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IMBEDDING THE IRRATIONAL ROTATION C^*-ALGEBRA INTO AN AF-ALGEBRA

M. PIMSNER and D. VOICULESCU

Let $\theta \in (0, 1)$ be an irrational number and A_θ the corresponding irrational rotation C^*-algebra (i.e. the cross-product of the continuous functions on the unit circle by the automorphism corresponding to the rotation of angle $2\pi\theta$). This C^*-algebra is simple and has a unique trace state τ. It has been shown by M. A. Rieffel [9] that for every $\alpha \in [0, 1] \cap (\mathbf{Z} + \theta\mathbf{Z})$ there is a projection $p \in A_\theta$ such that $\tau(p) = \alpha$ and he conjectured that $\tau(p)$ can take values only in $[0, 1] \cap (\mathbf{Z} + \theta\mathbf{Z})$ and even more that the map $K_0(A_\theta) \to \mathbf{R}$ induced by τ establishes an isomorphism between the ordered group $K_0(A_\theta)$ and the subgroup $\mathbf{Z} + \theta\mathbf{Z}$ of $\mathbf{R}$. The group $K_0(A_\theta)$ has also appeared in the context of A. Connes' work [2] on operator algebras associated with foliations, as the range of an index-map in the case of the Kronecker flow on the 2-torus.

In the present note we construct an imbedding of A_θ into an AF-algebra constructed by E. G. Effros and C. L. Shen ([3]), the K-group of which is $\mathbf{Z} + \theta\mathbf{Z}$. This has as a consequence the fact that $\tau(p) \in \mathbf{Z} + \theta\mathbf{Z}$ for all projections $p \in A_\theta$, or somewhat stronger, that the image of the homomorphism $K_0(A_\theta) \to \mathbf{R}$ induced by τ is contained in $\mathbf{Z}+\theta\mathbf{Z}$ thus confirming a part of the facts conjectured by M. A. Rieffel. This, together with Rieffel's result, implies that A_{θ_1} and A_{θ_2} are isomorphic if and only if $\theta_1 \in \{\theta_2, 1 - \theta_2\}$.

The irrational rotation algebra is generated by two unitaries $u, v \in A_\theta$ such that $u^*vu = e^{2\pi i\theta}v$ and every pair U, V of unitaries in some unital C^*-algebra B, satisfying $U^*VU = e^{2\pi i\theta}V$ determines a $*$-monomorphism $\psi: A_\theta \to B$, such that $\psi(v) = V$, $\psi(u) = U$. This fact will be used in what follows.

Let $[a_1, a_2, \ldots, a_n, \ldots]$ be the continued fraction expansion of θ (see [8]). The n'th convergent $\frac{p_n}{q_n}$ equals $[a_1, a_2, \ldots, a_n]$ and p_n, q_n may be computed recurrently by the formulae:

$$(1)\qquad \begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \quad (n \geqslant 2) \text{ with } p_0 = 0,\ p_1 = 1 \\ q_n &= a_n q_{n-1} + q_{n-2} \quad (n \geqslant 2) \text{ with } q_0 = 1,\ q_1 = a_1. \end{aligned}$$

For each $n \in \mathbf{N}$ let $\mathcal{M}_{q_n}$ denote the $q_n \times q_n$ matrices acting on the Hilbert space $\mathbf{C}^{q_n}$. The canonical basis in $\mathbf{C}^{q_n}$ will be denoted by $e(n;j)$, $1 \leqslant j \leqslant q_n$. The AF-algebra into which we shall imbed A_θ is the inductive limit

$$\to A_{n-1} \xrightarrow{\rho_n} A_n \to \tag{2}$$

of the finite-dimensional C^*-algebras $A_n = \mathcal{M}_{q_n} \oplus \mathcal{M}_{q_{n-1}}$, the imbeddings $\rho_n : A_{n-1} \to$ $\to A_n$ being defined by

$$\rho_n(x \oplus y) = W_n(\underbrace{x \oplus \ldots \oplus x}_{a_n\text{-times}} \oplus y)\, W_n^* \oplus x$$

where $x \in \mathcal{M}_{q_{n-1}}$, $y \in \mathcal{M}_{q_{n-2}}$ and

$$W_n\colon\ \underbrace{\mathbf{C}^{q_{n-1}} \oplus \ldots \oplus \mathbf{C}^{q_{n-1}}}_{a_n\text{-times}} \oplus \mathbf{C}^{q_{n-2}} \to \mathbf{C}^{q_n}$$

is a unitary operator we shall define below. Remark that (1) makes the definition of W_n possible.

To define W_n we fix an integer s, satisfying $q_{n-2}/4 \leqslant s \leqslant q_{n-2}/2$. This is always possible when $n \geqslant 6$, so that the inductive limit (2) will start at $A_5 \to A_6 \to A_7 \ldots$.

The operator W_n $(n \geqslant 6)$ is of the form:

$$W_n(\xi_1 \oplus \ldots \oplus \xi_{a_n} \oplus \eta) =$$
$$= W_n^{(1)}\xi_1 + \ldots + W_n^{(a_n)}\xi_{a_n} + \widetilde{W}_n \eta$$

where

$$W_n^{(k)}\colon\ \mathbf{C}^{q_{n-1}} \to \mathbf{C}^{q_n}, \quad (1 \leqslant k \leqslant a_n)$$

$$\widetilde{W}_n\colon\ \mathbf{C}^{q_{n-2}} \to \mathbf{C}^{q_n}$$

are isometric operators given by the formulae:

$$W_n^{(k)} e(n-1;j) = \alpha_j^{(k)} e(n; a(k,j)) + \beta_j^{(k)} e(n; b(k,j)) + \gamma_j^{(k)} e(n; c(k,j))$$

$$(1 \leqslant j \leqslant q_{n-1}, \quad 1 \leqslant k \leqslant a_n)$$

and

$$\widetilde{W}_n e(n-2;\, j) = \lambda_j e(n;\ l(j)) + \mu_j e(n;\ m(j))$$

$$(1 \leqslant j \leqslant q_{n-2})$$

with indices and coefficients given by formulae listed below.

The indices are defined modulo q_n by:

$$a(k,j) = (-1)^k \left[\frac{k}{2}\right] q_{n-1} + j$$

$$b(k,j) = (-1)^{k+1} \left[\frac{k}{2}\right] q_{n-1} + j$$

$$c(k,j) = (-1)^k \left[\frac{k}{2}\right] q_{n-1} - q_{n-1} + i$$

for $1 \leqslant k \leqslant a_n$, $1 \leqslant j \leqslant q_{n-1}$ and

$$l(j) = (-1)^{a_n+1} \left[\frac{a_n+1}{2}\right] q_{n-1} + j$$

$$m(j) = \left[\frac{a_n+1}{2}\right] q_{n-1} + j - \frac{(1-(-1)^{a_n})}{2} q_{n-2}$$

for $1 \leqslant j \leqslant q_{n-2}$ and with $[x]$ denoting the integer part of x.

The formulae for the coefficients are:

$\alpha_j^{(1)} = 0$ where $1 \leqslant j \leqslant q_{n-1}$
and for $1 < k \leqslant a_n$

$$\alpha_j^{(k)} = \begin{cases} \exp\left(\frac{\pi i}{s} \cdot j \cdot \frac{1-(-1)^k}{2}\right) \cos\left(\frac{\pi}{2s} j\right) & \text{when } 1 \leqslant j \leqslant s \\ 0 & \text{when } s < j \leqslant q_{n-1}; \end{cases}$$

$$\beta_j^{(1)} = \begin{cases} 1 & \text{when } 1 \leqslant j \leqslant q_{n-1} - s \\ \cos\left(\frac{\pi}{2s}(j - q_{n-1} + s)\right) & \text{when } q_{n-1} - s < j \leqslant q_{n-1} \end{cases}$$

and for $1 < k \leqslant a_n$

$$\beta_j^{(k)} = \begin{cases} (-1)^k \exp\left(\frac{\pi i}{s} \cdot j \cdot \frac{1-(-1)^k}{2}\right) \sin\left(\frac{\pi}{2s} j\right) & \text{when } 1 \leqslant j \leqslant s \\ 1 & \text{when } s < j \leqslant q_{n-1} - s \\ \exp\left(\frac{\pi i}{s}(j - q_{n-1} + s) \frac{1+(-1)^k}{2}\right) \cos\left(\frac{\pi}{2s}(j - q_{n-1} + s)\right) & \\ & \text{when } q_{n-1} - s < j \leqslant q_{n-1} \end{cases}$$

and for $1 \leqslant k \leqslant a_n$

$$\gamma_j^{(k)} = \begin{cases} 0 & \text{when } 1 \leqslant j \leqslant q_{n-1} - s \\ (-1)^{k+1} \exp\left(\frac{\pi i}{s}(j - q_{n-1} + s)\frac{1+(-1)^k}{2}\right) \sin\left(\frac{\pi}{2s}(j - q_{n-1} + s)\right) & \text{when } q_{n-1} - s < j \leqslant q_{n-1}. \end{cases}$$

The formulae for λ_j and μ_j are distinct according to the fact that a_n is even or odd.

For a_n an even number, we have:

$$\lambda_j = \begin{cases} \exp\left(\frac{\pi i}{s} j\right) \cos\left(\frac{\pi}{2s} j\right) & \text{when } 1 \leqslant j \leqslant s \\ 0 & \text{when } s < j \leqslant q_{n-2} \end{cases}$$

$$\mu_j = \begin{cases} -\exp\left(\frac{\pi i}{s} j\right) \sin\left(\frac{\pi}{2s} j\right) & \text{when } 1 \leqslant j \leqslant s \\ 1 & \text{when } s < j \leqslant q_{n-2}. \end{cases}$$

For a_n an odd number, we have:

$$\lambda_j = \begin{cases} 1 & \text{when } 1 \leqslant j \leqslant q_{n-2} - s \\ \exp\left(\frac{\pi i}{s}(j - q_{n-2} + s)\right) \cos\left(\frac{\pi}{2s}(j - q_{n-2} + s)\right) & \text{when } q_{n-2} - s < j \leqslant q_{n-2} \end{cases}$$

$$\mu_j = \begin{cases} 0 & \text{when } 1 \leqslant j \leqslant q_{n-2} - s \\ -\exp\left(\frac{\pi i}{s}(j - q_{n-2} + s)\right) \sin\left(\frac{\pi}{2s}(j - q_{n-2} + s)\right) & \text{when } q_{n-2} - s < j \leqslant q_{n-2}. \end{cases}$$

A careful inspection of the above formulae shows that $W_n^{(k)}$, $1 \leqslant k \leqslant a_n$ and $\widetilde{W}_n$ are isometries with pairwise orthogonal ranges. This together with (1) implies that W_n is a unitary operator.

For each $n \geqslant 4$ we shall consider the operators U_n, $V_n \in \mathcal{M}_{q_n}$ defined by:

$$U_n e(n; j) = \begin{cases} e(n; j+1) & \text{when } 1 \leqslant j < q_n \\ e(n; 1) & \text{when } j = q_n \end{cases}$$

and

$$V_n e(n; j) = \zeta_n^j e(n; j) \quad \text{where } \zeta_n = \exp\left(2\pi i \frac{p_n}{q_n}\right).$$

LEMMA 1. *For $n \geqslant 6$ the following inequalities hold:*

(i) $\|W_n^{(k)} U_{n-1} - U_n W_n^{(k)}\| \leqslant \dfrac{36\pi}{q_{n-2}}$

(ii) $\|W_n^{(k)} V_{n-1} - V_n W_u^{(k)}\| \leqslant \dfrac{6\pi}{q_{n-1}}$

(iii) $\|\widetilde{W}_n U_{n-2} - U_n \widetilde{W}_n\| \leqslant \dfrac{36\pi}{q_{n-2}}$

(iv) $\|\widetilde{W}_n V_{n-2} - V_n \widetilde{W}_n\| \leqslant 6\pi \left(\dfrac{1}{q_{n-1}} + \dfrac{1}{q_{n-2}}\right)$.

Proof: Applying the operators, to be estimated, to the canonical basis of $\mathbf{C}^{q_{n-1}}$ and $\mathbf{C}^{q_{n-2}}$ respectively we have:

(i′) $$(W_n^{(k)} U_{n-1} - U_n W_n^{(k)})\, e(n-1; j) =$$

$$= \begin{cases} (\alpha_{j+1}^{(k)} - \alpha_j^{(k)})\, e(n; a(k, j+1)) + (\beta_{j+1}^{(k)} - \beta_j^{(k)})\, e(n; b(k, j+1)) + \\ + (\gamma_{j+1}^{(k)} - \gamma_j^{(k)})\, e(n; c(k, j+1)) & \text{for } 1 \leqslant j < q_{n-1} \\ \alpha_1^{(k)}\, e(n; a(k, 1)) + \beta_1^{(k)}\, e(n; b(k, 1)) + \gamma_1^{(k)}\, e(n; c(k, 1)) - \\ - \alpha_{q_{n-1}}^{(k)} e(n; a(k, q_{n-1}) + 1) - \beta_{q_{n-1}}^{(k)} e(n; b(k, q_{n-1}) + 1) - \\ - \gamma_{q_{n-1}}^{(k)}\, e(n; c(k, q_{n-1}) + 1) & \text{for } j = q_{n-1}. \end{cases}$$

(ii′) $$(W_n^{(k)} V_{n-1} - V_n W_n^{(k)})\, e(n-1; j) =$$

$$= (\zeta_{n-1}^j - \zeta_n^{a(k,j)})\, \alpha_j^{(k)}\, e(n; a(k,j)) + (\zeta_{n-1}^j - \zeta_n^{b(k,j)}) \cdot$$

$$\cdot \beta_j^{(k)}\, e(n; b(k,j)) + (\zeta_{n-1}^j - \zeta_n^{c(k,j)})\, \gamma_j^{(k)}\, e(n; c(k,j)).$$

(iii′) $$(\widetilde{W}_n U_{n-2} - U_n \widetilde{W}_n)\, e(n-2; j) =$$

$$= \begin{cases} (\lambda_{j+1} - \lambda_j)\, e(n; l(j+1)) + (\mu_{j+1} - \mu_j)\, e(n; m(j+1)) & \text{for } 1 \leqslant j < q_{n-2} \\ \lambda_1 e(n; l(1)) + \mu_1 e(n; m(1)) - \lambda_{q_{n-2}} e(n; l(q_{n-2}) + 1) - \\ - \mu_{q_{n-2}} \cdot e(n; m(q_{n-2}) + 1) & \text{for } j = q_{n-2}. \end{cases}$$

(iv′) $$(\widetilde{W}_n V_{n-2} - V_n \widetilde{W}_n)\, e(n-2; j) =$$

$$= (\zeta_{n-2}^j - \zeta_n^{l(j)})\, \lambda_j e(n; l(j)) + (\zeta_{n-2}^j - \zeta_n^{m(j)})\, \mu_j e(n; m(j)).$$

The inequalities:

$$\left|\cos\left(\frac{\pi}{2s}(j+1)\right) - \cos\left(\frac{\pi}{2s} j\right)\right| \leqslant \frac{\pi}{2s} \leqslant \frac{2\pi}{q_{n-2}}$$

$$\left|\sin\left(\frac{\pi}{2s}(j+1)\right) - \sin\left(\frac{\pi}{2s} j\right)\right| \leqslant \frac{\pi}{2s} \leqslant \frac{2\pi}{q_{n-2}}$$

$$\left|\exp\left(\frac{\pi i}{s}(j+1)\right)\cos\left(\frac{\pi}{2s}(j+1)\right)-\exp\left(\frac{\pi i}{s}j\right)\cos\left(\frac{\pi}{2s}j\right)\right|\leqslant\frac{3\pi}{2s}\leqslant\frac{6\pi}{q_{n-2}}$$

$$\left|\exp\left(\frac{\pi i}{s}(j+1)\right)\sin\left(\frac{\pi}{2s}(j+1)\right)-\exp\left(\frac{\pi i}{s}j\right)\sin\left(\frac{\pi}{2s}j\right)\right|\leqslant\frac{3\pi}{2s}\leqslant\frac{6\pi}{q_{n-2}}$$

show that $|\alpha_{j+1}^{(k)}-\alpha_j^{(k)}|$, $|\beta_{j+1}^{(k)}-\beta_j^{(k)}|$, $|\gamma_{j+1}^{(k)}-\gamma_j^{(k)}|$ $(1\leqslant k\leqslant a_n,\ 1\leqslant j<q_{n-1})$ and $|\lambda_{j+1}-\lambda_j|$, $|\mu_{j+1}-\mu_j|$ $(1\leqslant j<q_{n-2})$ are all bounded by $\dfrac{6\pi}{q_{n-2}}$.

This implies

$$\|(W_n^{(k)}U_{n-1}-U_nW_n^{(k)})\,e(n-1;j)\|\leqslant\frac{18\pi}{q_{n-2}}\qquad\text{for } 1\leqslant k\leqslant a_n,\ 1\leqslant j<q_{n-1}$$

and

$$\|(\widetilde{W}_nU_{n-2}-U_n\widetilde{W}_n)\,e(n-2;j)\|\leqslant\frac{12\pi}{q_{n-2}}<\frac{18\pi}{q_{n-2}}\qquad\text{for } 1\leqslant j<q_{n-2}.$$

To see that both inequalities hold also for $j=q_{n-1}$ and respectively $j=q_{n-2}$, note that $a(k,1)=c(k,q_{n-1})+1$ and $|\alpha_1^{(k)}-\gamma_{q_{n-1}}^{(k)}|<\dfrac{6\pi}{q_{n-2}}$ for $k>1$, while for $k=1$, $b(1,1)=c(1,q_{n-1})+1$ and $|\lambda_1-\gamma_{q_{n-1}}^{(1)}|<\dfrac{6\pi}{q_{n-2}}$, respectively. Also $l(1)=$ $=m(q_{n-2})+1$ and $|\lambda_1-\mu_{q_{n-2}}|<\dfrac{6\pi}{q_{n-2}}$.

This shows that all vectors in (i′) and (iii′) have norm less than $\dfrac{18\pi}{q_{n-2}}$. To get the estimates (i) and (iii) remark that the vectors $e(n;a(k,j_1))$, $e(n;b(k,j_2))$, $e(n;c(k,j_3))$, $1\leqslant j_1,j_2,j_3\leqslant q_{n-1}$ are pairwise orthogonal for fixed k.

This shows that the operator $W_n^{(k)}U_{n-1}-U_nW_n^{(k)}$ maps the vectors $e(n-1;j)$ $(1\leqslant j\leqslant q_{n-1})$ into pairwise orthogonal vectors. This, now, easily gives

$$\|W_n^{(k)}U_{n-1}-U_nW_n^{(k)}\|\leqslant\frac{36\pi}{q_{n-2}}.$$

Similarly one gets

$$\|\widetilde{W}_nU_{n-2}-U_n\widetilde{W}_n\|\leqslant\frac{36\pi}{q_{n-2}}.$$

Let us now prove the inequalities (ii) and (iv). To this end, note that for every integer m such that $|mq_{n-1}+j| \leqslant q_n$ we have

$$|\zeta_{n-1}^j - \zeta_n^{mq_{n-1}+j}| =$$

$$= \left|\exp\left(2\pi i \frac{p_{n-1}}{q_{n-1}} j\right) - \exp\left(2\pi i \frac{p_n}{q_n}(mq_{n-1}+j)\right)\right| =$$

$$= \left|\exp\left(2\pi i \frac{p_{n-1}}{q_{n-1}}(mq_{n-1}+j)\right) - \exp\left(2\pi i \frac{p_n}{q_n}(mq_{n-1}+j)\right)\right| \leqslant$$

$$\leqslant 2\pi q_n \left|\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n}\right| = \frac{2\pi}{q_{n-1}}$$

where the last equality follows from 10.2. Thm. 150 of [8].

Similarly, if m and l are integers such that

$$|mq_{n-1}+lq_{n-2}+j| \leqslant q_n, \quad |lq_{n-2}+j| \leqslant q_{n-1},$$

then

$$|\zeta_{n-2}^j - \zeta_n^{mq_{n-1}+lq_{n-2}+j}| \leqslant$$

$$\leqslant |\zeta_{n-2}^j - \zeta_{n-1}^{lq_{n-2}+j}| + |\zeta_{n-1}^{lq_{n-2}+j} - \zeta_n^{mq_{n-1}+lq_{n-2}+j}| \leqslant$$

$$\leqslant 2\pi\left(\frac{1}{q_{n-2}} + \frac{1}{q_{n-1}}\right).$$

These inequalities imply that

$$|\zeta_{n-1}^j - \zeta_n^{a(k,j)}|, \ |\zeta_{n-1}^j - \zeta_n^{b(k,j)}|, \ |\zeta_{n-1}^j - \zeta_n^{c(k,j)}|$$

are bounded by $\dfrac{2\pi}{q_{n-1}}$ for $1 \leqslant k \leqslant a_n$, $1 \leqslant j \leqslant q_{n-1}$, and that

$$|\zeta_{n-2}^j - \zeta_n^{l(j)}|, \ |\zeta_{n-2}^j - \zeta_n^{m(j)}|$$

are bounded by $2\pi\left(\dfrac{1}{q_{n-1}} + \dfrac{1}{q_{n-2}}\right)$ for $1 \leqslant j \leqslant q_{n-2}$.

Looking at formulae (ii′) and (iv′) we have:

$$\|(W_n^{(k)}V_{n-1} - V_nW_n^{(k)})e(n-1;j)\| \leqslant \frac{6\pi}{q_{n-1}} \qquad \text{for } 1 \leqslant k \leqslant a_n,\ 1 \leqslant j \leqslant q_{n-1}$$

$$\|(\widetilde{W}_nV_{n-2} - V_n\widetilde{W}_n)\, e(n-2;j)\| \leqslant 6\pi\left(\frac{1}{q_{n-1}} + \frac{1}{q_{n-2}}\right) \qquad \text{for } 1 \leqslant j \leqslant q_{n-2}.$$

Remarking that the operators in (ii) and (iv) map the canonical basis of $\mathbf{C}^{q_{n-1}}$ and respectively $\mathbf{C}^{q_{n-2}}$ into pairwise orthogonal vectors, the preceding estimates give (ii) and (iv). Q.E.D.

LEMMA 2.

$$\sum_{n\geq 6} \|\rho_n(U_{n-1}\oplus U_{n-2}) - U_n\oplus U_{n-1}\| < \infty$$

$$\sum_{n\geq 6} \|\rho_n(V_{n-1}\oplus V_{n-2}) - V_n\oplus V_{n-1}\| < \infty.$$

Proof. It is easily seen that the ranges of the operators $W_n^{(k)}U_{n-1} - U_nW_n^{(k)}$ and $W_n^{(j)}U_{n-1} - U_nW_n^{(j)}$ are orthogonal whenever $|k-j| > 6$. Using the preceding lemma this implies that

$$\|W_n(\underbrace{U_{n-1}\oplus\ldots\oplus U_{n-1}}_{a_n\text{-times}}\oplus U_{n-2}) - U_nW_n\| \leqslant$$

$$\leqslant 6 \sup_{1\leqslant k\leqslant a_n} \|W_n^{(k)}U_{n-1} - U_nW_n^{(k)}\| + \|\widetilde{W}_nU_{n-2} - U_n\widetilde{W}_n\| \leqslant 7\cdot\frac{36\pi}{q_{n-2}}$$

so that

$$\|\rho_n(U_{n-1}\oplus U_{n-2}) - U_n\oplus U_{n-1}\| \leqslant \frac{300\pi}{q_{n-2}}.$$

The same kind of argument shows that

$$\|\rho_n(V_{n-1}\oplus V_{n-2}) - V_n\oplus V_{n-1}\| \leqslant \frac{42\pi}{q_{n-1}} + \frac{7\pi}{q_{n-2}}.$$

But $q_n = a_nq_{n-1} + q_{n-2} > 2q_{n-2} \geqslant \ldots \geqslant 2^{\left[\frac{n-1}{2}\right]}$ implies the convergence of the series $\sum \frac{1}{q_n}$ which ends the proof. Q.E.D.

The preceding lemma shows that the operators $U_n\oplus U_{n-1}\in A_n \subset A$ and $V_n\oplus V_{n-1}\in A_n \subset A$ are norm-convergent to unitary elements U and respectively V of the AF-algebra A. Since $U_n^*V_nU_n = \exp\left(2\pi i\,\frac{p_n}{q_n}\right)V_n$ it follows that $U^*VU = \exp(2\pi i\theta)\,V$. Thus there is a unital $*$-monomorphism $\rho\colon A_\theta \to A$. We thus have proved the following theorem.

THEOREM. *Let $\theta\in(0,1)$ be an irrational number. There exists a $*$-monomorphism*

$$\rho\colon A_\theta \to A$$

where A is an AF-algebra defined by an inductive limit of finite-dimensional C^*-algebras for which the corresponding limit of K-groups is $\ldots \to \mathbf{Z}^2 \xrightarrow{\varphi_n} \mathbf{Z}^2 \to \to \ldots$ *where each* $\mathbf{Z}^2$ *is endowed with its natural ordering and*

$$\varphi_n = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

where $[a_1, a_2, \ldots, a_n, \ldots]$ *is the continuous fraction expansion of* θ.

By results of E. G. Effros and C. L. Shen [3] we have that $K_0(A)$ is isomorphic to the ordered group $\mathbf{Z} + \theta\mathbf{Z} \subset \mathbf{R}$. In fact, we shall need only the fact that there is a trace-state τ_1 on A such that the range of the induced homomorphism $K_0(A) \to \mathbf{R}$ is contained in $\mathbf{Z} + \theta\mathbf{Z}$, a fact which can be seen directly as follows. In order to define τ_1 it is sufficient to find positive numbers α_n, β_n corresponding to the values of τ_1 on minimal projections of $\mathcal{M}_{q_n}$ and $\mathcal{M}_{q_{n-1}}$, the two summands of A_n. These numbers must satisfy the relations: $a_n\alpha_n + \beta_n = \alpha_{n-1}$, $\beta_{n-1} = \alpha_n$ and $a_1\alpha_1 + \beta_1 = 1$. Thus it will be sufficient to determine the α_n's so that $a_1\alpha_1 + \alpha_2 = 1$ and $\alpha_{n+1} + a_n\alpha_n = \alpha_{n-1}$. Taking $\alpha_1 = \theta$ and $\alpha_2 = 1 - a_1\theta$ we can take $\alpha_n = \theta_1 \ldots \theta_n$ where $\theta_1 = \theta$ and $\theta_{j+1} = \dfrac{1}{\theta_j} - a_j$. Since $\alpha_1, \alpha_2 \in \mathbf{Z} + \theta\mathbf{Z}$ we shall have $\alpha_n \in \mathbf{Z} + \theta\mathbf{Z}$ for all $n \in \mathbf{N}$ and thus the range of the homomorphism $K_0(A) \to \mathbf{R}$ induced by τ_1 is contained in $\mathbf{Z} + \theta\mathbf{Z}$.

Remarking that $\tau_1 \circ \rho = \tau$ we infer that the range of the homomorphism $K_0(A_\theta) \to \mathbf{R}$ induced by τ is contained in $\mathbf{Z} + \theta\mathbf{Z}$. Thus, using Rieffel's result mentioned in the introduction we have:

COROLLARY 1. *Let* $\psi: K_0(A_\theta) \to \mathbf{R}$ *be the homomorphism induced by* τ. *Then* $\psi(K_0(A_\theta)) = \mathbf{Z} + \theta\mathbf{Z}$.

Let now $\theta_1, \theta_2 \in (0, 1)$ be irrational numbers and remark that $\mathbf{Z} + \theta_1\mathbf{Z} = \mathbf{Z} + \theta_2\mathbf{Z}$ as subsets of $\mathbf{R}$ if and only if $\theta_1 \in \{\theta_2, 1 - \theta_2\}$. This together with the isomorphism $A_{\theta_2} \simeq A_{1-\theta_2}$ gives:

COROLLARY 2. *Let* $\theta_1, \theta_2 \in (0, 1)$ *be irrational numbers. Then* A_{θ_1} *and* A_{θ_2} *are isomorphic if and only if* $\theta_1 \in \{\theta_2, 1 - \theta_2\}$.

Added in Proof

1) After this paper had been circulated as a preprint (INCREST-preprint no. 45/1979) several advances have been made concerning K-groups of crossed products (see the Added in Proof of our paper "Exact sequences for K-groups and *Ext*-groups of certain cross-product C^*-algebras", *J. Operator Theory* **4** (1980), 93–118).

2) The authors thank Professor A. M. Vershik for pointing out that the imbedding result of the present paper (used together with a non-commutative Weyl-von Neumann type theorem obtained by the second author) provides an affirmative answer to a problem raised by Professor

Vershik concerning approximability in norm of the pair of unitaries corresponding to the usual representation in $L^2(\mathrm{T})$ of A_θ, by pairs of unitaries contained in finite-dimensional C^*-algebras (see Zapiski nauchnih seminarov LOMI vol. 81, "99 unsolved problems of linear and complex analysis", pp. 77–81).

3) We would like to mention that our construction for the imbedding $A_\theta \hookrightarrow A$ is related to I. D. Berg's technique for the approximation of weighted shifts.

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M. PIMSNER and D. VOICULESCU
Department of Mathematics,
National Institute for Scientific and Technical Creation,
Bd. Păcii 220, 79622 Bucharest,
Romania

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