Since $G_j^{-1}(z_i) = w_i$ for all $j > i$, we have $F(z_i) = w_i$ for all i. Thus $E \subset F(\mathbf{C}^n)$.

To finish, assume, to reach a contradiction, that $w = F(z)$ for some $w \in K$, $z \in \mathbf{C}^n$. Let β be a ball with center z. For all sufficiently large j it follows from (4) that there are points $p_j \in \beta$ so that $G_j^{-1}(p_j) = w$, i.e., $p_j = G_j(w)$. But our construction shows that $|G_j(w)| \to \infty$ as $j \to \infty$, because $w \in K$, whereas $\{p_j\}$ is bounded. This contradiction shows that $F(\mathbf{C}^n)$ contains no point of K.

8.6. REMARK. In [**9**, **10**], J. A. Morrow has classified the nonsingular compact complex manifolds M of complex dimension 2 that contain a nonempty nowhere dense closed *analytic* subset A so that $M\backslash A$ is biholomorphic to $\mathbf{C}^2$.

One may ask whether "analytic" is redundant in this statement. Theorem 8.5 shows that it is not:

Take $n = 2$, K a point (say $K = \{0\}$), E dense in $\mathbf{C}^2$, and construct F as in the proof of Theorem 8.5, as the limit of a sequence of automorphisms of $\mathbf{C}^2$. This implies that $\Omega = F(\mathbf{C}^2)$ is a Runge domain [**2**, p. 141].

Let L be a complex line in $\mathbf{C}^2\backslash\{0\}$ which intersects Ω, and put $L_w = \{\lambda w : \lambda \in \mathbf{C}\}$ for each $w \in L \cap \Omega$. Since $0 \notin \Omega$, no L_w lies in Ω. Since Ω is a Runge domain, each component of $\Omega \cap L_w$ is simply connected (otherwise polynomial approximation would fail; see, for example, [**20**]) and its boundary relative to L_w must therefore have positive one-dimensional Hausdorff measure. This holds for each $w \in L \cap \Omega$. A Fubini-type argument shows now that *the Hausdorff dimension of* $\mathbf{C}^2\backslash\Omega$ *is at least* 3.

We may regard F as a biholomorphic map from $\mathbf{C}^2$ into (for example) complex projective space P^2, with Ω dense in P^2. Put $A = P^2\backslash\Omega$. Then $A \supset \mathbf{C}^2\backslash\Omega$, so that A has Hausdorff dimension ≥ 3, and this shows that A is not an analytic subset of P^2. (The Hausdorff dimension of analytic subsets of P^2 is at most 2.)

We thank E. L. Stout for drawing our attention to this question.

9. Regions attracted to a fixed point. We begin with a simple case of the basic theorem that was mentioned in the Introduction.

9.1. THEOREM. *Suppose* $F \in \mathrm{Aut}(\mathbf{C}^n)$, $p \in \mathbf{C}^n$, $F(p) = p$, *and the eigenvalues* λ_i *of* $A = F'(p)$ *satisfy* $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ *and*

$$|\lambda_1|^2 < |\lambda_n|. \tag{1}$$

Define

$$\Omega = \left\{ z \in \mathbf{C}^n : \lim_{k\to\infty} F^k(z) = p \right\}. \tag{2}$$

Then Ω *is a region, and there is a biholomorphic map* Ψ *from* Ω *onto* $\mathbf{C}^n$, *given by*

$$\Psi = \lim_{k\to\infty} A^{-k}F^k. \tag{3}$$

The convergence in (3) *is uniform on compact subsets of* Ω.

Recall that $F^k = F \circ F^{k-1}$, $F^1 = F$. Note that (1) implies that $0 < |\lambda_i| < 1$ for all i. We may describe Ω as *the region that is attracted to* p *by* F.

One immediate consequence of (3) is the functional equation

$$\Psi = A^{-1}\Psi F. \tag{4}$$

Another is that $J\Psi \equiv 1$ whenever JF is constant.

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PROOF. Take $p = 0$, without loss of generality. Pick constants α, β_1, β_2, β so that $\alpha < |\lambda_n|$, $|\lambda_1| < \beta_1 < \beta_2 < \beta$, and $\beta^2 < \alpha$. The spectral radius formula gives an m so that $\|A^{-N}\| < \alpha^{-N}$ and $\|A^N\| < \beta_1^N$ for all $N \geq m$. Approximating F^m by A^m shows that there is an $r > 0$ so that (for our fixed m) $z \in rB$ implies

(5) $$|F^m(z)| \leq \beta_2^m |z|.$$

Put $C = \sup\{|F^j(z)|/|z| : 0 \leq j < m, 0 < |z| < r\}$.

If $N = km + j$, $k = 1, 2, 3, \ldots$, $0 \leq j < m$, and if $|z| < r$, then iteration of (5) yields

$$|F^N(z)| = |F^j(F^{km}(z))| \leq C|F^{km}(z)| \leq C\beta_2^{km}|z|.$$

Thus, for all sufficiently large $N \geq N_0$ (where $N_0 \geq m$ depends only on m and r) we have

(6) $$|F^N(z)| < \beta^N \quad \text{for all } z \in rB.$$

It follows from (6) that $rB \subset \Omega$ (because $\beta < 1$), hence that

(7) $$\Omega = \bigcup_{-\infty}^{\infty} F^k(rB).$$

This shows that Ω is a region and that $F(\Omega) = \Omega$.

Now pick a compact set $K \subset \Omega$. For some s, $F^s(K) \subset rB$. Hence (6) shows that

(8) $$|F^N(z)| \leq \beta^{N-s} = a\beta^N \qquad (z \in K,\ N \geq s + N_0)$$

where $a = \beta^{-s}$. Since $(A^{-1}F)'(0) = I$, there is a constant b so that

(9) $$|w - A^{-1}F(w)| \leq b|w|^2 \qquad (|w| \leq a).$$

Thus, if $z \in K$ and if we set $w_N = F^N(z)$, we get the estimate

$$\begin{aligned} |A^{-N}F^N(z) - A^{-N-1}F^{N+1}(z)| &\leq \|A^{-N}\| \cdot |w_N - A^{-1}F(w_N)| \\ &\leq \alpha^{-N} b|w_N|^2 \leq a^2 b(\beta^2/\alpha)^N \end{aligned}$$

for all $N \geq s + N_0$.

Since $\beta^2/\alpha < 1$, it follows that (3) holds. It is clear that Ψ (being a limit of a sequence of automorphisms) is holomorphic and one-to-one in Ω. (Note that $\Psi'(0) = I$.) Since $F(\Omega) = \Omega$ and $\Psi = A^{-1}\Psi F$, we see that Ψ and $A^{-1}\Psi$ have the same range. Since the linear operator A^{-1} is an expansion, it follows that $\Psi(\Omega)$ is all of $\mathbf{C}^n$.

9.2. EXAMPLE. Define $F \in \mathrm{Aut}(\mathbf{C}^2)$ by $F(z, w) = (\alpha z, \beta w + z^2)$, where $0 < \beta < \alpha < 1$. This F fixes the origin, and $A = F'(0,0) = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$. By induction

$$F^k(z, w) = (\alpha^k z, \beta^k w + \beta^{k-1}(1 + c + \cdots + c^{k-1})z^2),$$

where $c = \alpha^2/\beta$. Thus

$$(A^{-k}F^k)(z, w) = (z, w + \beta^{-1}(1 + c + \cdots + c^{k-1})z^2).$$

The coefficient of z^2 in the second component of $A^{-k}F^k$ tends to infinity, except when $c < 1$, i.e., when $\alpha^2 < \beta$.

Conclusion: The sequence (3) *may fail to converge (even locally, and even on the level of formal power series) if assumption* (1) *of Theorem* 9.1 *is violated.*

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The region that is attracted to the origin by this F is all of $\mathbf{C}^2$. To get away from this, put

$$G(z,w) = (\alpha z + (\beta w + z^2)^2, \beta w + z^2).$$

Again, $G \in \operatorname{Aut}(\mathbf{C}^2)$; $G'(0,0) = F'(0,0)$; the coefficients in G^k are at least as large as those in F^k. Therefore $A^{-k}G^k$ will still diverge when $\alpha^2 \geq \beta$. But now the region Ω that is attracted to $(0,0)$ by G is not all of $\mathbf{C}^2$, because G has three other fixed points, given by $z^3 = (1-\alpha)(1-\beta)^2$, $w = z^2/(1-\beta)$.

9.3. NOTATION. In the examples that follow, we shall use the abbreviation *F. B. region* (for Fatou-Bieberbach) to denote regions $\Omega \subset \mathbf{C}^n$, $\Omega \neq \mathbf{C}^n$, which are biholomorphically equivalent to $\mathbf{C}^n$.

Actually, the examples will all be in $\mathbf{C}^2$.

9.4. *Example of an F.B. region $\Omega \subset \mathbf{C}^2$ whose intersection with every complex line is bounded.* Define $F(z,w) = (u,v)$ by

$$u = \alpha w, \qquad v = \alpha z + w^2 \tag{1}$$

for some fixed α, $0 < |\alpha| < 1$. Then $F \in \operatorname{Aut}(\mathbf{C}^2)$, F fixes $(0,0)$, the eigenvalues of $F'(0,0)$ are $\pm\alpha$. Let Ω be the region attracted to $(0,0)$ by F, as in Theorem 9.1.

If (z,w) lies in the set E defined by $|w| > 1 + 2|\alpha| + |z|$, then (1) shows that

$$\begin{aligned}|v| &\geq |w|^2 - |\alpha z| > |w|^2 - |\alpha w| = |w|(|w| - |\alpha|) \\ &> |w|(1+|\alpha|) > 1 + 2|\alpha| + |u|\end{aligned}$$

so that $(u,v) \in E$. Thus $F(E) \subset E$. This shows that no point of E lies in Ω.

Now let L be a complex line in $\mathbf{C}^2$. Parametrize L by $z = a + b\lambda$, $w = c + d\lambda$, where a, b, c, d are constants and λ ranges over $\mathbf{C}$. If we substitute these expressions for z and w into (1) we see that $F(z,w) \in E$ as soon as $|\lambda|$ is large enough. (Note that $d = 0$ implies $b \neq 0$.) For such λ, it follows that (z,w) is not in Ω.

9.5. EXAMPLE. The automorphism $F(z,w) = (u,v)$ given by

$$u = z + w, \qquad v = \tfrac{1}{2}(1 - w - e^{z+w}) \tag{1}$$

leads to several interesting phenomena.

Its fixed points are

$$p_m = (2m\pi i, 0), \tag{2}$$

one for each integer m. The eigenvalues of $F'(p_m)$ are $\pm 1/\sqrt{2}$. Theorem 9.1 can therefore be applied:

There exist pairwise disjoint F.B. regions $\Omega_m \subset \mathbf{C}^2$ $(m = 0, \pm1, \pm2, \dots)$, attracted to p_m by F, which are translates of each other:

$$\Omega_m = \Omega_0 + p_m. \tag{3}$$

To see (3), note that $F((z,w) + p_m) = F(z,w) + p_m$. Hence

$$\lim_{k\to\infty} F^k((z,w) + p_m) = p_m \quad \text{if } \lim_{k\to\infty} F^k(z,w) = p_0.$$

It follows from (3) and the disjointness of $\{\Omega_m\}$ that the map E given by

$$E(u,v) = (e^u, ve^{-u}) \tag{4}$$

is one-to-one on each Ω_m, and that

$$\Omega^* = E(\Omega_m) \tag{5}$$

is independent of m.

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This gives an F. B. region Ω^* in $\mathbf{C}^2$ which does not intersect the line $\{z=0\}$.

Moreover, since $JF \equiv -1/2$ (a constant), the Ω_m's as well as Ω^* are biholomorphic images of $\mathbf{C}^2$ via *volume-preserving maps.* (This is why we defined E by (4), rather than by the simpler formula $E(u,v)=(e^u,v)$.)

It is known [5] that the range of a nondegenerate holomorphic map from $\mathbf{C}^2$ into $\mathbf{C}^2$ cannot avoid 3 complex lines. We shall now see that this is not so if complex lines are replaced by translates of R^2. Here R^2 denotes the set of points of $\mathbf{C}^2$ both of whose coordinates are real. Define

$$\Pi_k = R^2 + ((2k+1)\pi i, 0) \tag{6}$$

for $k=0,\pm1,\pm2,\ldots$. Then $F(\Pi_k)=\Pi_k$, and no p_m lies in any Π_k. Therefore no point of any Π_k is attracted to any p_m by F.

Conclusion: No Π_k intersects any Ω_m.

Finally, we modify the regions Ω_m so as to obtain disjoint F. B. regions $\tilde{\Omega}_m$ with the following property:

For each m, $\tilde{\Omega}_m \cap \{w=0\}$ has infinitely many components.

Picard's theorem shows that at most one line $u=$ const. misses Ω_0. Therefore Ω_0 contains points (u_s,v_s) with $u_s=s+iy_s$, $2s\pi < y_s < (2s+1)\pi$, for every integer s. Since the numbers $\exp u_s$ are not real, and no two of them are complex conjugates of each other, there is an entire function $h\colon \mathbf{C}\to\mathbf{C}$ so that $h(R)\subset R$ and $h(\exp(u_s))=v_s$. Define a shear Φ by

$$\Phi(u,v) = (u, v-h(e^u)) \tag{7}$$

and put $\tilde{\Omega}_m = \Phi(\Omega_m)$.

Since $\Phi(\Pi_k)=\Pi_k$, no Π_k intersects any $\tilde{\Omega}_m$. Each $\tilde{\Omega}_m$ contains the points

$$(u_s+2m\pi i, v_s - h(e^{u_s})) = (s+(y_s+2m\pi)i, 0), \tag{8}$$

one in each strip bounded by the (real) lines

$$(x+(2k+1)\pi i, 0) \qquad (-\infty < x < \infty) \tag{9}$$

which lie in Π_k. Thus $\tilde{\Omega}_m$ has at least one component in each of these strips.

9.6. EXAMPLE. We just saw that there exist F. B. regions Ω_m in $\mathbf{C}^2$ which miss infinitely many translates of R^2. *The same can be done with finitely many rotated copies of R^2:*

Let N be a positive integer, put $\alpha=\exp(\pi i/2N)$, and put $E_k=\alpha^k R^2$ for $k=0,1,\ldots,2N-1$. Define $F(z,w)=(u,v)$ by

$$u = z+w, \qquad v = \frac{1}{2N+1}[z+(z+w)^{2N+1}]. \tag{1}$$

Then $F\in \mathrm{Aut}(\mathbf{C}^2)$, $F(E_k)=E_k$ for all k, the fixed points of F are $(0,0)$ and $p_m=(\alpha^m,0)$ for *odd* m. The eigenvalues of $F'(p_m)$ are $\pm(2N+1)^{-1/2}$. It follows from Theorem 9.1 that there are N pairwise disjoint F. B. regions Ω_m, attracted to p_m by F, and

$$\Omega_m \subset \mathbf{C}^2 \backslash (E_0 \cup E_2 \cup \cdots \cup E_{2N-2}). \tag{2}$$

Note also that $F(\alpha^2 z, \alpha^2 w) = \alpha^2 F(z,w)$, by (1). From this one can deduce that the rotation $(z,w)\to(\alpha^2 z,\alpha^2 w)$ permutes the regions Ω_m.

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9.7. *Example of an F.B. region $\Omega_0 \subset \mathbf{C}^2$ whose closure misses a complex line.* (We do not know whether the region Ω^* in Example 9.5 also has this property.)

Pick $\alpha \in \mathbf{C}$, $0 < |\alpha| < 1$, find an entire function $f\colon \mathbf{C} \to \mathbf{C}$ so that

(1) $$e^{f(0)} = 1/\alpha, \quad f'(0) = 0, \quad f(1) = 0, \quad f'(1) = (1+\alpha^2/(1-\alpha^2),$$

and define $F(z,w) = (u,v)$ by

(2) $$u = 1 - \alpha^2 + \alpha^2 z e^{f(zw)}, \qquad v = we^{-f(zw)}.$$

Then $F \in \operatorname{Aut}(\mathbf{C}^2)$, $JF \equiv \alpha^2$, $F(1,1) = (1,1)$, and the eigenvalues of $F'(1,1)$ are $\pm\alpha i$. Let Ω_0 be the region attracted to $(1,1)$ by F.

Let Ω_1 be the region attracted to the fixed point $(1+\alpha, 0)$, where $F' = \alpha I$. Since

(3) $$F(z,0) = (1 - \alpha^2 + \alpha z, 0)$$

for all $z \in \mathbf{C}$, we see that Ω_1 contains the line $\{w = 0\}$. Therefore $\bar{\Omega}_0$ does not intersect this line.

This example is quite similar to one of Nishimura's [11]. He does not, however, derive it from a theorem about *fixed points* of automorphisms, but from a more difficult one that involves *pointwise fixed analytic subvarieties.*

9.8. REMARK. All the F. B. regions Ω obtained in Examples 9.4 to 9.7 were ranges of biholomorphic maps $\Phi\colon \mathbf{C}^2 \to \Omega$ with $J\Phi \equiv 1$, because the automorphisms that were used in the constructions had constant Jacobians. (Here $\Phi = \Psi^{-1}$, where Ψ is given by Theorem 9.1.)

Our next example will use automorphisms of the kind that we mentioned at the end of the Introduction. That the resulting map Φ does not have constant Jacobian follows from Theorem I of [13], which states:

If $\Phi\colon \mathbf{C}^2 \to \mathbf{C}^2$ is holomorphic and one-to-one, $J\Phi \equiv c$, and Φ preserves the lines $\{z = 0\}$ and $\{w = 0\}$, then $\Phi \in \operatorname{Aut}(\mathbf{C}^2)$; in fact

$$\Phi(z,w) = (cze^{f(zw)}, we^{-f(zw)})$$

for some entire $f\colon \mathbf{C} \to \mathbf{C}$.

9.9. *Example of an F. B. region Ω in $\mathbf{C}^2$ which contains the set $\{zw = 0\}$ and is not dense in $\mathbf{C}^2$.* Let $g, h\colon \mathbf{C} \to \mathbf{C}$ be entire functions, so that

(1) $$\exp g(0) = 2, \qquad \exp h(0) = 1/4$$

and

(2) $$\exp g(2^{4p}) = 1/2, \qquad \exp h(2^{2p+2}) = 4$$

for $p = 0, 1, 2, \ldots$, define

(3) $$G(z,w) = (z \exp g(z^3 w), w \exp[-3g(z^3 w)]),$$

(4) $$H(u,v) = (u \exp h(uv), v \exp[-h(uv)])$$

and put $F = H \circ G$. Then $F \in \operatorname{Aut}(\mathbf{C}^2)$,

(5) $$F(z,0) = \left(\tfrac{1}{2}z, 0\right), \qquad F(0,w) = \left(0, \tfrac{1}{2}w\right)$$

and

(6) $$F(2^p, 2^p) = (2^{p+1}, 2^{p+1}) \qquad (p = 0, 1, 2, \ldots).$$

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Setting $A = F'(0,0)$, we have $A = \frac{1}{2}I$. Hence (5) shows that each $A^{-k}F^k$ fixes every point of $\{zw = 0\}$. If $\Phi = \Psi^{-1}$, where Ψ is as given by Theorem 9.1, we conclude:

Φ is a biholomorphic map from $\mathbf{C}^2$ onto the region Ω that is attracted to the origin by F; every point of $\{zw = 0\}$ lies in Ω because Φ fixes it; by (6), Ω contains none of the points $(2^p, 2^p)$.

In particular, $\Omega \neq \mathbf{C}^2$.

But we claimed more, namely that Ω is not dense in $\mathbf{C}^2$. To achieve this, we have to choose g and h with more care; specifically, we strengthen (2) by requiring that g and h are almost constant on discs centered at 2^{4p} and 2^{2p+2}, respectively. Here are the details:

Choose constants c_p and c so that

(7) $$0 < c_0 < c_1 < \cdots < c, \qquad (1+c)^4 - 1 < 1/4.$$

Writing $D(a, r)$ for the open disc in $\mathbf{C}$ with center at a and radius r, consider the discs

(8) $$D_p = 2^p D(1, c_p), \qquad X_p = 2^{4p} D\left(1, \tfrac{1}{4}\right), \qquad Y_p = 2^{2p+2} D\left(1, \tfrac{1}{4}\right)$$

and the polydiscs

(9) $$\Delta_p = D_p \times D_p$$

for $p = 0, 1, 2, \ldots$.

The X_p's have disjoint closures; the same is true of the Y_p's. Therefore, given $\varepsilon_p > 0$, we can find entire functions g and h so that (1) holds and

(10) $$\left|\tfrac{1}{2} - e^g\right| < \varepsilon_p \quad \text{on } X_p, \qquad |4 - e^h| < \varepsilon_p \quad \text{on } Y_p,$$

for $p = 0, 1, 2, \ldots$. (The existence of g and h can be proved by repeated applications of Runge's theorem, followed by a passage to the limit.)

Our choice of c in (7) guarantees that $z^3 w \in X_p$ and $4zw \in Y_p$ for all $(z, w) \in \bar{\Delta}_p$.

Therefore, if $(z, w) \in \bar{\Delta}_p$ and $(u, v) = G(z, w)$, then $(u, v) \approx (z/2, 8w)$ since $e^g \approx 1/2$ on X_p. So if ε_p is small enough, it follows that $uv \in Y_p$, and therefore

(11) $$F(z, w) = H(u, v) \approx (4u, v/4) \approx (2z, 2w).$$

We conclude: If ε_p is small enough (depending on the choices made in (7)) then (10) will ensure that

(12) $$F(\Delta_p) \subset \Delta_{p+1} \qquad (p = 0, 1, 2, \ldots).$$

Thus $|F^k(z, w)| \to \infty$ as $k \to \infty$, for (z, w) in any Δ_p. This shows that Ω intersects no Δ_p.

Open questions.

1. Consider the following properties which an infinite discrete set $E \subset \mathbf{C}^n$ may or may not have:

(a) E is tame in $\mathbf{C}^n$.

(b) E is avoidable by biholomorphic maps.

(c) E is permutable: every permutation of E extends to an automorphism of $\mathbf{C}^n$.

(d) E is the set of all fixed points of some automorphism of $\mathbf{C}^n$.

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We know that (a) implies the other three. (For (a) $\Rightarrow$ (d) see Example 9.5.) What other implications hold among these four properties?

2. Suppose $\{\Omega_j\}$ is an infinite disjoint collection of F. B. regions in $\mathbf{C}^n$, E is discrete in $\mathbf{C}^n$, and E has exactly one point in each Ω_j. (So E is obviously avoidable by biholomorphic maps.) Must E be tame in $\mathbf{C}^n$?

3. Suppose that the distance between any two points of a set $E \subset \mathbf{C}^n$ is at least 1. Must E be tame in $\mathbf{C}^n$?

4. If E is discrete in $\mathbf{C}^2$ and $|z_1| > 1$ for every $(z_1, z_2) \in E$, must E be tame in $\mathbf{C}^2$? (Compare with Theorem 3.8.) The proof of Theorem 6.4 shows that E need not be very tame.

5. If a discrete set $E \subset \mathbf{C}^n$ is unavoidable (by whatever class of maps), must E stay unavoidable after removal of one point?

6. Is there a biholomorphic map from $\mathbf{C}^n$ into $\mathbf{C}^n$ which is not a limit of automorphisms? Yes

No

Some related questions: If F is biholomorphic, must $F(\mathbf{C}^n)$ be a Runge domain?

Is the region Ω^* in Example 9.5 a Runge domain? Anderson

Is the union of every expanding sequence of F. B. regions an F. B. region? + Anderson Lempert,

7. Is there a holomorphic $F\colon \mathbf{C}^2 \to \mathbf{C}^2$ with $JF \equiv 1$ (or with $JF \not\equiv 0$) so that the *closure* of $F(\mathbf{C}^2)$ has finite volume?

8. Is every $F \in \mathrm{Aut}(\mathbf{C}^n)$ with $JF \equiv 1$ a limit of a sequence of compositions of shears?

A more specific question: Is the map $(z, w) \to (ze^{zw}, we^{-zw})$ a limit of a sequence of compositions of shears in $\mathbf{C}^2$?

9. Let $n = 2$ for simplicity. Do the transformations described at the end of the Introduction generate the group Γ of all automorphisms of $\mathbf{C}^2$ that fix every point of $\{z_1 z_2 = 0\}$? (One needs to have $f(0) = 0$.)

Does every $F \in \Gamma$ satisfy

$$(JF)(z_1, z_2) = w_1 w_2 / z_1 z_2$$

if $(w_1, w_2) = F(z_1, z_2)$ and $z_1 z_2 \neq 0$?

Peschl **[14]** claims that the answer to the second question is yes. We believe that there may be a gap in his proof. To be specific, we do not see how one can justify the claim (made on line 10 of p. 1838) that $G_n^m \overset{m}{=} G$.

10. Is there a biholomorphic map from $\mathbf{C}^2$ into the set $\{zw \neq 0\}$, i.e., into the complement of the union of two intersecting complex lines?

(Nishimura's papers **[12** and **13]** contain several results about biholomorphic maps from $\mathbf{C}^2$ into the complement of one complex line.)

11. If Ω is an F. B. region and L is a complex line, is it possible that

(a) $L \cap \Omega$ is connected (and not empty)?

(b) $L \cap \Omega$ has finitely many components?

(c) $L \cap \Omega$ is a circular disc?

12. How many complex lines can an F. B. region in $\mathbf{C}^2$ contain? Examples 9.4, 9.7, and 9.9 show that 0, 1, and 2 are possible.

13. Are there two disjoint F. B. regions in $\mathbf{C}^n$ whose union is dense in $\mathbf{C}^n$? What if "two" is replaced by "finitely many" or by "infinitely many"?

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Appendix. As mentioned earlier, it is the purpose of this Appendix to give a proof of the theorem concerning attracting fixed points of automorphisms that was stated in the Introduction.

We begin with some facts about holomorphic maps $G = (g_1, \dots, g_m)$ from $\mathbf{C}^n$ into $\mathbf{C}^n$ of the form

$$\begin{aligned} g_1(z) &= c_1 z_1, \\ g_2(z) &= c_2 z_2 + h_2(z_1), \\ &\vdots \\ g_n(z) &= c_n z_n + h_n(z_1, \dots, z_{n-1}) \end{aligned}$$

where $c_1, \dots, c_n$ are scalars and each h_i is a holomorphic function of $(z_1, \dots, z_{i-1})$ which vanishes at the origin. We call such maps *lower triangular.*

The matrix that represents the linear operator $G'(0)$ is then lower triangular. Thus $G'(0)$ is invertible if and only if no c_i is 0. It follows that G is an automorphism of $\mathbf{C}^n$ (a composition of an invertible linear map and $n-1$ shears) if and only if no c_i is 0.

If $g_1, \dots, g_n$ are polynomials, the degree of $G = (g_1, \dots, g_n)$ is defined to be $\deg G = \max_i \deg g_i$.

LEMMA 1. *Let G be a lower triangular polynomial automorphism of $\mathbf{C}^n$.*

(a) *The degrees of the iterates G^k of G are then bounded, and there is a constant $\beta < \infty$ so that*

$$G^k(U^n) \subset \beta^k U^n \qquad (k = 1, 2, 3, \dots). \tag{1}$$

Here U^n is the unit polydisc in $\mathbf{C}^n$.

(b) *If also $|c_i| < 1$ for $1 \le i \le n$, then $G^k(z) \to 0$, uniformly on compact subsets of $\mathbf{C}^n$, and*

$$\bigcup_{k=1}^{\infty} G^{-k}(V) = \mathbf{C}^n \tag{2}$$

for every neighborhood V of 0.

PROOF. Let $G = (g_1, \dots, g_n)$, $G^k = (g_1^{(k)}, \dots, g_n^{(k)})$, put $\mu_i = \deg g_i$, and let $S(m,k)$ be the statement

$$\deg g_i^{(k)} \le \mu_1 \cdots \mu_i \quad \text{for } 1 \le i \le m. \tag{3}$$

We want to prove $S(n,k)$ for $k = 1, 2, 3, \dots$.

Since $G^{k+1} = G \circ G^k$, we have

$$g_i^{(k+1)} = c_i g_i^{(k)} + h_i(g_1^{(k)}, \dots, g_{i-1}^{(k)}) \qquad (2 \le i \le n). \tag{4}$$

This shows that $S(m, k+1)$ follows from $S(m,k)$ and $S(m-1,k)$. Since $S(1,k)$ and $S(m,1)$ are obviously true for all k and m (note that $\mu_1 = 1$, and $\mu_i \ge 1$ for all i), $S(n,k)$ follows by induction.

Putting $d = \mu_1 \cdots \mu_n$ we have thus proved that $\deg G^k \le d$ for $k = 1, 2, 3, \dots$.

Next, let M be the number of multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ that have $|\alpha| \le d$. (As usual, a multi-index α is an ordered n-tuple of nonnegative integers $\alpha_1, \dots, \alpha_n$,

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and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.) Choose $C \geq 1$ so that $|g_i| \leq C$ on U^n for $1 \leq i \leq n$, and put $\beta = M \cdot C^d$. We claim that then

$$|g_i^{(k)}(z)| \leq \beta^k \qquad (z \in U^n,\ 1 \leq i \leq n,\ k = 1, 2, 3, \ldots). \tag{5}$$

Since $C \leq \beta$, (5) holds when $k = 1$. Assume (5) for some $k \geq 1$. The coefficients a_α in

$$g_i^{(k)}(z) = \sum_{|\alpha| \leq d} a_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n} = \sum_{|\alpha| \leq d} a_\alpha z^\alpha \tag{6}$$

are equal to the integrals of $g_i^{(k)}(z)\bar{z}^\alpha$ over the unit torus T^n. Thus (5) implies $|a_\alpha| \leq \beta^k$.

Since $G^{k+1} = G^k \circ G$, (6) shows that

$$g_i^{(k+1)} = g_i^{(k)}(g_1, \ldots, g_n) = \sum_{|\alpha| \leq d} a_\alpha g_1^{\alpha_1} \cdots g_n^{\alpha_n}. \tag{7}$$

Our choice of M and C implies now that

$$|g_i^{(k+1)}| \leq M\beta^k C^{|\alpha|} \leq M\beta^k C^d = \beta^{k+1} \tag{8}$$

which is (5) with $k+1$ in place of k.

Thus (1) holds, and part (a) of the lemma is proved.

We turn to (b). Let $E \subset \mathbf{C}^n$ be compact. Note that $g_1^{(k)}(z) = c_1^k z_1$. Thus $\|g_1^{(k)}\|_E \to 0$ as $k \to \infty$. (We use $\|\cdot\|_E$ to denote the sup-norm over E.) Assume now that $1 < i \leq n$ and that

$$\lim_{k\to\infty} \|g_j^{(k)}\|_E = 0 \quad \text{for } 1 \leq j < i. \tag{9}$$

Since $h_i(0) = 0$, it follows that

$$\lim_{k\to\infty} \|h_i(g_1^{(k)}, \ldots, g_{i-1}^{(k)}\|_E = 0. \tag{10}$$

Therefore, given $\varepsilon > 0$, (4) shows that

$$|g_i^{(k+1)}| \leq |c_i|\,|g_i^{(k)}| + \varepsilon \tag{11}$$

on E, for all sufficiently large k. This implies

$$\limsup_{k\to\infty} \|g_i^{(k)}\|_E \leq \frac{\varepsilon}{1 - |c_i|}, \tag{12}$$

for all $\varepsilon > 0$. Hence (9) holds with $i+1$ in place of i.

The first assertion in part (b) follows now by induction on i. The second assertion is an immediate consequence of the first.

This completes the proof of Lemma 1.

From now on we shall deal with a fixed invertible linear transformation $A\colon \mathbf{C}^n \to \mathbf{C}^n$, all of whose eigenvalues λ_i are less than 1 in absolute value. We order them so that

$$0 < |\lambda_n| \leq \cdots \leq |\lambda_1| < 1$$

and then choose coordinates in $\mathbf{C}^n$ in such a way that the matrix representation of A is *lower triangular*. If $A = (a_{ij})$ then $a_{ii} = \lambda_i$ and $a_{ij} = 0$ when $i < j$.

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In preparation for our next lemma, we let $\mathscr{H}_m$ denote the vector space of all holomorphic maps $H: \mathbf{C}^n \to \mathbf{C}^n$, $H = (h_1, \dots, h_n)$, whose components h_i are homogeneous polynomials of degree m.

A convenient basis $\mathscr{B}$ for $\mathscr{H}_m$ consists of those maps H that have only one component different from 0, and that one, say h_j, is a monomial z^α (with $|\alpha| = m$, of course). Among the members of $\mathscr{B}$ we call those *special* in which this h_j has the form

$$h_j(z) = z_1^{\alpha_1} \cdots z_{j-1}^{\alpha_{j-1}}$$

and the relation

$$\lambda_j = \lambda_1^{\alpha_1} \cdots \lambda_{j-1}^{\alpha_{j-1}}$$

holds.

This notion of "special" depends of course on our operator A; more precisely, it depends on the *spectrum* of A. Note that no such relation can exist when m is so large that $|\lambda_1|^m < |\lambda_n|$; in that case, no member of $\mathscr{B}$ is special. Note also that the special members of $\mathscr{B}$ are lower triangular.

We let X_m be the subspace of $\mathscr{H}_m$ that is spanned by these special basis elements. ($X_m = \{0\}$ when there are none.)

We let Γ_A be the "commutator map" defined by $\Gamma_A(H) = A \circ H - H \circ A$. For each m, Γ_A is thus a linear operator on $\mathscr{H}_m$.

LEMMA 2. *For* $m \geq 2$, $\mathscr{H}_m = X_m + \Gamma_A(\mathscr{H}_m)$.

PROOF. In place of A, we begin with the diagonal matrix D which has $\lambda_1, \lambda_2, \dots, \lambda_n$ down its main diagonal.

If $H = (0, \dots, 0, z^\alpha, 0, \dots, 0)$ is in $\mathscr{B}$, with z^α in the jth spot, then

$$\Gamma_D(H) = DH - HD = (\lambda_j - \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n})H.$$

This shows that Γ_D annihilates precisely those members of $\mathscr{B}$ that are special, and that Γ_D acts as an invertible linear operator on the space Y_m that is spanned by the other members of $\mathscr{B}$.

(Note that $\lambda_j - \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}$ *cannot* be 0 if $\alpha_k > 0$ for some $k \geq j$, because $|\alpha| = m \geq 2$, so that $|\lambda_j| > |\lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}|$.)

Let π be the projection in $\mathscr{H}_m$ whose range is X_m and whose nullspace is Y_m. The preceding observations can then be summarized by saying *that* $\pi + \Gamma_D$ *is an invertible linear operator on* $\mathscr{H}_m$.

We now return to our given A. For any $\varepsilon > 0$, let $S = S_\varepsilon$ be the diagonal matrix that has $\varepsilon^n, \varepsilon^{n-1}, \dots, \varepsilon$ down its main diagonal. Since A is lower triangular, so is $S^{-1}AS$; if $i \geq j$ then $\varepsilon^{i-j}a_{ij}$ stands in the ith row and jth column of $S^{-1}AS$. Thus $S^{-1}AS$ converges to D as $\varepsilon \to 0$. The invertible operators form an open set in the algebra of all linear operators on $\mathscr{H}_m$. We conclude from this that *there is an* $\varepsilon > 0$, *so small that* $\pi + \Gamma_{S^{-1}AS}$ *is invertible on* $\mathscr{H}_m$.

In other words, to each $G \in \mathscr{H}_m$ corresponds some $H_0 \in X_m$ and some $H \in \mathscr{H}_m$ so that

$$S^{-1}GS = H_0 + (S^{-1}AS)H - H(S^{-1}AS)$$

or

$$G = SH_0S^{-1} + A(SHS^{-1}) - (SHS^{-1})A.$$

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The fact that S is diagonal shows that SHS^{-1} is a scalar multiple of H, for every $H \in \mathscr{B}$. Since $H_0 \in X_m$, it follows that $SH_0S^{-1} \in X_m$. Thus $G \in X_m + \Gamma_A(\mathscr{H}_m)$. This completes the proof of Lemma 2.

LEMMA 3. *Suppose that V is a neighborhood of* 0 *in* $\mathbf{C}^n$, *that* $F: V \to \mathbf{C}^n$ *is holomorphic,* $F(0) = 0$, *and that all eigenvalues* λ_i *of* $A = F'(0)$ *satisfy* $0 < |\lambda_i| < 1$.

Then there exist

(i) *a lower triangular polynomial automorphism* G *of* $\mathbf{C}^n$, *with* $G(0) = 0$, $G'(0) = A$, *and*

(ii) *polynomial maps* $T_m: \mathbf{C}^n \to \mathbf{C}^n$, *with* $T_m(0) = 0$, $T'_m(0) = I$, *so that*

$$G^{-1} \circ T_m \circ F - T_m = O(|z|^m) \qquad (m = 2, 3, 4, \ldots). \tag{1}$$

In other words, the conclusion is that the power series expansion of the left side of (1), about the origin of $\mathbf{C}^n$, contains no terms of degree less than m.

PROOF. We choose coordinates, as before, so that A is lower triangular and $|\lambda_1| \geq \cdots \geq |\lambda_n|$.

Suppose that the following induction hypothesis holds for some $m \geq 2$: T_m is as in (ii), G_m is a lower triangular polynomial automorphism of $\mathbf{C}^n$ with $G'_m(0) = A$, and

$$T_m \circ F - G_m \circ T_m = O(|z|^m). \tag{2_m}$$

Note that this is true when $m = 2$, with $G_2 = A$, $T_2 = I$.

Now (2_m) can be rewritten in the form

$$T_m \circ F - G_m \circ T_m - P_m = O(|z|^{m+1}) \tag{3_m}$$

for some $P_m \in \mathscr{H}_m$. Lemma 2 allows us to decompose P_m:

$$P_m = Q + A \circ H - H \circ A \tag{4}$$

for some $Q \in X_m$, $H \in \mathscr{H}_m$. Define

$$G_{m+1} = G_m + Q, \qquad T_{m+1} = T_m + H \circ T_m. \tag{5}$$

We have to prove that (2_{m+1}) holds.

Let the symbol $\sim$ indicate that the difference between the two terms on either side of it is $O(|z|^{m+1})$.

Then $Q \circ T_{m+1} \sim Q$, $T_{m+1} - T_m \sim H$, and the difference Δ between the left sides of (2_{m+1}) and (3_m) satisfies therefore

$$\begin{aligned}\Delta &= (H \circ T_m \circ F) + (G_m \circ T_m) - (G_m \circ T_{m+1}) - (Q \circ T_{m+1}) + P_m \\ &\sim (H \circ A) + (G_m \circ T_m) - (G_m \circ (T_m + H)) + (A \circ H) - (H \circ A)\end{aligned}$$

so that

$$-\Delta \sim G_m \circ (T_m + H) - G_m \circ T_m - G'_m(0)H,$$

or, equivalently

$$-\Delta(z) \sim \int_0^1 \{G'_m[T_m(z) + tH(z)] - G'_m(0)\} H(z)\, dt.$$

Observe now that $H(z) = O(|z|^m)$, $T_m(z) = O(|z|)$, and that the norm of the linear operator in $\{\cdots\}$ is therefore $O(|z|)$. This shows that $\Delta(z) = O(|z|^{m+1})$ and proves (2_{m+1}).

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As soon as m is large enough, $X_m = \{0\}$, hence $G_{m+1} = G_m$. This gives G, as in (i) satisfying

$$T_m \circ F - G \circ T_m = O(|z|^m) \tag{6}$$

for all $m \geq 2$. (Note that anything that is $O(|z|^m)$ is also $O(|z|^{m-1})$, etc.) Finally we apply G^{-1} to (6) to obtain (1).

We are now ready for the main result:

THEOREM. *Suppose that $F \in \mathrm{Aut}(\mathbf{C}^n)$, $F(0) = 0$, and all eigenvalues λ_i of $F'(0)$ satisfy $|\lambda_i| < 1$.*

Then there exists a biholomorphic map Φ from $\mathbf{C}^n$ onto the region

$$\Omega = \left\{ z \in \mathbf{C}^n \colon \lim_{k \to \infty} F^k(z) = 0 \right\}.$$

Moreover, Φ can be chosen so that $J\Phi \equiv 1$ if JF is constant.

PROOF. As before, we choose coordinates so that $A = F'(0)$ is lower diagonal, and $|\lambda_1| \geq \cdots \geq |\lambda_n|$. We can then find a diagonal operator S, as in the proof of Lemma 2, which makes $A_0 = S^{-1}AS$ so close to being diagonal that $|A_0 z| \leq c|z|$ holds for some $c < 1$ and all $z \in \mathbf{C}^n$. (This uses the assumption $|\lambda_1| < 1$.) If we put $F_0 = S^{-1}FS$ and prove the theorem for F_0, obtaining Φ_0 and Ω_0, then it holds also for F, with $\Phi = S\Phi_0 S^{-1}$ and $\Omega = S(\Omega_0)$.

So we may assume, in addition to the stated hypotheses, that $||A|| < 1$.

Fix α, $||A|| < \alpha < 1$. Then there exists $r > 0$ so that

$$|F(z)| \leq \alpha |z| \quad \text{if } |z| \leq r. \tag{1}$$

It follows, as in the proof of Theorem 9.1, that $rB \subset \Omega$, that Ω is a region, and that $F(\Omega) = \Omega$.

Next, we associate G to F as in Lemma 3, and apply Lemma 1(a) to G^{-1} in place of G to conclude (with the aid of the Schwarz lemma) that there is a constant $\gamma < \infty$ so that

$$|G^{-k}(w) - G^{-k}(w')| \leq \gamma^k |w - w'| \qquad (k = 1, 2, 3, \ldots) \tag{2}$$

for all $w, w' \in \mathbf{C}^n$ with $|w| \leq 1/2$, $|w'| \leq 1/2$.

Fix a positive integer m so that $\alpha^m < 1/\gamma$.

Lemma 3 gives us a polynomial map $T = T_m$, with $T(0) = 0$, $T'(0) = I$, and it gives us constants $\delta > 0$, $C_1 < \infty$, so that $|w| \leq \delta$ implies

$$|G^{-1}TF(w) - T(w)| \leq C_1 |w|^m. \tag{3}$$

Now let $E \subset \Omega$ be compact. Then $F^s(E) \subset rB$ for some integer s. Hence $F^{s+k}(E) \subset F^k(rB) \subset \alpha^k rB$, for all $k \geq 0$, by (1). It follows that there is a constant $C_2 < \infty$ so that

$$|F^k(z)| \leq C_2 \alpha^k < \delta \tag{4}$$

for all $z \in E$ and all $k \geq k_0$. For such z and k, (3) and (4) show that

$$|G^{-1}TF^{k+1}(z) - TF^k(z)| \leq C_1 |F^k(z)|^m \leq C_1 C_2^m \alpha^{mk}. \tag{5}$$

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For large k, $|G^{-1}TF^{k+1}(z)|$ and $|TF^k(z)|$ are $< 1/2$, for all $z \in E$. Hence (2) can be applied to (5), and we conclude that for $k \geq k_1$ and for all $z \in E$,

$$|G^{-k-1}TF^{k+1}(z) - G^{-k}TF^k(z)| \leq C_1 C_2^m (\gamma\alpha^m)^k. \tag{6}$$

Since $\gamma\alpha^m < 1$, we have proved:

The limit

$$\Psi(z) = \lim_{k\to\infty} (G^{-k} \circ T \circ F^k)(z) \tag{7}$$

exists, uniformly on every compact subset of Ω, and defines a holomorphic map $\Psi\colon \Omega \to \mathbf{C}^n$ which satisfies $\Psi(0) = 0$, $\Psi'(0) = I$, as well as the functional equation

$$G^{-1} \circ \Psi \circ F = \Psi. \tag{8}$$

Since $F(\Omega) = \Omega$, (8) shows that Ψ has the same range as $G^{-1} \circ \Psi$. Thus

$$\Psi(\Omega) = G^{-1}(\Psi(\Omega)) = \cdots = G^{-k}(\Psi(\Omega)) = \cdots \tag{9}$$

and since $\Psi(\Omega)$ contains a neighborhood of 0, Lemma 1(b) shows that $\Psi(\Omega) = \mathbf{C}^n$.

Assume next that $x, y \in \Omega$ and $\Psi(x) = \Psi(y)$. By (8), $\Psi \circ F = G \circ \Psi$. Hence $\Psi(F(x)) = \Psi(F(y))$. Continuing, we see that $\Psi(F^k(x)) = \Psi(F^k(y))$ for all positive k. But when k is sufficiently large, both $F^k(x)$ and $F^k(y)$ are in a neighborhood of 0 in which Ψ is one-to-one. Thus $F^k(x) = F^k(y)$, and this implies $x = y$. So Ψ is one-to-one in Ω.

We have now proved that Ψ is a biholomorphic map from Ω onto $\mathbf{C}^n$.

The first conclusion of the theorem is therefore satisfied by $\Phi = \Psi^{-1}$.

Finally, assume that JF is constant. Since G is a *polynomial automorphism* of $\mathbf{C}^n$, the polynomial JG has no zero in $\mathbf{C}^n$, hence is also constant. In fact, $JG = JF$ because $G'(0) = F'(0)$. If we apply the chain rule to $\Psi \circ F = G \circ \Psi$, we obtain, for $z \in \Omega$,

$$(J\Psi)(F(z))(JF)(z) = (JG)(\Psi(z))(J\Psi)(z). \tag{10}$$

Hence

$$(J\Psi)(z) = (J\Psi)(F(z)) = \cdots = (J\Psi)(F^k(z)) = \cdots. \tag{11}$$

Since $F^k(z) \to 0$ as $k \to \infty$ we conclude that

$$(J\Psi)(z) = (J\Psi)(0) = 1 \tag{12}$$

for all $z \in \Omega$. Hence $J\Phi \equiv 1$ on $\mathbf{C}^n$.

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REMARK. In the generic case, the eigenvalues of $F'(0)$ satisfy none of the relations that give rise to the "special" basis elements of $\mathscr{H}_m$. In that case, $X_m = \{0\}$ for all m, the proof of Lemma 3 gives $G = A$, and the functional equation (8) can be written in the form

$$\Psi \circ F \circ \Psi^{-1} = A. \tag{13}$$

One refers to this as "linearizing" the map F, by a biholomorphic change of variables.

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