Math. Ann. 270, 285–296 (1985)

Mathematische Annalen


# Plurisubharmonic Defining Functions of Weakly Pseudoconvex Domains in $\mathbb{C}^2$

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## 1. Introduction and Notation

*1.1.* It is well known that strongly pseudoconvex $C^k$-smooth domains $\subset\subset\mathbb{C}^n$, $n, k \geqq 2$, have strongly plurisubharmonic defining functions. If one asks whether non-strongly pseudoconvex, but pseudoconvex, $C^k$-domains $\subset\subset\mathbb{C}^n$ have a similar property, i.e. whether they admit plurisubharmonic defining functions, the answer is unfortunately no as is shown in the $C^\infty$-case in $\mathbb{C}^2$ by Diederich and Fornæss [3] and in the $C^\omega$-case in $\mathbb{C}^2$ by Fornæss [5]. The domains constructed in these papers do even not admit defining functions with positive semidefinite Leviform only in points $p$ on the boundary.

*1.2.* In some applications the local existence of such functions would already help. Let $\Omega\subset\subset\mathbb{C}^n$ be a $C^k$-smooth pseudoconvex non-strongly pseudoconvex domain, $p\in b\Omega$. Does there exist an open neighborhood $U=U(p)$ of $p$ and a $C^l$-function $\varrho: U\to\mathbb{R}$, $2\leqq l\leqq k$, such that $d\varrho_{|U\cap b\Omega}\neq 0$, $\Omega\cap U=\{\varrho<0\}$ and $\varrho$ is plurisubharmonic or has at least a positive semidefinite Leviform in points $q\in b\Omega\cap U$.

The domains constructed in [3] and [5] still satisfy this condition. But in [5] Fornæss gives a pseudoconvex $C^\infty$-domain $\subset\subset\mathbb{C}^3$ for which the answer to the weaker question is also no.

In this paper we will show that the answer to the local problem also is no in the $C^\omega$-case in $\mathbb{C}^2$.

The results published in this article are from the doctoral thesis of the author [1]. It is a pleasure to thank Prof. Dr. Klas Diederich both for drawing my attention to this problem and for his encouragement during the completion of this work.

*1.3. Notation.* The coordinates in $\mathbb{C}^2$ will be $z=x+iy$ and $w=u+iv$. For partial derivatives $\frac{\partial}{\partial z}, \frac{\partial}{\partial w}\dots$ we often simply write lower indices $z, w\dots$.

If $\varrho:\mathbb{C}^2\to\mathbb{R}$ is a $C^2$-function, the Leviform $L_\varrho(p)$ of $\varrho$ in a point $p\in\mathbb{C}^2$ is given by the hermitian matrix

$$\begin{pmatrix}\varrho_{z\bar z}(p) & \varrho_{z\bar w}(p)\\ \varrho_{w\bar z}(p) & \varrho_{w\bar w}(p)\end{pmatrix}.$$

## 2. Results and Applications

*2.1.* **Theorem 1.** *There is a domain $\Omega_1 \subset\subset \mathbb{C}^2$, $0 \in b\Omega_1$, with real-analytic smooth boundary that satisfies the following conditions:*

a) *$b\Omega_1$ is pseudoconvex, the origin is the only non-strongly pseudoconvex boundary point.*

b) *The origin is of strict type 3 in the sense of Kohn [6].*

c) *If $U = U(0) \subset \mathbb{C}^2$ is an open neighborhood of the origin and $\sigma: U \to \mathbb{R}$ a defining $C^k$-function, $k \geqq 6$, for $b\Omega_1 \cap U$ then there exist points $p \in b\Omega_1 \cap U$ (arbitrarily close to 0) such that the Leviform $L_\sigma(p)$ has a negative eigenvalue on $\mathbb{C}^2$.*

*Remark.* According to Kohn [6] property b) implies that there is, nevertheless, an open neighborhood $U = U(0)$ and a holomorphic function $h: U \to \mathbb{C}$ such that

$$\bar{\Omega}_1 \cap \{(z, w) \in U | h(z, w) = 0\} = \{0\}.$$

This is a kind of convexity property.

*2.2.* **Theorem 2.** *There is a pseudoconvex domain $\Omega_2 \subset\subset \mathbb{C}^2$, $0 \in b\Omega_2$, with real-analytic smooth boundary satisfying the following conditions:*

a) *The set of all non-strongly pseudoconvex boundary points is a smooth real-analytic curve through the origin.*

b) *If $U = U(0) \subset \mathbb{C}^2$ is an open neighborhood of the origin and $\sigma: U \to \mathbb{R}$ a $C^k$-defining function for $b\Omega_2 \cap U$, $k \geqq 9$, then there exist points $p \in b\Omega_2 \cap U$ (arbitrarily close to 0) such that $L_\sigma(p)$ has a negative eigenvalue on $\mathbb{C}^2$.*

*2.3.* Recently Cafarelli et al. [2] have obtained strong results concerning Dirichlet problems on strongly pseudoconvex $C^\infty$-domains $\subset\subset \mathbb{C}^n$ for the complex Monge-Ampere-operator

$$J := \det\left(\frac{\partial^2}{\partial \bar{z}_i \partial z_j}\right)_{1 \leqq i, j \leqq n}.$$

They get for a strongly pseudoconvex $C^\infty$-smooth domain $\Omega \subset\subset \mathbb{C}^n$ and a given positive function $\psi \in C^\infty(\bar{\Omega})$ a solution $u \in C^\infty(\bar{\Omega})$, i.e. $J(u) = \psi$ on $\bar{\Omega}$ and $u_{|b\Omega} = 0$.

Because of ellipticity requirements for the operator $J$ the solution $u$ must be plurisubharmonic and the positivity of $\psi$ even forces $u$ to be strongly plurisubharmonic. The Hopf lemma implies that this $u$ is automatically a $C^\infty$-smooth strongly plurisubharmonic defining function of $\Omega$.

We want to consider the situation in which $\Omega \subset\subset \mathbb{C}^n$ is a $C^\infty$-smooth pseudoconvex non-strongly pseudoconvex domain, $\psi \in C^{l-2}(\bar{\Omega})$, $l \geqq 2$, a non-negative function and ask for a function $u \in C^l(\bar{\Omega})$ with $J(u) = \psi$ on $\bar{\Omega}$ and $u_{|b\Omega} = 0$. Again because of ellipticity requirements for $J$ the solution $u$ must be supposed to be plurisubharmonic and with the Hopf lemma we get that it is a $C^l$-defining function, $l \geqq 2$, for $\Omega$ that is plurisubharmonic in the interior. In particular, $L_u(p)$ is positive semidefinite in all points $p \in b\Omega$. Therefore, the domain given in [5] shows that already for weakly pseudoconvex $C^\omega$-domains $\subset\subset \mathbb{C}^n$ and $\psi \in C^{l-2}(\bar{\Omega})$, $\psi \geqq 0$, $l \geqq 2$, in general, such a solution $u \in C^l(\bar{\Omega})$ does not exist. (The fact that $u$ is plurisubharmonic on $\Omega$, of course, already gives certain restrictions on $\psi$.)

If we localize the problem, i.e. take a point $p \in b\Omega$, an open neighborhood $U = U(p)$, a function $\psi \in C^{l-2}(U \cap \bar{\Omega})$, $l \geqq 2$, and ask for a solution $u \in C^l(\bar{\Omega} \cap U)$, the above considerations for $u$ obviously remain true.

Therefore, we get:

**Theorem 3.** *Let $\Omega$ be the domain $\Omega_1$ from Theorem 1 in 2.1. Let $U = U(0) \subset \mathbb{C}^2$ be an open neighborhood of the origin and $\psi$ a non-negative $C^{k-2}$-function on $U \cap \bar{\Omega}$, $k \geq 6$. Then there is on no open neighborhood $V = V(0) \subset\subset U$ a function $u \in C^k(V \cap \bar{\Omega})$ such that*

a) *$u$ is plurisubharmonic on $\Omega \cap V$;*
b) *$J(u) = \psi$ on $\bar{\Omega} \cap V$;*
c) $u_{|b\Omega \cap V} = 0$.

*2.4.* Another application of the existence of the domain $\Omega_2$ from 2.2 concerns the following question. Let $\Omega \subset\subset \mathbb{C}^n$, $n > 1$, be a pseudoconvex but non-strongly pseudoconvex $C^\infty$-smooth domain. When does there exist a strongly pseudoconvex $C^\infty$-smooth domain $\Omega' \subset\subset \mathbb{C}^n$ and a proper holomorphic mapping $\Phi: \Omega \to \Omega'$? In [4] Diederich and Fornæss show that the set $E_{b\Omega}$ of the non-strongly pseudoconvex boundary points of $b\Omega$ in this case cannot be too small, i.e. the $2n-3$-dimensional Hausdorff-measure $\Lambda_{2n-3}(E_{b\Omega}) > 0$. This result is local.

For the domain $\Omega_2$ from 2.2 we, of course, have $\Lambda_1(E_{b\Omega_2}) > 0$ since $E_{b\Omega_2}$ is a curve. But in spite of the real-analyticity of $\Omega_2$ and $E_{b\Omega_2}$ we have:

**Theorem 4.** *Let $\Omega, \Omega' \subset\subset \mathbb{C}^2$ be domains, $0 \in b\Omega'$ and $b\Omega$, such that there exist neighborhoods $U = U(0)$ and $V = V(0)$ of the origin with $b\Omega \cap U = b\Omega_2 \cap U$ and $b\Omega' \cap V$ strongly pseudoconvex and $C^\infty$-smooth. Then there is no proper holomorphic mapping*

$$\Phi: U \cap \Omega \to V \cap \Omega'$$

*such that $\Phi \in C^\infty(U \cap \bar{\Omega})$, $\Phi(0) = 0$, and $\Phi(U \cap b\Omega) = V \cap b\Omega'$.*

*Proof.* Suppose there is a proper holomorphic mapping $\Phi: U \cap \Omega \to V \cap \Omega'$ as in the theorem. Let $\varrho'$ be a strongly plurisubharmonic defining function of $\Omega'$ near 0. On $\Omega$ near 0 we define $\varrho$ to be the $C^\infty$-function $\varrho' \circ \Phi$. It vanishes on $b\Omega$ near 0 and has positive semidefinite Leviform on $b\Omega$. Because of the Hopf lemma $\varrho$ is a $C^\infty$-defining function of $b\Omega$ near 0. This contradicts Theorem 2 since $b\Omega = b\Omega_2$ near 0. □

## 3. Proof of Theorem 1

*3.1.* First, a real hypersurface $H \subset \mathbb{C}^2$, $0 \in H$, is constructed such that on an open neighborhood $U = U(0)$ of the origin $H$ is defined by a smooth $C^\omega$-function $\varrho$ and $H$ satisfies a) to c) of Theorem 1 from the side $\varrho < 0$. If we set

$$P_6(z) := \frac{1}{2}|z|^6 + 2\,\mathrm{Re}\left(-\frac{1}{20}\bar{z}^5 z + \frac{i}{4}\bar{z}^4 z^2\right),$$

$$Q_4(z) := \frac{1}{2}|z|^4 - \frac{i}{6}z^3\bar{z} + \frac{i}{6}\bar{z}^3 z,$$

$$R(z,v) := P_6(z) + 2vQ_4(z) + |z|^2v^2 + |z|^2v^4 + |z|^{10} + |z|^6v^2$$

and

$$\varrho(z,w) := u + R(z,v)$$

and choose $U=U(0)$ so small that $d\varrho \neq 0$ on $U$ then

$$H:=\{(z,w)\in U \mid \varrho(z,w)=0\}$$

is a smooth $C^\omega$-hypersurface.

*3.2.* We want to mention the following obvious but essential properties of $P_6$ and $Q_4$:

**Lemma.** a) *For all* $z\in\mathbb{C}$ $P_6(z)\geqq\frac{1}{10}|z|^6$.

b) *For all* $z\in\mathbb{C}$ $\dfrac{\partial^2}{\partial z\partial\bar z}P_6(z)\geqq|z|^4$.

c) *For all* $z\in\mathbb{C}$ *and all* $v\in\mathbb{R}$

$$\frac{\partial^2}{\partial z\partial\bar z}(P_6(z)+2vQ_4(z)+|z|^2v^2)=\left(2|z|^2-\frac{i}{2}(z^2-\bar z^2)+v\right)^2.$$

*3.3.* **Proposition.** *There is a* $\tau_0>0$ *such that* $U_{\tau_0}(0)\subset U(0)$ *and for all* $\tau<\tau_0$ $H\cap H_{\tau_0}(0)$ *is pseudoconvex from the side* $\varrho<0$ *and the origin is the only non-strongly pseudoconvex point.*

*Proof.* We calculate all derivatives of $\varrho$ that we need:

$$R_z(z,v)=\frac{\partial}{\partial z}P_6(z)+2v\frac{\partial}{\partial z}Q_4(z)+\bar z v^2+\bar z v^4+5\bar z^5z^4+3\bar z^3z^2v^2,$$

$$R_{z\bar z}(z,v)=\left(2|z|^2-\frac{i}{2}(z^2-\bar z^2)+v\right)^2+v^4+25|z|^8+9|z|^4v^2,$$

$$R_v(z,v)=2Q_4(z)+2v|z|^2+4v^3|z|^2+2v|z|^6,$$

$$R_{vv}(z,v)=2|z|^2+12v^2|z|^2+2|z|^6,$$

$$R_{\bar z v}(z,v)=2\frac{\partial}{\partial\bar z}Q_4(z)+2vz+4v^3z+6v\bar z^2z^3.$$

For $(z,w)\in\mathbb{C}^2$ we get:

$$\begin{aligned}4\mathscr{L}_{H,\varrho}(z,w)&=\{R_{z\bar z}(R_v^2+1)+R_{vv}|R_z|^2+2\operatorname{Re}(R_{\bar z v}R_z(i-R_v))\}(z,v)\\&\geqq(v^4+25|z|^8+9|z|^4v^2)(1+O(|z|)+O(|v|))\\&\quad+2\operatorname{Re}\left\{\left(2\frac{\partial}{\partial\bar z}Q_4(z)\frac{\partial}{\partial z}P_6(z)+4v\left|\frac{\partial}{\partial z}Q_4(z)\right|^2+2\bar z\frac{\partial}{\partial\bar z}Q_4(z)v^2\right.\right.\\&\quad\left.\left.+2vz\frac{\partial}{\partial z}P_6(z)+4v^2z\frac{\partial}{\partial z}Q_4(z)+2|z|^2v^3\right)(i-O(|z|^2))\right\}\\&\geqq(v^4+11|z|^8+6|z|^4v^2)(1+O(|z|)+O(|v|)).\end{aligned}$$

With suitable $\tau_0>0$ we get for all $\tau<\tau_0$ and $(z,w)\in U_\tau(0)$

$$\mathscr{L}_{H,\varrho}(z,w)\geqq\tfrac{1}{2}(v^4+11|z|^8+6|z|^4v^2)\geqq0$$

and

$$\mathscr{L}_{H,\varrho}(z,-R(z,v)+iv)=0$$

only in the origin. □

*3.4.* Because of 3.2a) and 3.3 the origin is of strict type 3 in the sense of Kohn [6]. Hence b) of Theorem 1 is satisfied.

*3.5.* To prove c) of Theorem 1 we look at $H$ and its defining functions along the curves ($0<\delta$ suitably small, $\theta\in[-\pi,\pi)$)

$$\phi_\theta:[0,\delta]\to H\cap U_\tau(0)$$

with

$$\phi_\theta(r):=(re^{i\theta}, -R(re^{i\theta}, -r^2(2+\sin 2\theta))-ir^2(2+\sin 2\theta)).$$

Because of 3.2 the function $R$ and its derivatives have the following order of vanishing in $r$ along $\operatorname{Im}\phi_\theta$:

$$\begin{aligned}
&R(\phi_\theta(r))=O(r^6),\\
&R_z(\phi(r))=O(r^5),\\
&R_{z\bar z}(\phi_\theta(r))=O(r^8),\\
&R_v(\phi_\theta(r))=O(r^4),\\
&R_{\bar z v}(\phi_\theta(r))=\left(2\frac{\partial}{\partial\bar z}Q_4(z)+2zv\right)_{|\operatorname{Im}\phi_\theta}+O(r^7)\\
&\qquad=r^3e^{i\theta}\left(-2-\frac{2}{3}\sin 2\theta+\frac{2i}{3}\cos 2\theta\right)+O(r^7).
\end{aligned}$$

*3.6.* **Lemma.** *The Leviform $L_\varrho$ of $\varrho$ has a negative eigenvalue in all points $p\in\operatorname{Im}\phi_\theta$ for all $\theta$.*

*Proof.* Because of 3.5

$$4|\varrho_{\bar z w}|^2_{|\operatorname{Im}\phi_\theta}=r^6((2+\tfrac{2}{3}\sin 2\theta)^2+\tfrac{4}{9}\cos^2 2\theta)+O(r^{10})$$

and

$$\varrho_{z\bar z|\operatorname{Im}\phi_\theta}=O(r^8).$$

Therefore,

$$(\varrho_{z\bar z}\varrho_{w\bar w}-|\varrho_{\bar z w}|^2)_{|\operatorname{Im}\phi_\theta}=-r^6((1+\tfrac{1}{3}\sin 2\theta)^2+\tfrac{1}{9}\cos^2 2\theta)+O(r^8).$$

This is negative for small $r\neq 0$ and for all $\theta$. □

*3.7.* In 3.8 the existence of a local defining function of $H$ with positive semidefinite Leviform near 0 is reduced to the existence of a real solution of a differential equation. To simplify the proof of 3.8 we first show in the following lemma that such a real solution does not exist.

**Lemma.** *There is no open neighborhood $V=V(0)\subset\mathbb{C}^2$ and no $C^5$-function $h:V\to\mathbb{R}^+$ such that*

$$h_z(0)=h_{zz}(0)=h_{z\bar z}(0)=0$$

*and for all $\theta$ along $\operatorname{Im}\phi_\theta$*

$$\frac{\partial}{\partial\bar z}h(z,w)=2i\left(\frac{\partial}{\partial\bar z}Q_4(z)+vz\right).$$

*Proof.* Suppose there is an open neighborhood $V=V(0)\subset\mathbb{C}^2$ and a $C^5$-function $h:V\to\mathbb{R}^+$ that fulfills the equations above. Taylor expansion gives:

$$h(z,w)=h(0)+h_{3,0}(z)+h_{4,0}(z)+vh_v(0)+vh_{1,1}(z)$$
$$+vh_{2,1}(z)+\text{higher order terms}$$

with homogeneous real polynomials $h_{j,0}, h_{j,1}$ in $z,\bar{z}$ of degree $j$ resp. $i$, $j=3$, 4; $i=1,2$.

According to 3.5 the term $2i\left(\frac{\partial}{\partial\bar{z}}Q_4(z)+vz\right)$ has only terms of order 3 in $r$ along $\operatorname{Im}\phi_\theta$. If we look at $\frac{\partial}{\partial\bar{z}}h(z,w)_{|\operatorname{Im}\phi_\theta}$ we see that

$$\frac{\partial}{\partial\bar{z}}(h_{3,0}(z)+vh_{1,1}(z)) \quad \text{has order 2,}$$

$$\frac{\partial}{\partial\bar{z}}(h_{4,0}(z)+vh_{2,1}(z)) \quad \text{has order 3}$$

and the rest have higher order in $r$ along $\operatorname{Im}\phi_\theta$. Therefore,

$$h_{3,0}(z)=h_{1,1}(z)=0.$$

The equation

$$-2i\left(\frac{\partial}{\partial\bar{z}}Q_4(z)+vz\right)+\frac{\partial}{\partial\bar{z}}(h_{4,0}(z)+vh_{2,1}(z))_{|\operatorname{Im}\phi_\theta}=0$$

comes down to an inhomogeneous system of linear equations for the coefficients of $h_{4,0}$ and $h_{2,1}$ which has no solution. □

*3.8.* Now we can show that $H$ satisfies c) of Theorem 1:

**Proposition.** *Let $U=U(0)\subset\mathbb{C}^2$ be an open neighborhood of the origin and $\sigma:U\to\mathbb{R}$ a defining $C^6$-function of $H\cap U$. Then there are points $p\in H\cap U$ (arbitrarily close to 0) such that the Leviform $L_\sigma(p)$ has a negative eigenvalue on $\mathbb{C}^2$.*

*Proof.* Suppose there is an open neighborhood $U=U(0)$ and a defining $C^6$-function $\sigma:U\to\mathbb{R}$ such that the Leviform

$$L_\sigma(p) \text{ is positive semidefinite for all } p\in H\cap U. \qquad (*)$$

Then there exists a positive $C^5$-function $h:U\to\mathbb{R}^+$ with $\sigma=\varrho\cdot h$. Set $h(0)=1$.

a) If we look at $\sigma$ on

$$M=\{(z,w)\in H\cap U \mid v=0,\ u=-R(z,0)\}$$

we get for $p\in M$

$$\sigma_{z\bar{z}}(p)=O(|z|^4)$$

and

$$2\sigma_{\bar{z}w}(p)=h_{\bar{z}}(z,-R(z,0))+O(|z|^3).$$

Because of $(*)$ $\sigma_{\bar{z}w}(p)$ must vanish at least to order 2 in $|z|$, and therefore,

$$h_z(0)=h_{zz}(0)=h_{z\bar{z}}(0)=0.$$

b) For fixed $\theta$ we look at $\sigma$ along $\operatorname{Im}\phi_\theta$. With the help of 3.5 we get that $\sigma_{z\bar{z}|\operatorname{Im}\phi_\theta} = O(r^7)$ and again (*) implies that $|\sigma_{\bar{z}w}|^2_{|\operatorname{Im}\phi_\theta} = O(r^7)$, and therefore,

$$\frac{\partial}{\partial \bar{z}} h(z,w) = 2i\left(\frac{\partial}{\partial \bar{z}} Q_4(z) + vz\right)$$

for all $\theta$ along $\operatorname{Im}\phi_\theta$. This contradicts the assumption because of Lemma 3.7. □

*3.9.* Tedious but straightforward calculations show that for $\tau > 0$ sufficiently small we get by defining

$$\Phi_\tau(z,w) := \varrho(z,w) + \frac{2}{\tau^{11}}(|w|^{12} + |w|^2|z|^{10} + |z|^{12} + |z|^2|w|^{10})$$

and

$$\Omega_\tau := \{(z,w) \in \mathbb{C}^2 | \Phi_\tau(z,w) < 0\}$$

a bounded $C^\omega$-smooth domain. Because $H$ is of strict type in the origin and the pollynomials added to $\varrho$ are of high enough order in $z$ and $w$ the domains $\Omega_\tau$ are pseudoconvex and satisfy properties a) to c) from Theorem 1. We will omit the details.

This finishes the proof of Theorem 1.

## 4. Proof of Theorem 2

*4.1.* First, we define for $0 < \tau < \frac{1}{2}$ unbounded domains

$$\Omega_\tau =: \{(z,w) \in \mathbb{C}^2 | \varrho_\tau(z,w) < 0\}$$

by

$$P_{10}(z) := \tfrac{1}{16}(2\operatorname{Re}(\tfrac{1}{16}\bar{z}^8 z^2 + \tfrac{2}{21}\bar{z}^7 z^3 + \tfrac{1}{8}\bar{z}^6 z^4) + \tfrac{4}{25}|z|^{10});$$

$$Q_7(z) := 2\operatorname{Re}(\tfrac{1}{20}\bar{z}^5 z^2 + \tfrac{1}{24}\bar{z}^4 z^3);$$

and

$$\varrho_\tau(z,w) := u + P_{10}(z) + vQ_7(z) + \frac{1}{4}|z|^4 v^2 + \frac{2}{\tau^{15}}|z|^{16} + |w|^4|z|^4 + \frac{2}{\tau^{11}}|w|^2|z|^{10}.$$

*4.2.* We mention the important properties of $P_{10}$ and $Q_7$:

**Lemma.** a) *For all* $z \in \mathbb{C}$

$$\frac{\partial^2}{\partial z \partial \bar{z}} P_{10}(z) = \tfrac{1}{16}|z|^2(\bar{z}^3 + z^3 + \bar{z}z^2 + z\bar{z}^2)^2 \geqq 0;$$

b) *for all* $z \in \mathbb{C}$ *and* $v \in \mathbb{R}$

$$\frac{\partial^2}{\partial z \partial \bar{z}}\left(P_{10}(z) + vQ_7(z) + \frac{1}{4}|z|^4 v^2\right) = |z|^2\left(\frac{1}{4}(\bar{z}^3 + z^3 + \bar{z}z^2 + z\bar{z}^2) + v\right)^2. \quad \square$$

*4.3.* If we define

$$G_\tau := \{u+iv \in \mathbb{C} \mid u < (\tau^5 + \tfrac{1}{2}\tau^2|v|)^2\}$$

easy estimates show

**Lemma.** a) $\Omega_\tau \subset \{z \in \mathbb{C} \mid |z| < \tau\} \times G_\tau$;
b) *for* $p \in b\Omega_\tau$ *the gradient* $d\varrho_\tau(p) \neq 0$. □

*Remark.* Lemma 4.3 obviously implies that $\Omega_\tau$ is biholomorphically equivalent to a bounded smooth $C^\omega$-domain in $\mathbb{C}^2$, for instance via the mapping $F(z,w) := (z, T(w))$, where $T: \mathbb{C} \to \mathbb{C}$ is biholomorphic with $T(G_\tau) \subset \{|w| < 1\}$. Thus it suffices to show that some $\Omega_\tau$ fulfills Theorem 2.

*4.4.* **Proposition.** *There is an* $\tau_1 < \tau_0$, *such that* $\Omega_\tau$ *is pseudoconvex for all* $\tau < \tau_1$. *The set of the non-strongly pseudoconvex boundary points is*

$$E := \{(z,w) \in \mathbb{C}^2 \mid z = 0 = u\}.$$

*Proof.* We calculate the derivatives of $\varrho_\tau = \varrho$ that we need:

$$\varrho_z = \frac{\partial}{\partial z} P_{10}(z) + v \frac{\partial}{\partial z} Q_7(z) + \frac{1}{2}\bar{z}^2 z v^2 + \frac{16}{\tau^{15}} \bar{z}^8 z^7 + 2|w|^4 \bar{z}^2 z + \frac{10}{\tau^{11}} |w|^2 \bar{z}^5 z^4,$$

$$\varrho_{z\bar{z}} = |z|^2 \left(\frac{1}{4}(\bar{z}^3 + z^3 + \bar{z}z^2 + z\bar{z}^2) + v\right)^2 + \frac{128}{\tau^{15}} |z|^{14} + 4|w|^4|z|^2 + \frac{50}{\tau^{11}} |w|^2 |z|^8,$$

$$\varrho_{\bar{w}} = \frac{1}{2}\left(1 + iQ_7(z) + \frac{i}{2}|z|^4 v\right) + 2w^2\bar{w}|z|^4 + \frac{2}{\tau^{11}} w|z|^{10},$$

$$\varrho_{w\bar{w}} = \frac{1}{8}|z|^4 + 4|w|^2|z|^4 + \frac{2}{\tau^{11}}|z|^{10},$$

$$\varrho_{\bar{z}w} = -\frac{i}{2}\frac{\partial}{\partial \bar{z}} Q_7(z) - \frac{i}{2} z^2 \bar{z} v + 4\bar{w}^2 w \bar{z} z^2 + \frac{10}{\tau^{11}} \bar{w} \bar{z}^4 z^5.$$

To simplify the notation we introduce

$$\varrho^1(z,w) := u + P_{10}(z) + vQ_7(z) + \tfrac{1}{4}|z|^4 v^2;$$

$$\varrho^2(z,w) := \frac{2}{\tau^{15}}|z|^{16} + |w|^4|z|^4 + \frac{2}{\tau^{11}}|w|^2|z|^{10}.$$

Because $\varrho^1_{z\bar{z}}$ and $\varrho^1_{w\bar{w}}$ are positive

$$\begin{aligned}\mathscr{L}_{b\Omega_\tau,\varrho} \geq\ & \varrho^2_{z\bar{z}}|\varrho^2_w|^2 + \varrho^2_{w\bar{w}}|\varrho^2_z|^2 - 2\operatorname{Re}(\varrho^2_{\bar{z}w}\varrho^2_z\varrho^2_{\bar{w}}) \\ & + \varrho^1_{z\bar{z}}(2\operatorname{Re}(\varrho^1_w\varrho^2_{\bar{w}}) + |\varrho^2_w|^2) + \varrho^2_{z\bar{z}}(|\varrho^1_w|^2 + 2\operatorname{Re}(\varrho^1_w\varrho^2_{\bar{w}})) \\ & + \varrho^2_{w\bar{w}}(|\varrho^1_z|^2 + 2\operatorname{Re}(\varrho^1_z\varrho^2_{\bar{z}})) \\ & - 2\operatorname{Re}[\varrho^1_{\bar{z}w}(\varrho^1_z + \varrho^2_z)\cdot(\varrho^1_{\bar{w}} + \varrho^2_{\bar{w}}) + \varrho^2_{\bar{z}w}(\varrho^1_z\varrho^1_{\bar{w}} + \varrho^1_z\varrho^2_{\bar{w}} + \varrho^2_z\varrho^1_{\bar{w}})]. \qquad (1)\end{aligned}$$

We only have to calculate $\mathscr{L}_{b\Omega_{\tau,\varrho}}$ up to terms of order

$$O\left(\frac{1}{\tau^{15}}|z|^{15}\right)+O(|w|^4|z|^3)+O\left(\frac{1}{\tau^{11}}|w|^2|z|^9\right)+O\left(\frac{1}{\tau^{11}}|w|^8|z|^{17}\right).$$

Terms of this order we abbreviate by the symbol $O(S)$.

We split the sum in (1) into

$$\begin{aligned}&\varrho^2_{z\bar z}|\varrho^2_w|^2+\varrho^2_{w\bar w}|\varrho^2_z|^2-2\operatorname{Re}[\varrho^2_{\bar z w}\varrho^2_z\varrho^2_{\bar w}]\\&\quad=128\left\{\frac{4}{\tau^{41}}|z|^{40}+\left(\frac{4}{\tau^{37}}+\frac{8}{\tau^{30}}\right)|w|^6|z|^{34}+\frac{12}{\tau^{26}}|w|^4|z|^{28}\right.\\&\qquad\left.+\left(\frac{4}{\tau^{15}}+\frac{2}{\tau^{22}}\right)|w|^6|z|^{22}+\frac{1}{\tau^{11}}|w|^8|z|^{16}\right\};\end{aligned}\tag{2}$$

$$\begin{aligned}&\varrho^1_{z\bar z}(2\operatorname{Re}(\varrho^1_w\varrho^2_{\bar w})+|\varrho^2_w|^2)\\&\quad=4|w|^6v|z|^8\frac{\partial^2}{\partial z\partial\bar z}Q_7(z)+2|w|^2v^2|z|^6u+4|w|^6v^2|z|^{10}+O(S);\end{aligned}\tag{3}$$

$$\begin{aligned}&\varrho^2_{z\bar z}(|\varrho^1_w|^2+2\operatorname{Re}(\varrho^1_w\varrho^2_{\bar w}))\\&\quad\geqq\frac{1}{4}\left(\frac{128}{\tau^{15}}|z|^{14}+4|w|^4|z|^2+\frac{50}{\tau^{11}}|w|^2|z|^8\right)\\&\qquad+\frac{50}{\tau^{22}}|w|^2v^2|z|^{22}+4|w|^6v^2|z|^{10}\\&\qquad+u\left(\frac{256}{\tau^{26}}|z|^{24}+8|w|^6|z|^6+\frac{108}{\tau^{11}}|w|^4|z|^{12}+\frac{100}{\tau^{22}}|w|^2|z|^{18}\right)+O(S);\end{aligned}\tag{4}$$

$$\begin{aligned}&\varrho^2_{w\bar w}(|\varrho^1_z|^2+2\operatorname{Re}(\varrho^1_z\varrho^2_{\bar z}))\\&\quad\geqq 8|w|^6v^2|z|^{10}+16|w|^6|z|^6v\operatorname{Re}\left(z\frac{\partial}{\partial z}Q_7(z)\right)+O(S);\end{aligned}\tag{5}$$

$$\begin{aligned}&-2\operatorname{Re}[\varrho^1_{\bar z w}(\varrho^1_z+\varrho^2_z)(\varrho^1_{\bar w}+\varrho^2_{\bar w})]\\&\quad=16|w|^6|z|^6\operatorname{Re}\left(iw\bar z\frac{\partial}{\partial\bar z}Q_7(z)\right)\\&\qquad-\frac{20}{\tau^{22}}|w|^2v^2|z|^{22}-4|w|^6v^2|z|^{10}+O(S);\end{aligned}\tag{6}$$

$$\begin{aligned}&-2\operatorname{Re}[\varrho^2_{\bar z w}(\varrho^1_z\varrho^2_{\bar w}+\varrho^1_z\varrho^2_{\bar w}+\varrho^2_z\varrho^1_{\bar w})]\\&\quad=-16v|wz|^6\operatorname{Re}\left(z\frac{\partial}{\partial z}Q_7(z)\right)\\&\qquad-8|w|^6v|z|^6Q_7(z)-\frac{20}{\tau^{22}}|w|^2v^2|z|^{22}\\&\qquad-12|w|^6v^2|z|^{10}-u\left(\frac{160}{\tau^{26}}|z|^{24}+\frac{100}{\tau^{22}}|w|^2|z|^{18}+2|w|^2v^2|z|^6\right.\\&\qquad\left.+\frac{60}{\tau^{11}}|w|^4|z|^{12}+8|w|^6|z|^6\right)+O(S).\end{aligned}\tag{7}$$

Adding (2) – (7) and taking into account that for $(z,w)\in b\Omega_\tau$ $u=-R(z,v)$ and that for suitable constants $C$, $\tilde{C}>O$ we have

$$4|w|^6|z|^6\left(v|z|^2\frac{\partial^2}{\partial z\partial\bar{z}}Q_7(z)+4\mathrm{Re}\left(iw\bar{z}\frac{\partial}{\partial\bar{z}}Q_7(z)\right)-2vQ_7(z)\right)$$
$$\geqq -C(|w|^8|z|^{16}+|w|^6|z|^{10})\geqq -\tilde{C}(|w|^8|z|^{16}+|w|^4|z|^4),$$

we get for all $(z,w)\in b\Omega_\tau$:

$$\mathscr{L}_{b\Omega_\tau,\varrho_\tau}(z,w)\geqq\frac{1}{4}\left(\frac{128}{\tau^{15}}|z|^{14}+4|w|^4|z|^2+\frac{50}{\tau^{11}}|w|^2|z|^8\right)$$
$$+\left(\frac{80}{\tau^{11}}-\tilde{C}\right)|w|^8|z|^{16}+O(S).$$

Because for all $(z,w)\in b\Omega_\tau$ we have $|z|<\tau$ (Lemma 4.3), the definition of $O(S)$ implies that for suitable $\tau_0$ we get for all $\tau\leqq\tau_0$ and all $(z,w)\in b\Omega_\tau$:

$$\mathscr{L}_{b\Omega_\tau,\varrho_\tau}(z,w)\geqq\frac{1}{8}\left(\frac{128}{\tau^{15}}|z|^{14}+4|w|^4|z|^2+\frac{50}{\tau^{11}}|w|^2|z|^8\right)\geqq 0.$$

Then (1) shows that

$$\mathscr{L}_{b\Omega_\tau,\varrho_\tau}(z,w)=0\quad\text{iff}\quad z=0.$$

But on $b\Omega_\tau$ $z=0$ implies $u=0$. □

*4.5.* From now on we keep $\tau\leqq\tau_0$ fixed and write $\Omega:=\Omega_\tau$ and $\varrho:=\varrho_\tau$. With the help of the implicit function theorem applied to $\varrho$ at 0 we see that in a small open neighborhood $U=U(0)$ of the origin $\Omega$ is defined by the graph $u=-R(z,v)$ with

$$R(z,v):=P_{10}(z)+vQ_7(z)+\tfrac{1}{4}v^2|z|^4+O(|z|^{15})+O(|v|\,|z|^{12})$$
$$+O(v^2|z|^9)+O(|v|^3|z|^6)+O(v^4|z|^3)+O(|v|^5).$$

Denote $u+R(z,v)$ by $\tilde{\varrho}(z,w)$.

*4.6.* To prove b) of Theorem 2 we look at $\Omega\cap U$ along the curves ($0<\delta$ suitably small, $\theta\in[-\pi,\pi)$):

$$\phi_\theta:[0,\delta]\to b\Omega\cap U$$

defined by

$$\phi_\theta(r):=\left(re^{i\theta},-R\left(re^{i\theta},-\frac{r^3}{2}(\cos 3\theta+\cos\theta)\right)-i\frac{r^3}{2}(\cos 3\theta+\cos\theta)\right).$$

For $p\in\mathrm{Im}\,\phi_\theta$ we have:

$$R(p)=O(r^{10});\qquad R_z(p)=O(r^9);$$
$$R_{z\bar{z}}(p)=O(r^{13});\qquad R_v(p)=O(r^7);$$

$$R_{\bar{z}v}(p)=\frac{\partial}{\partial\bar{z}}Q_7(z)+vz^2\bar{z}_{|p}+O(r^7)$$
$$=-\tfrac{1}{2}e^{i\theta}r^6\{\tfrac{3}{10}\cos 3\theta+\tfrac{5}{12}\cos\theta+i(\tfrac{3}{10}\sin 3\theta+\tfrac{1}{12}\sin\theta)\}+O(r^7).$$

*4.7.* By a method similar to that in the proof of 3.6 one shows:

**Lemma.** *There is no open neighborhood $U=U(0)$ such that for all $p\in U\cap b\Omega$ the Leviform $L_{\tilde{\varrho}}(p)$ is positive semidefinite on $\mathbb{C}^2$.* □

*4.8.* The idea in the proof that $\Omega$ satisfies condition b) of Theorem 2 is the same as in 3.8: reduction of the problem to a system of differential equations that has no real solution.

We put some tedious but simple calculations from the proof into the following two lemmas whose verification only involve inhomogeneous systems of linear equations.

**Lemma.** *For $5\leqq v\leqq 7$ let $H_{v0}(z)$ be a homogeneous real polynomial in $z$, $\bar{z}$ of degree $v$ such that for all $z=x+iy$ with $x=0$, $x=y$ or $x=-y$*

$$\frac{\partial}{\partial\bar{z}}H_{50}(z)=\frac{\partial}{\partial\bar{z}}H_{60}(z)=0$$

*and*

$$\frac{\partial}{\partial\bar{z}}H_{70}(z)=i\frac{\partial}{\partial\bar{z}}Q_7(z)\,.$$

*Then*

$$H_{50}(z)=0\,,$$

$$H_{60}(z)=c_1\left(\operatorname{Re}\left(\frac{1}{2}\bar{z}^6+\bar{z}^5z+\frac{3}{2}\bar{z}^4z^2\right)+|z|^6\right),$$

$$H_{70}(z)=\operatorname{Re}\left(\left(\frac{1}{7}c_2+ic_3\right)\bar{z}^7+\frac{3}{7}\left(c_2+ic_3\right)\bar{z}^6z\right.$$
$$\left.+\left(\frac{5}{7}c_2+\frac{3i}{70}+ic_3\right)\bar{z}^5z^2+\left(c_2+\frac{i}{84}+ic_3\right)\bar{z}^4z^3\right)$$

*with $c_i\in\mathbb{R}$, $1\leqq i\leqq 3$.* □

*4.9.* **Lemma.** *There are no homogeneous real polynomials $H_{70}(z)$, $H_{41}(z)$, and $H_{12}(z)$ of degree 7, 4, and 1 such that*

a) *$H_{70}$ is the polynomial in 4.8;*

b) *for all $\theta\in[-\frac{\pi}{2},\frac{\pi}{2})$ we have along $\operatorname{Im}\phi_\theta$:*

$$-i\frac{\partial}{\partial\bar{z}}Q_7(z)-ivz^2\bar{z}+\frac{\partial}{\partial\bar{z}}(H_{70}(z)+vH_{41}(z)+v^2H_{12}(z))=0\,. \quad \square$$

Now we can show:

*4.10.* **Proposition.** *Let $U=U(0)$ be an open neighborhood of the origin and $\sigma: U\to\mathbb{R}$ a defining $C^k$-function, $k\geqq 9$, of $b\Omega\cap U$. Then there exist points $p\in b\Omega\cap U$ (arbitrarily close to 0) such that the Leviform $L_\sigma(p)$ has a negative eigenvalue on $\mathbb{C}^2$.*

*Proof.* Suppose there exist $U$ and $\sigma$ as described in the proposition such that $L_\sigma(p)$ is positive semidefinite for all $p\in b\Omega\cap U$. Then there is a positive $C^{k-1}$-function $H: U\to\mathbb{R}^+$ with $\sigma=\tilde{\varrho}H$. Set $H(0)=1$.

Taylor expansion gives:

$$H(z,w)=1+\sum_{v=1}^{7} H_{v0}(z)+v\sum_{v=0}^{6} H_{v1}(z)+v^2\sum_{v=0}^{5} H_{v2}(z)$$

$$+v^3\sum_{v=0}^{1} H_{v3}(z)+\text{higher order terms}$$

with real homogeneous polynomials $H_{v\mu}$ in $z$, $\bar{z}$ of degree $v$, $0\leqq\mu\leqq 3$.

a) Looking at $\sigma$ in points $p=(z, -R(z,0))$ we find that

$$\sigma_{z\bar{z}}(p)=O(|z|^{13})$$

and thus that $\sigma_{\bar{z}w}$ must vanish at least to order 6 in $|z|$. Since

$$\sigma_{\bar{z}w}(p)=-\frac{i}{2}\frac{\partial}{\partial\bar{z}}Q_7(z)+\frac{\partial}{\partial\bar{z}}\left(\sum_{v=5}^{7} H_{v0}(z)\right)+O(r^7),$$

it follows that the $H_{v0}(z)$ must be the polynomials in 4.8 for $5\leqq v\leqq 7$.

b) Looking at $\sigma$ in points $p=(0, -R(0,v)+iv)$ one sees that

$$H_{11}(0)=H_{12}(0)=0.$$

c) Finally, we take points $p\in\operatorname{Im}\phi_\theta$ for arbitrary $\theta$. There again $\sigma_{z\bar{z}}(p)=O(r^{13})$. Because

$$2\sigma_{\bar{z}w}(p)=-i\frac{\partial}{\partial\bar{z}}Q_7(z)-ivz^2\bar{z}+\frac{\partial}{\partial\bar{z}}\{H_{60}(z)+H_{70}(z)$$

$$+v(H_{21}(z)+H_{31}(z)+H_{41}(z))$$

$$+v^2H_{12}(z)\}_{|\operatorname{Im}\phi_\theta}+O(r^7),$$

we get that – independent of the choice of $H_{60}$, $H_{21}$, $H_{21}$, and $H_{31}$ –

$$0=-i\frac{\partial}{\partial\bar{z}}Q_7(z)-ivz^2\bar{z}+\frac{\partial}{\partial\bar{z}}(H_{70}(z)+vH_{41}(z)+v^2H_{12}(z))_{|\operatorname{Im}\phi_\theta}.$$

This contradicts 4.9, and therefore, the assumption. □

With $\Omega_2:=\Omega$ we have found a domain satisfying the conditions of Theorem 2.

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Received June 29, 1984