Suppose that $\lambda_1, \ldots, \lambda_k$ are Brjuno rotations. Let

$$\hat{F} = (z_1(\lambda_1 + a_1 z_1 \cdots z_k), \ldots, z_k(\lambda_k + a_k(z_1 \cdots z_k)).$$

We assume that $\lambda_1 \cdots \lambda_k = 1$.

Also assume that

$$\frac{a_j}{\lambda_j} = -\frac{1}{k}.$$

Then

$$z_1 \cdots z_k \to (z_1 \cdots z_k) \left[1 - z_1 \cdots z_k + \mathcal{O}((z_1 \cdots z_k)^2)\right].$$

Set $z_1 \cdots z_k =: X$. Hence we get

$$X \to X[1 - X + \mathcal{O}(X^2)].$$

Now choose a large integer M and use Weickert to find an automorphism F such that

$$\|F - \hat{F}\| = \mathcal{O}(\|(z_1, \ldots, z_k)\|^M)$$

Further let α be a small positive integer such that $M \cdot \alpha > 8$.

Then let

$$B = \{(z_1, \ldots, z_k) \in \mathbb{C}^k; |z_j| < |X|^\alpha, -\frac{\pi}{8} < \arg(X) < \frac{\pi}{8} \text{ and } |X| < R\}$$

Lemma 0.1. *If $R > 0$ is chosen well, then*

1. $F(B) \subset B$

2. $F^n(X_0) \sim \frac{1}{x_0+n}$ *when* $x_0 \in B$

3.

$$\begin{aligned} \pi_j F^n(x_0) &=: z_j(n) \\ z_j(n) &= e^{in\theta_j} - \sum_{m=0}^{n-1} \frac{1}{x_0+m} + \sum_{m=0}^{n-1} \mathcal{O}((\frac{1}{x_0+m})^2) \\ &\lambda_j = e^{i\theta_j}, j = 1, \ldots, k \end{aligned}$$

Hence $\|z_j(n)\| \sim \left(\frac{1}{n}\right)^{1/k}$.

Lemma 0.2. *There exist local coordinates $(\tilde{z}_1, \ldots \tilde{z}_k)$ such that $\tilde{F}$ acts like an irrational rotation on the set where $\tilde{z}_1 \cdots \tilde{z}_k = 0$ and $\|\vec{\tilde{z}} - \vec{z}\| < (\|z\|^M)$*

Let $\Omega = \cup_{n=0}^{\infty} F^{-n}(B)$.

Lemma 0.3. *Ω is a Fatou component for F.*

Proof. Assume not. Then there is a Fatou component $W \supset \Omega$, $W \setminus \Omega \neq \emptyset$. Moreover $W \cap \mathbb{B}(0,R)$ is contained in $\mathbb{B}^* = \mathbb{B} \setminus \{\tilde{z}_1 \cdots \tilde{z}_k = 0\}$.

For an arbitrarily small $\delta > 0$ there exists two points $p \in B$ and $q \in W \setminus B$, both goint uniformly to 0 $K_{\mathbb{B}^*(p,q)} < \delta$ but by passing to a subsequence if need be then $F^n(q) \notin B$.Let $q = (x_1, \ldots, x_n)$ then at least one $|x_j(n)| > |X|^\alpha\ \forall\ n$.

Again by passing to a subsequence of $\{n\}$ we may assume that $|y_1(n)| > |X|^\alpha$ for each n. Again this means that $|y_1(n)|^{1-\alpha} > |y_j(n)|^{(k-1)\alpha}$ for some $j = 2, \ldots, k$.

By passing to a subsequence of $\{n\}$ we may assume that

$$|y_1|^{1-\alpha}(n) > |y_2(n)|^{(k-1)\alpha}$$

So

$$K_{\Delta^*}(y_1(n), y_2(n)) \geq |\ln\left(\frac{1-\alpha}{(k-1)\alpha}\right)|$$

Choose α such that $> \log(2)$.

For the same subsequence

$$K_{\mathbb{B}^*}(F^n(p), F^n(q)) < \delta$$

for all n. So

$$K_{\Delta^*}(z_j(n), y_j(n)) < \delta$$

for all $p = (z_1, \ldots, z_k)$ and for all n.

From Lemma 1 part 3 it follows that

$$K_{\Delta^*(z_1(n), z_2(n))} \to 0,$$

$$K_{\Delta^*(y_1(n), 2_2(n))} \leq K_{\Delta^*(y_1(n), z_1(n))} + K_{\Delta^*(z_1(n), z_2(n))} + K_{\Delta^*(z_2(n), y_2(n))} < 3\delta$$

when n is large. This gives a contradiction if $3\delta < \ln 2$.

□

We now want to produce a biholomorphism from Ω to $\mathbb{C} \times (\mathbb{C}^*)^{k-1}$. To do this we need to understand the shape of B very well.

We start by changing coordinates in B to

$$(z_1 \cdots z_k, z_2 \cdots z_k, z_3 \cdots z_k, \ldots, z_k) =: (X, Y_2, \ldots, Y_{k-1}, Y_k)$$

where $\tilde{B}$ is given by

$$\begin{aligned}
X &\in V = \{X \in \mathbb{C}; -\frac{\pi}{8} < \arg(X) < \frac{\pi}{8}, 0 < |X| < R\} \\
|Y_k| &< |X|^\alpha \\
\frac{|Y_{k-1}|}{|Y_k|} &< |X|^\alpha, \\
&\cdots \\
\frac{|Y_j|}{|Y_{j+1}|} &< |X|^\alpha, \\
&\cdots \\
\frac{|Y_2|}{|Y_3|} &< |X|^\alpha, \\
\frac{|X|}{|Y_2|} &< |X|^\alpha
\end{aligned}$$

To build the map we start by changing coordinates in $\tilde{B}$.

X will be replaced by

$$\Psi(X, Y_2, \dots, Y_{k-1}, W) = \frac{1}{X} + c \log \frac{1}{X} + v(X, Y_2, \dots, Y_{k-1}, W)$$

where $v(X, \dots) = Xg(X, Y_2, \dots)$ where $g : \tilde{B} \to \mathbb{C}$ is bounded and holomorphic and $\Psi \circ F = \Psi + 1$.

For the remaining coordinates σ_j, where $j = 2, \dots, k$ we use:

$$\begin{aligned}
\sigma_j &:= \lim_{n\to\infty} (\lambda_j \cdots \lambda_k)^n \Pi_j F^n(z_1, \dots, z_k) \cdot \exp\left(- \sum_{m=0}^{n-1} \frac{\frac{a_j}{\lambda_j} + \cdots + \frac{a_k}{\lambda_k}}{\Psi + m} \right) \\
&= \lim_{n\to\infty} (\lambda_j \cdots \lambda_k)^n \Pi_j F^n(z_1, \dots, z_k) \cdot \exp\left(- \sum_{m=0}^{n-1} \frac{\frac{k-j}{k}}{\Psi + m} \right)
\end{aligned}$$

where $\Pi_j(z_1, \dots, z_k) = (z_j \cdots z_k)$.

The map $G : \Omega \to \mathbb{C} \times (\mathbb{C}^*)^{k-1}$ is given by $G = (g_1, \dots, g_k)$. Given a point $p \in \Omega$, apply F n times until $F^n(p) \in B$.

Let

$$\begin{aligned}
g_1 &= \Psi \circ F^n - n \\
g_2 &= \lambda_2^n \cdots \lambda_k^n \exp\left(\sum_{m=0}^{n-1} \frac{\frac{k-1}{k}}{\Psi(F^n(p)+m-n}\right) \cdot \sigma_2(F^n(p)) \\
&\cdots \\
g_j &= \lambda_j^n \cdots \lambda_k^n \exp\left(\sum_{m=0}^{n-1} \frac{\frac{k-1+j}{k}}{\Psi(F^n(p)+m-n}\right) \cdot \sigma_j(F^n(p)) \\
&\cdots \\
g_k &= \lambda_k^n \exp\left(\sum_{m=0}^{n-1} \frac{\frac{1}{k}}{\Psi(F^n(p)+m-n}\right)
\end{aligned}$$

1. Estimates to give that the map from Ω to $\mathbb{C} \times (\mathbb{C}^*)^{k-1}$

We want to show that any point

$$(A_1, \ldots, A_k) \in \mathbb{C} \times (\mathbb{C}^*)^{k-1}$$

is in the image of G.

The way we obtain the image of a point p is to go forward with F n times until the image $F^n(p)$ is in B. Then

$A_1 = \Psi(F^n(p)) - n$, in fact after a certain n off we obtain A_1 this way. Next

$$\begin{aligned}
A_k &= \lambda_k^n \sigma_k(F^n(p)) \cdot \exp\left(\frac{1}{k} \sum_{m=0}^{n-1} \frac{1}{\Psi(F^n(p)) + n - m}\right) \\
&\sim \lambda_k^n Y_k n^{1/k}
\end{aligned}$$

In general

$$A_j \sim \lambda_j^n \cdots \lambda_k^n Y_j n^{\frac{k-j+1}{k}}$$

Now our domain is defined by:

$$\begin{array}{rcl} X & \in & V \\ |Y_k| & < & |X|^\alpha \\ \dfrac{|Y_{k-1}|}{|Y_k|} & < & |X|^\alpha \\ & \cdots & \\ \dfrac{|Y_j|}{|Y_{j-1}|} & < & |X|^\alpha \\ & \cdots & \\ \dfrac{|Y_2|}{|Y_3|} & < & |X|^\alpha \end{array}$$

and finally

$$\frac{|X|}{|Y_2|} < |X|^\alpha$$

or

$$\begin{array}{rcl} |Y_k n^{1/k}| & < & |X|^\alpha n^{1/k} \\ \dfrac{|Y_{k-1} n^{2/k}|}{|Y_k n^{1/k}|} & < & |X|^\alpha n^{1/k} \\ & \cdots & \\ \dfrac{|Y_j| n^{\frac{k-j+1}{k}}}{|Y_{j-1}| n^{k-j} k} & < & |X|^\alpha n^{1/k} \\ & \cdots & \\ \dfrac{|Y_2| n^{\frac{k-1}{k}}}{|Y_3| n^{\frac{k-2}{2}}} & < & |X|^\alpha n^{1/k} \\ \dfrac{|X|}{|Y_2| n^{\frac{k-1}{k}}} & < & |X|^\alpha \dfrac{1}{n^{\frac{k-1}{k}}} \end{array}$$

Then the challenge is to show that any nonzero A_j can be obtained in this way for some n. Hence we need to solve the inequalities

$$
\begin{aligned}
|A_k| &< |X|^\alpha n^{1/k} \\
\frac{|A_{k-1}|}{|A_k|} &< |X|^\alpha n^{1/k} \\
&\cdots \\
\frac{|A_j|}{|A_{j-1}|} &< |X|^\alpha n^{1/k} \\
&\cdots \\
\frac{|A_2|}{|A_3|} &< |X|^\alpha n^{1/k} \\
\frac{|X|}{|A_2|} &< |X|^\alpha \frac{1}{n^{\frac{k-1}{k}}}
\end{aligned}
$$

The last inequality is equivalent to

$$|X|^{1-\alpha} < \frac{A_2}{n^{\frac{k-1}{k}}}$$

We show that these inequalities can all be solved: If we set $X = 1/n$ and $\alpha = 1/(2k)$, then $X^\alpha n^{1/k} \to \infty$. Moreover $X^{1-\alpha} n^{\frac{k-1}{k}} \to 0$. This shows that all inequalities are satisfied for large enough n.

References

[H] Hormander, Lars,: An introduction to complex analysis in several variables. D. Van Nostrand. (1966).