k on the same set $f(V)$.

Before proving it, let us show how to use it to prove the implicit function theorem:

Theorem 5.10 (Implicit function theorem)**.** *Let H_1, H_2 and H_3 be three real Banach spaces. Let Φ be a smooth map from an open neighborhood, say U, of $(x_0, y_0) \in H_1 \times H_2$ to H_3. Assume that there exists $A \in L(H_3, H_2)$ such that*

$$A\Phi'_{H_2}(x_0, y_0) = id_{H_2}, \ \Phi'_{H_2}(x_0, y_0)A = id_{H_3}, \tag{5.33}$$

then there exists a neighborhood, say V, of x_0 and a unique smooth map, say f, from V to H_2 such that $f(x_0) = y_0$ and

$$\Phi(x, f(x)) \equiv \Phi(x_0, y_0), \ \forall\, x \in V. \tag{5.34}$$

Proof. Notice that we can assume x_0, y_0 are at the origin. Put

$$\Psi(x,y) = (x, \Phi(x,y)). \tag{5.35}$$

Then

$$||\Psi(a,b) - \Psi(0) - (a, \Phi'_{H_1}(0)a + \Phi'_{H_2}(0)b)|| = o||(a,b)||. \tag{5.36}$$

Thus

$$\Psi'(0)(a,b) = (a, \Phi'_{H_1}(0)a + \Phi'_{H_2}(0)b). \tag{5.37}$$

Let us consider the linear map, say B, from $H_1 \times H_3$ to $H_1 \times H_2$ such that

$$B(a,c) = (a, A(c - \Phi'_{H_1}(0)a)). \tag{5.38}$$

Thus

$$B\Psi'(0) = id_{H_1 \times H_2}, \ \Psi'(0)B = id_{H_1 \times H_3}. \tag{5.39}$$

Put

$$\tilde{\Psi} := B\Psi. \tag{5.40}$$

Then $\tilde{\Psi}'(0) = id_{H_1 \times H_2}$. By the inverse function theorem, the inverse, $\tilde{\Psi}^{-1}$ of $\tilde{\Psi}$ is smooth in a neighborhood of the origin. Thus $\tilde{\Psi}^{-1}B$ is the inverse map of Ψ near the origin. Let us denote it by Ψ^{-1} and write

$$\Psi^{-1}(x,z) = (x, g(x,z)). \tag{5.41}$$

Thus

$$\Phi(x, g(x,z)) \equiv z \tag{5.42}$$

near the origin, which implies that $f(x) = g(x,0)$. □

Proof of the inverse function theorem. We shall use the following lemma.

Lemma 5.11. *Let f be a C^1 map from an open set U in a Banach space H to H. Then*

$$||f(x) - f(y)|| \leq ||x - y|| \cdot \sup_{0 \leq t \leq 1} ||f'(x + t(y-x))||, \ \ \forall\, x, y \in H. \tag{5.43}$$

Remark. *Put $g(t) = f(x + t(y-x))$, then $g'(t) = f'(x + t(y-x))(y-x)$ and the above estimate follows directly from the following Newton-Lebnitz formula*

$$f(y) - f(x) = g(1) - g(0) = \int_0^1 g'(t)\, dt = \int_0^1 f'(x + t(y-x))(y-x)\, dt.$$

Fix y near the origin, we want to find x^* such that $f(x^*) = y$. Put

$$g(x) = x - f(x) + y,$$

it suffices to find the fixed point of g. We claim that the fixed point of g exists and is given by the limit of

$$x_{k+1} := y - f(x_k) + x_k = g(x_k), \ k \geq 0, \ x_0 := 0. \quad \text{(24th October)}$$

In fact, since $g'(0) = 0$ and g is C^1, there exists a small $\delta > 0$ such that

$$||g'(x)|| < \frac{1}{2}, \ \forall \ ||x|| \leq 2\delta. \tag{5.44}$$

Applying Lemma 5.11 to g, we know that

$$||g(x') - g(x)|| \leq \frac{1}{2}||x' - x||, \tag{5.45}$$

if both $||x||$ and $||x'||$ are $< 2\delta$. Note that $x_1 = y$, $x_0 = 0$. Apply (5.45) to x_1, x_0, we get

$$||x_2 - x_1|| = ||g(x_1) - g(x_0)|| \leq \frac{1}{2}||x_1 - x_0|| = \frac{1}{2}||y||.$$

Assume that $||y|| < \delta$, we obtain which gives

$$||x_2|| \leq ||x_1|| + \frac{1}{2}||y|| = \left(1 + \frac{1}{2}\right)||y|| < 2\delta.$$

By induction on k, we have

$$||x_{k+1} - x_k|| \leq \frac{1}{2}||x_k - x_{k-1}|| \leq \cdots \leq \frac{\delta}{2^k}, \ \forall \ k \geq 2 \tag{5.46}$$

and

$$||x_{k+1}|| \leq \left(1 + \cdots + \frac{1}{2^k}\right)||y|| \leq 2||y|| < 2\delta, \ \forall \ k \geq 0, \tag{5.47}$$

thus $\{x_k\}$ is a Cauchy sequence with limit x^* such that $||x^*|| \leq 2||y|| < 2\delta$. Now we know that $f(x^*) = y$. If there is another $\hat{x}$ such that $||\hat{x}|| < 2\delta$ and $f(\hat{x}) = y$ then (5.45) implies that

$$||\hat{x} - x^*|| = ||g(\hat{x}) - g(x^*)|| \leq \frac{1}{2}||\hat{x} - x^*||. \tag{5.48}$$

Thus $x^* = x$. Thus for every y with $||y|| < \delta$ there exists a unique point, say x^*, with $||x^*|| < 2\delta$ such that $f(x^*) = y$. We shall write $x^* = f^{-1}(y)$. Put

$$V = \{x : ||x|| < 2\delta\} \cap f^{-1}\{y : ||y|| < \delta\}. \tag{5.49}$$

Then $V \subset U$ is an open neighborhood of the origin such that $f|_V$ is a bijection onto

$$f(V) = \{y \in H : ||y|| < \delta\}.$$

The final step is to prove that f^{-1} is C^1 on $f(V)$. Fix $y_0 = f(x_0)$, $x_0 \in V$. Since f is differentiable at x_0, we have

$$||f(x) - f(x_0) - f'(x_0)(x - x_0)|| = o||x - x_0||. \tag{5.50}$$

Notice that (5.44) implies that $f'(x_0)$ is invertiable. And (5.45) implies that

$$\frac{1}{2}||x - x_0|| \leq ||y - y_0|| \leq 2||x - x_0||, \ \ \forall \ x \in V, \ y = f(x). \tag{5.51}$$

Thus (5.50) gives

$$||(f'(x_0))^{-1}(y - y_0) - (x - x_0)|| = o||y - y_0||, \tag{5.52}$$

which implies that f^{-1} is differentiable at y_0 with derivative $(f'(f^{-1}(y_0)))^{-1}$. Thus f^{-1} is differentiable on $f(V)$ and its derivative

$$y \mapsto (f'(f^{-1}(y)))^{-1} \tag{5.53}$$

is continuous since f is C^1. Using (5.53) inductively, we know that if f is C^k on V then f^{-1} is also C^k on $f(V)$. The proof is complete. □

5.3. **Convexity of Bergman kernels.** Let ψ be a plurisubharmonic function on a domain $\Omega \subset \mathbb{C}^n$. Fix $z_0 \in \Omega$, we call

$$K_\psi(z_0) := \sup_{u \text{ holomorphic on } \Omega} \frac{|u(z_0)|^2}{\int_\Omega |u|^2 e^{-\psi}} \tag{5.54}$$

the Bergman kernel with respect to (Ω, ψ, z_0). Apply Theorem 5.8 to the functional

$$f : u \mapsto u(z_0),$$

we obtain the following convexity property of the Bergman kernels.

Theorem 5.12. *Let $G \leq 0$ be a function on a pseudoconvex domain $\Omega \subset \mathbb{C}^n$ such that e^G is smooth and*

$$\phi + \lambda G \text{ is smooth and strictly plurisubarmonic} \tag{5.55}$$

on $\{G \neq -\infty\}$ for some constant $\lambda > 0$, where ϕ is a given smooth strictly plurisubarmonic function on Ω. Put

$$H_0 := \left\{ u \text{ is holomorphic on } \Omega : ||u||^2_\phi := \int_\Omega |u|^2 e^{-\phi} < \infty \right\}, \quad \phi^t := \phi + \lambda \max\{G - t, 0\},$$

then $\log K_{\phi^t}(z_0)$ is convex in $t \in \mathbb{R}$.

Exercise 25: Replace ϕ by $\phi + \varepsilon|z|^2$ and show that the above theorem can be generalized to the following case.

Theorem 5.13. *Let $G \leq 0$ be a function on a pseudoconvex domain $\Omega \subset \mathbb{C}^n$ such that e^G is smooth and*

$$\phi + \lambda G \text{ is smooth and plurisubarmonic} \tag{5.56}$$

on $\{G \neq -\infty\}$ for some constant $\lambda > 0$, where ϕ is a given smooth plurisubarmonic function on Ω. Put

$$H_0 := \left\{ u \text{ is holomorphic on } \Omega : ||u||^2_\phi := \int_\Omega |u|^2 e^{-\phi} < \infty \right\}, \quad \phi^t := \phi + \lambda \max\{G - t, 0\},$$

then $\log K_{\phi^t}(z_0)$ is convex in $t \in \mathbb{R}$.

5.4. **Suita conjecture.** Let us apply Theorem 5.12 to the case that $\Omega \subset \mathbb{C}$ is smoothly bounded (in fact, we only need the Green function of Ω exists), $\phi = 0$ and

$$G(z) := 2G_\Omega(z, z_0).$$

Note that

$$K_{\phi^t}(z_0) \leq \sup_{u \text{ holomorphic on } \Omega} \frac{|u(z_0)|^2}{\int_{G<t} |u|^2} \leq Ce^{-2t}$$

for some constant C does not depend on t, we know that the convex function $\log K_{\phi^t}(z_0) + 2t$ is bounded near $t = -\infty$, thus it is increasing in t, which gives

$$K_\Omega(z_0) = K_{\phi^0}(z_0) \geq \lim_{t\to-\infty} e^{2t} K_{\phi^t}(z_0) \geq \frac{1}{\limsup_{t\to-\infty} e^{-2t} \int_\Omega e^{-\lambda \max\{G-t,0\}}}.$$

Put

$$A(s) := \text{the Lebesgue measure of } \{G < s\}.$$

Then for every $t < 0$, we have

$$\int_\Omega e^{-\lambda \max\{G-t,0\}} = \int_{-\infty}^0 e^{-\lambda \max\{s-t,0\}} dA(s) = e^{\lambda t} A(0) - \int_{-\infty}^0 A(s)\, d\, e^{-\lambda \max\{s-t,0\}}.$$

Thus for $\lambda > 2$, we have

$$\begin{aligned}
\frac{1}{K_\Omega(z_0)} &\leq \limsup_{t\to-\infty} \left(-e^{-2t} \int_{-\infty}^0 A(s)\, d\, e^{-\lambda \max\{s-t,0\}}\right) \\
&= \limsup_{t\to-\infty} \left(-e^{-2t} \int_t^0 A(s)\, d\, e^{-\lambda(s-t)}\right) \\
&= \limsup_{t\to-\infty} \left(\lambda \int_t^0 A(s) e^{-2s} e^{-(\lambda-2)(s-t)}\, ds\right) \\
&= \limsup_{t\to-\infty} \left(\lambda \int_0^{-t} A(x+t) e^{-2(x+t)} e^{-(\lambda-2)x}\, dx\right) \\
&\leq \left(\lambda \int_0^\infty e^{-(\lambda-2)x}\, dx\right) \limsup_{s\to-\infty} \left(A(s)e^{-2s}\right) \\
&= \frac{\lambda}{\lambda-2} \pi e^{-2\rho(z_0)},
\end{aligned}$$

where

$$\rho(z_0) := \lim_{z\to z_0} \{G(z) - \log|z - z_0|\}$$

denotes the Robin constant of Ω at z_0. Letting $\lambda \to \infty$, the Suita conjecture (0.1) follows.

Exam project: Use the methods in section 5.2 to prove the following Berndtsson's subharmonicity property of the Bergman kernel (try to read the original paper [B06]!).

Subharmonicity property of the Bergman kernel. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Let ϕ be a smooth strictly plurisubarmonic function on $\mathbb{D}\times U$, where $\mathbb{D}:=\{t\in\mathbb{C}:|t|<1\}$ and U is an open neighborhood of the closure of Ω. Put*

$$\phi^t(z):=\phi(t,z),\ \ (t,z)\in\mathbb{D}\times\Omega.$$

Then

$$(t,z)\mapsto\log K_{\phi^t}(z)$$

is plurisubarmonic in $(t,z)\in\mathbb{D}\times\Omega$. In particular, for every fixed $z_0\in\Omega$, $\log K_{\phi^t}(z_0)$ is subharmonic in t (compare with Theorem 5.13).

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