Commun. Math. Phys. 217, 181 – 201 (2001)

Communications in
Mathematical Physics

From Large N Matrices to the Noncommutative Torus

G. Landi[1,3], F. Lizzi[2,3], R. J. Szabo[4]

[1] Dipartimento di Scienze Matematiche, Università di Trieste, P.le Europa 1, 34127 Trieste, Italy. E-mail: landi@mathsun1.univ.trieste.it
[2] Dipartimento di Scienze Fisiche, Università di Napoli Federico II, Mostra d'Oltremare Pad. 20, 80125 Napoli, Italy. E-mail: fedele.lizzi@na.infn.it
[3] INFN, Sezione di Napoli, Napoli, Italy
[4] The Niels Bohr Institute, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark. E-mail: szabo@nbi.dk

Received: 24 December 1999 / Accepted: 7 October 2000

Abstract: We describe how and to what extent the noncommutative two-torus can be approximated by a tower of finite-dimensional matrix geometries. The approximation is carried out for both irrational and rational deformation parameters by embedding the C^*-algebra of the noncommutative torus into an approximately finite algebra. The construction is a rigorous derivation of the recent discretizations of noncommutative gauge theories using finite dimensional matrix models, and it shows precisely how the continuum limits of these models must be taken. We clarify various aspects of Morita equivalence using this formalism and describe some applications to noncommutative Yang–Mills theory.

1. Introduction

The relationship between large N matrix models and noncommutative geometry in string theory was suggested early on in studies of the low energy dynamics of D-branes, where it was observed [1] that a system of N coincident D-branes has collective coordinates which are described by mutually noncommuting $N \times N$ matrices. Various aspects of the large N limit of such systems have been important to the Matrix theory conjecture [2] and the representation of branes in terms of large N matrices [3]. The connection between finite dimensional matrix algebras and noncommutative Riemann surfaces is the basis for the fact that large N Matrix theory contains M2-branes. A more precise connection to noncommutative geometry came with the observation [4] that the most general solutions to the quotient conditions for toroidal compactification of the IKKT matrix model [5] are given by connections of vector bundles over a noncommutative torus. The resulting large N matrix model is noncommutative Yang–Mills theory which is dual to the low-energy dynamics of open strings ending on D-branes in the background of a constant Neveu–Schwarz two-form field [6].

The description of noncommutative tori and their gauge bundles as the large N limit of some sort of tower of finite-dimensional matrix geometries is therefore an important, yet elusive, problem. This correspondence was described at a very heuristic level in [7], while a definition of noncommutative gauge theory as the large N limit of a matrix model has been made more precise recently in [8,9]. In particular, in [9] it was shown how the standard projective modules [10,11] over the noncommutative two-torus can be discretized in terms of finite-dimensional matrix algebras. This immediately raises an apparent paradox. A standard result asserts that the noncommutative torus cannot be described by any approximately finite dimensional algebra. This means that it cannot be written explicitly as the large N limit of some sequence of finite dimensional matrix algebras. One way to understand this is in terms of K-theory. K-theory groups are stable under deformations of algebras, and those of the ordinary torus $\mathbf{T}^2$ are non-trivial. The deformation of the algebra of functions on $\mathbf{T}^2$ to the noncommutative torus therefore preserves this non-trivial K-theory structure. On the other hand, the K_1 group of any approximately finite dimensional algebra is trivial (see for instance [12]). In fact, it is precisely this K-theoretic stability which immediately implies that there is a canonical map between gauge bundles on ordinary $\mathbf{T}^2$ and gauge bundles on the noncommutative torus. This canonical map is constructed explicitly in [6].

However, this mathematical reasoning would seem to put very stringent restrictions on the allowed observables of field theories defined on the noncommutative torus. The generators of a noncommutative torus with a deformation parameter θ that is a rational number can be represented by finite dimensional (clock and shift) matrices. There is no such matrix description in the case that θ is irrational. However, an irrational (or rational) θ can always be represented as the limit of a sequence θ_n of rational numbers. From a physical standpoint, we would expect any correlation function C of a field theory on such noncommutative tori to be a continuous function of θ, so that $C(\theta) = \lim_n C(\theta_n)$. This means that there must be some sense in which observables of noncommutative Yang–Mills theory can be approximated as the large N limit of a sequence of those for finite dimensional matrix models. Such an approximation scheme is reminescent of fuzzy spaces [13], whereby the multiplication law of the algebra of functions is approximated by a particular matrix multiplication. Although the space of functions on a manifold is not an approximately finite dimensional algebra, its product is approximated arbitrarily well as $N \to \infty$. However, the algebras which are deformations of function algebras are somewhat distinct from fuzzy spaces which are typically finite dimensional [14], and the algebraic approximation in the case of the noncommutative torus must come about in a different way.

In this paper we will show precisely how to do this. The main point is that although the algebra of the noncommutative torus is not approximately finite, it can be realized as a subalgebra of an algebra which is built from a certain tower of finite dimensional matrix algebras [15]. As an important byproduct we solve what has been a problem for the physical interpretation of the deformation parameter of the algebra of the torus. The mathematical properties of the noncommutative torus depend crucially on whether or not the parameter θ is a rational number. Certain distinct values of θ are connected by Morita equivalence, and the set of equivalent θ's is dense on the real line. This is similar (and in some cases equivalent) to the phenomenon of T-duality in string theory [6,16]. Nevertheless, with a particular choice of background fields, θ is in principle an observable variable, and it would be wrong to expect that the fact that θ is rational or not could have measurable physical consequences. In what follows we will see how it is possible to approximate the algebra with irrational or rational θ by a sequence of finite dimensional matrix algebras. As an immediate corollary, the physical quantities that one calculates as the limit (which we show exists) are continuous functions of θ. In fact, we will show that all Morita equivalent noncommutative tori can be embedded into the same approximately finite algebra, so that the present construction shows that all noncommutative gauge theories can be approximated within a unifying framework. This description is therefore useful for analysing the phase structure of noncommutative Yang–Mills theory, as a function of θ, using matrix models. The results presented in the following give a very precise meaning to the definition of noncommutative Yang–Mills theory as the large N limit of a matrix model, and at the same time clarify in a rigorous manner the way that the field content, observables and correlators of the matrix model must be mapped to the continuum gauge theory. This is particularly important for numerical computations in which the interest is in determining quantities in noncommutative Yang–Mills theory in terms of those of large matrices at finite N. Such large N limits are also important for describing the dynamics of Matrix theory, whereby the $N \times N$ matrix geometries coincide with the parameter spaces of systems of N D0-branes.

This paper is organized as follows. In Sect. 2 we shall describe this construction, and discuss exactly in what sense the generators of any noncommutative torus can be approximated by large N matrices. In Sect. 3 we will then show that this procedure can be used to approximate correlation functions for field theories on the noncommutative torus in terms of expectation values constructed from matrices acting on a finite dimensional vector space. In Sect. 4 we show how to express geometries on the noncommutative torus, including gauge bundles, in terms of a tower of matrix geometries. Section 5 contains some concluding remarks.

2. AF-Algebras and the Noncommutative Torus

The algebra $\mathcal{A}_\theta$ of smooth functions on the "noncommutative two-torus" $\mathbf{T}^2_\theta$ is the unital $*$-algebra generated by two unitary elements U_1, U_2 with the relation

$$U_1 U_2 = \mathrm{e}^{2\pi i\theta}\, U_2 U_1. \tag{2.1}$$

A generic element $a \in \mathcal{A}_\theta$ is written as a convergent series of the form

$$a = \sum_{(m,n)\in\mathbb{Z}^2} a_{mn}\, (U_1)^m (U_2)^n, \tag{2.2}$$

where a_{mn} is a complex-valued Schwarz function on $\mathbb{Z}^2$, i.e. a sequence of complex numbers $\{a_{mn} \in \mathbb{C} \mid (m,n) \in \mathbb{Z}^2\}$ which decreases rapidly at "infinity". When the deformation parameter $\theta = M/N$ is a rational number, with M and N positive integers which we take to be relatively prime, the algebra $\mathcal{A}_{M/N}$ is intimately related to the algebra $C^\infty(\mathbf{T}^2)$ of smooth functions on the ordinary torus $\mathbf{T}^2$. Precisely, $\mathcal{A}_{M/N}$ is Morita equivalent to $C^\infty(\mathbf{T}^2)$, i.e., $\mathcal{A}_{M/N}$ is a twisted matrix bundle over $C^\infty(\mathbf{T}^2)$ of topological charge M whose fibers are $N \times N$ complex matrix algebras. Physically, this implies that noncommutative $U(1)$ Yang–Mills theory with rational deformation parameter $\theta = M/N$ is dual to a conventional $U(N)$ Yang–Mills theory with M units of 't Hooft flux.

The algebra $\mathcal{A}_{M/N}$ has a "huge" center $\mathcal{C}(\mathcal{A}_{M/N})$ which is generated by the elements $(U_1)^N$ and $(U_2)^N$. One identifies $\mathcal{C}(\mathcal{A}_{M/N})$ with the algebra $C^\infty(\mathbf{T}^2)$, while the appearence of finite dimensional matrix algebras can be seen as follows. With

$\omega = \mathrm{e}^{2\pi i M/N}$, one introduces the $N \times N$ clock and shift matrices

$$\widetilde{U}_1 = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{N-1} \end{pmatrix}, \qquad \widetilde{U}_2 = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix}. \tag{2.3}$$

These matrices are traceless (since $\sum_{k=0}^{N-1} \omega^k = 0$), they obey the relation (2.1), and they satisfy

$$\big(\widetilde{U}_1\big)^N = \big(\widetilde{U}_2\big)^N = \mathbb{I}_N. \tag{2.4}$$

Since M and N are relatively prime, the matrices (2.3) generate the finite dimensional algebra $\mathbb{M}_N(\mathbb{C})$ of $N \times N$ complex matrices [17].[1] Furthermore, there is a surjective algebra morphism

$$\pi : \mathcal{A}_{M/N} \to \mathbb{M}_N(\mathbb{C}) \tag{2.5}$$

given by

$$\pi\left(\sum_{(m,n)\in\mathbb{Z}^2} a_{mn}\, (U_1)^m (U_2)^n\right) = \sum_{(m,n)\in\mathbb{Z}^2} a_{mn}\, \big(\widetilde{U}_1\big)^m \big(\widetilde{U}_2\big)^n , \tag{2.6}$$

under which the whole center $\mathcal{C}(\mathcal{A}_{M/N})$ is mapped to $\mathbb{C}$.

When $\mathbb{M}_N(\mathbb{C})$ is thought of as the Lie algebra $gl(N, \mathbb{C})$, a basis is provided by the $N \times N$ matrices

$$\mathcal{T}_p^{(N)} = \frac{i}{2\pi}\,\frac{N}{M}\,\omega^{p_1 p_2/2}\,\big(\widetilde{U}_1\big)^{p_1}\big(\widetilde{U}_2\big)^{p_2} , \tag{2.7}$$

where $p_a \in \{-\frac{N-1}{2}, -\frac{N-3}{2}, \dots, \frac{N-1}{2}\}$. These matrices obey the commutation relations

$$\Big[\mathcal{T}_p^{(N)} , \mathcal{T}_q^{(N)}\Big] = \frac{N}{\pi M}\,\sin\left(\frac{\pi M}{N}\,(p_1 q_2 - p_2 q_1)\right)\mathcal{T}^{(N)}_{p+q\ (\mathrm{mod}\ N)} \tag{2.8}$$

which in the limit $N \to \infty$ with $M/N \to 0$ become

$$\Big[\mathcal{T}_p^{(\infty)} , \mathcal{T}_q^{(\infty)}\Big] = \Big(p_1 q_2 - p_2 q_1\Big)\,\mathcal{T}^{(\infty)}_{p+q}. \tag{2.9}$$

Equation (2.9) is recognized as the Poisson-Lie algebra of functions on $\mathbf{T}^2$ with respect to the usual Poisson bracket. In a unitary representation of the algebra (2.8), anti-Hermitian combinations of the traceless matrices $\mathcal{T}_p^{(N)}$ span the Lie algebra $su(N)$. This identifies the symplectomorphism algebra (2.9) of the torus with $su(\infty)$ [18] which is an example of a universal gauge symmetry algebra [7]. This identification has been exploited recently in [19] to study the perturbative renormalizability properties of noncommutative Yang–Mills theory. For finite N, $su(N)$ may be regarded as the Lie algebra of infinitesimal reparametrizations of the algebra described by (2.7) and (2.8). Given these connections,

[1] If M and N are not coprime then the generated algebra would be a proper subalgebra of $\mathbb{M}_N(\mathbb{C})$.

it follows that the noncommutative two-torus coincides with the parameter space of Matrix theory.

In what follows we shall be interested in taking the limit where both $N, M \to \infty$ with the ratio M/N approaching a fixed irrational or rational number. This is the type of limit considered in [9], and it yields the appropriate embeddings of matrix algebras into the infinite dimensional C^*-algebra which describes the noncommutative spacetime of D0-branes in Matrix theory [2]. For finite N, the matrix model consists of maps of a quantum Riemann surface (the noncommutative toroidal M2-brane) into a noncommutative transverse space. In the case where θ is an irrational number, the algebra (2.1) cannot be mapped to any subalgebra of $su(\infty)$. We would like to investigate how and to what extent the geometries for $\mathcal{A}_\theta$ can be approximated by towers of matrix geometries. Naively, one could think of considering the algebra $\mathcal{A}_\theta$ as the inductive limit of a sequence of finite dimensional $*$-algebras. This would be tantamount to (the closure of) $\mathcal{A}_\theta$ being an approximately finite dimensional C^*-algebra. As we mentioned in the previous section, this is not the case, as can be easily seen for any value of θ using cohomological arguments. The K-theory groups of $\mathbf{T}^2_\theta$ are $\mathrm{K}_n(\mathbf{T}^2_\theta) = \mathbb{Z} \oplus \mathbb{Z}$, $n = 0, 1$, just as for the ordinary torus $\mathbf{T}^2$. On the other hand, the group K_1 of any approximately finite algebra is necessarily trivial [12].

2.1. AF-algebras. In [15], Pimsner and Voiculescu have shown that there is the possibility to realize the C^*-algebra A_θ, which is the norm closure of the algebra of smooth functions $\mathcal{A}_\theta$, as a subalgebra of a larger, approximately finite dimensional C^*-algebra. In a classical sense, this would mean that an embedded submanifold of $\mathbf{T}^2_\theta$ is induced by the parameter space geometries. This is analogous to what happens in Matrix theory, whereby the noncommutative target space is realized as a "submanifold" of the matrix parameter space of N D0-branes. Before describing this embedding, we shall in this subsection briefly describe some general properties of the class of approximately finite algebras [20].

A unital C^*-algebra A is said to be approximately finite dimensional (AF for short) if there exists an increasing sequence

$$A_0 \overset{\rho_1}{\hookrightarrow} A_1 \overset{\rho_2}{\hookrightarrow} A_2 \overset{\rho_3}{\hookrightarrow} \cdots \overset{\rho_n}{\hookrightarrow} A_n \overset{\rho_{n+1}}{\hookrightarrow} \cdots \tag{2.10}$$

of finite dimensional C^*-subalgebras of A such that A is the norm closure of the union $\bigcup_n A_n$, $A = \overline{\bigcup_n A_n}$. The maps ρ_n are injective $*$-morphisms. Without loss of generality one may assume that each A_n contains the unit $\mathbb{I}$ of A and that the maps ρ_n are unital. The algebra A is the inductive limit of the inductive system of algebras $\{A_n, \rho_n\}_{n\in\mathbb{Z}^+}$ [12]. As a set, $\bigcup_n A_n$ is made of coherent sequences,

$$\bigcup_{n=0}^{\infty} A_n = \left\{ a = (a_n)_{n\in\mathbb{Z}^+},\ a_n \in A_n \;\middle|\; \exists N_0,\ a_n = \rho_n(a_{n-1})\ \forall n > N_0 \right\}. \tag{2.11}$$

The sequence $(\|a_n\|_{A_n})_{n\in\mathbb{Z}^+}$ is eventually decreasing since $\|a_{n+1}\| \le \|a_n\|$ (the maps ρ_n are norm decreasing) and is therefore convergent. The norm on A is given by

$$\|(a_n)_{n\in\mathbb{Z}^+}\| = \lim_{n\to\infty} \left\| a_n \right\|_{A_n}. \tag{2.12}$$

Since the maps ρ_n are injective, the expression (2.12) gives a true norm directly and not merely a semi-norm, and there is no need to quotient out the zero norm elements.

Since each subalgebra A_n is finite dimensional, it is a direct sum of matrix algebras,

$$A_n = \bigoplus_{k=1}^{k_n} \mathbb{M}_{d_k^{(n)}}(\mathbb{C}), \tag{2.13}$$

where $\mathbb{M}_d(\mathbb{C})$ is the algebra of $d \times d$ matrices with complex entries and endowed with its usual Hermitian conjugation and operator norm. On the other hand, given a unital embedding $A_1 \hookrightarrow A_2$ of the algebras $A_1 = \bigoplus_{j=1}^{n_1} \mathbb{M}_{d_j^{(1)}}(\mathbb{C})$ and $A_2 = \bigoplus_{k=1}^{n_2} \mathbb{M}_{d_k^{(2)}}(\mathbb{C})$, one can always choose suitable bases in A_1 and A_2 in such a way as to identify A_1 with a subalgebra of A_2 having the form

$$A_1 \cong \bigoplus_{k=1}^{n_2} \bigoplus_{j=1}^{n_1} N_{kj}\, \mathbb{M}_{d_j^{(1)}}(\mathbb{C}). \tag{2.14}$$

Here, for any two non-negative integers p, q, the symbol $p\,\mathbb{M}_q(\mathbb{C})$ denotes the algebra

$$p\,\mathbb{M}_q(\mathbb{C}) \cong \mathbb{M}_q(\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{I}_p, \tag{2.15}$$

and one identifies $\bigoplus_{j=1}^{n_1} N_{kj}\, \mathbb{M}_{d_j^{(1)}}(\mathbb{C})$ with a subalgebra of $\mathbb{M}_{d_k^{(2)}}(\mathbb{C})$. The non-negative integers N_{kj} satisfy the condition

$$\sum_{j=1}^{n_1} N_{kj}\, d_j^{(1)} = d_k^{(2)}. \tag{2.16}$$

One says that the algebra $\mathbb{M}_{d_j^{(1)}}(\mathbb{C})$ is partially embedded in $\mathbb{M}_{d_k^{(2)}}(\mathbb{C})$ with multiplicity N_{kj}. A useful way of representing the algebras A_1, A_2 and the embedding $A_1 \hookrightarrow A_2$ is by means of a diagram, the so-called Bratteli diagram [20], which can be constructed out of the dimensions $d_j^{(1)},\ j = 1, \dots, n_1$ and $d_k^{(2)},\ k = 1, \dots, n_2$ of the diagonal blocks of the two algebras, and out of the numbers N_{kj} that describe the partial embeddings. One draws two horizontal rows of vertices, the top (bottom resp.) one representing A_1 (A_2 resp.) and consisting of n_1 (n_2 resp.) vertices, one for each block which are labeled by the corresponding dimensions $d_1^{(1)}, \dots, d_{n_1}^{(1)}$ ($d_1^{(2)}, \dots, d_{n_2}^{(2)}$ resp.). Then, for each $j = 1, \dots, n_1$ and $k = 1, \dots, n_2$, one has a relation $d_j^{(1)} \searrow^{N_{kj}} d_k^{(2)}$ to denote the fact that $\mathbb{M}_{d_j^{(1)}}(\mathbb{C})$ is partially embedded in $\mathbb{M}_{d_k^{(2)}}(\mathbb{C})$ with multiplicity N_{kj}.

For any AF-algebra A one repeats this procedure for each level, and in this way one obtains a semi-infinite diagram which completely defines A up to isomorphism. This diagram depends not only on the collection of A_n's but also on the particular sequence $\{A_n, \rho_n\}_{n\in\mathbb{Z}^+}$ which generates A. However, one can obtain an algorithm which allows one to construct from a given diagram all diagrams which define AF-algebras that are isomorphic to the original one [20]. The problem of identifying the limit algebra or of determining whether or not two such limits are isomorphic can be very subtle. In [21] an invariant for AF-algebras has been devised in terms of the corresponding K-theory which completely distinguishes among them. Note that the isomorphism class of an AF-algebra $\overline{\bigcup_n A_n}$ depends not only on the collection of algebras A_n but also on the way that they are embedded into one another.

2.2. Embedding the noncommutative torus in an AF-algebra: Irrational case. We are now ready to describe the realization [15] of the algebra A_θ as a subalgebra of a larger, AF algebra A_∞ which is determined by the K-theory of A_θ (to be precise $\mathrm{K}_0(A_\theta)$). While in [15] the values of θ are taken to be irrational and to lie in the interval (0, 1), we shall repeat the construction for an arbitrary real-valued deformation parameter. In this subsection we shall take θ to be irrational. The case of rational θ will be described in the next subsection.

It is known [22] that any $\theta \in \mathbb{R} - \mathbb{Q}$ has a unique representation as a simple continued fraction expansion

$$\theta = \lim_{n\to\infty} \theta_n \tag{2.17}$$

in terms of positive integers $c_k > 0$ ($k \geq 1$) and $c_0 \in \mathbb{Z}$. The n^{th} convergents θ_n of the expansion are given by

$$\theta_n \equiv \frac{p_n}{q_n} = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots\; c_{n-1} + \cfrac{1}{c_n}}}}. \tag{2.18}$$

One also writes this as

$$\theta = [c_0, c_1, c_2, \dots\ ]. \tag{2.19}$$

The relatively prime integers p_n and q_n may be computed recursively using the formulae

$$\begin{aligned} p_n &= c_n p_{n-1} + p_{n-2}, \quad p_0 = c_0, \quad p_1 = c_0 c_1 + 1, \\ q_n &= c_n q_{n-1} + q_{n-2}, \quad q_0 = 1, \quad\ \ q_1 = c_1 \end{aligned} \tag{2.20}$$

for $n \geq 2$. Note that all q_n's are strictly positive, $q_n > 0$, while $p_n \in \mathbb{Z}$, and that both q_n and $|p_n|$ are strictly increasing sequences which therefore diverge as $n \to \infty$.

For each positive integer n, we let $\mathbb{M}_{q_n}(\mathbb{C})$ denote the finite dimensional C^*-algebra of $q_n \times q_n$ complex matrices acting on the finite dimensional Hilbert space $\mathbb{C}^{q_n}$ which is endowed with its usual inner product and its canonical orthonormal basis $\boldsymbol{e}_j^{(n)}$, $1 \leq j \leq q_n$. Then, for any integer n, consider the semi-simple algebra

$$A_n = \mathbb{M}_{q_n}(\mathbb{C}) \oplus \mathbb{M}_{q_{n-1}}(\mathbb{C}) \tag{2.21}$$

and introduce the embeddings $A_{n-1} \overset{\rho_n}{\hookrightarrow} A_n$ defined by[2]

$$\begin{pmatrix} \mathcal{M} & \\ & \mathcal{N} \end{pmatrix} \overset{\rho_n}{\longmapsto} \begin{pmatrix} \left.\begin{matrix} \mathcal{M} & & \\ & \ddots & \\ & & \mathcal{M} \end{matrix}\right\} c_n & & \\ & \mathcal{N} & \\ & & \mathcal{M} \end{pmatrix}, \tag{2.22}$$

[2] In [15], in order to explicitly construct the embedding of the noncommutative torus algebra in the limit AF-algebra, the embeddings (2.22) are conjugated with suitable (and rather involved) unitary operators

$$W_n : \underbrace{\mathbb{C}^{q_{n-1}} \oplus \cdots \oplus \mathbb{C}^{q_{n-1}}}_{c_n \text{ times}} \longrightarrow \mathbb{C}^{q_n}.$$

Since the two embeddings are the same up to an inner automorphism, the limit algebra is the same [20].

where $\mathcal{M}$ and $\mathcal{N}$ are $q_{n-1} \times q_{n-1}$ and $q_{n-2} \times q_{n-2}$ matrices, respectively, and we have used (2.20). The norm closure of the inductive limit

$$A_\infty = \overline{\bigcup_{n=0}^{\infty} A_n} \tag{2.23}$$

is the AF-algebra that we are looking for. As mentioned in the previous subsection, the elements of A_∞ are coherent sequences $\{\mathcal{G}_n\}_{n\in\mathbb{Z}^+}$, $\mathcal{G}_n \in A_n$, with $\mathcal{G}_n = \rho_n(\mathcal{G}_{n-1})$ for n sufficiently large, or limits of coherent sequences. It is useful to visualize them as infinite matrices and we shall also loosely write $A_\infty \cong \mathbb{M}_\infty(\mathbb{C})$.

From the discussion of the previous subsection it follows that the embeddings $A_{n-1} \overset{\rho_n}{\hookrightarrow} A_n$ are completely determined by the collection of partial embeddings $\{c_n\}$. The corresponding Bratteli diagram is shown in Fig. 1. Associated with them we have positive maps $\varphi_n : \mathbb{Z}^2 \to \mathbb{Z}^2$ defined by

$$\begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix} = \varphi_n \begin{pmatrix} q_{n-1} \\ q_{n-2} \end{pmatrix}, \qquad \varphi_n = \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.24}$$

As a consequence, the group $\mathrm{K}_0(A_\infty)$ can be obtained as the inductive limit of the inductive system $\{\varphi_n : \mathrm{K}_0(A_{n-1}) \to \mathrm{K}_0(A_n)\}_{n\in\mathbb{Z}^+}$ of ordered groups. Since $\mathrm{K}_0(A_n) = \mathbb{Z} \oplus \mathbb{Z}$ (with the canonical ordering $\mathbb{Z}^+ \oplus \mathbb{Z}^+$) it follows that [23]

$$\mathrm{K}_0(A_\infty) = \mathbb{Z} + \theta\mathbb{Z} \tag{2.25}$$

with ordering defined by taking the cone of non-negative elements to be

$$\mathrm{K}_0^+(A_\infty) = \Big\{(z, w) \in \mathbb{Z}^2 \;\Big|\; z + \theta w \geq 0\Big\}. \tag{2.26}$$

This is a total ordering since for all pairs of integers (z, w), one has either $z + \theta w \geq 0$ or $z + \theta w < 0$. We shall comment more on the K-theory group (2.25) later on. Furthermore, these K-theoretic properties will enable us in Sect. 4 to map a gauge bundle over a matrix algebra to a gauge bundle over the noncommutative torus.

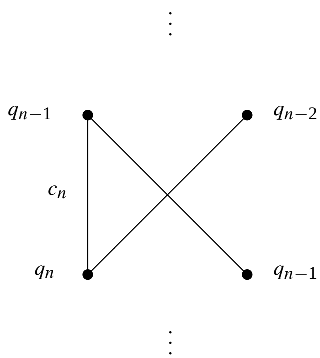

Fig. 1. Bratteli diagram for the algebra A_∞ in the case of irrational θ. The labels of the vertices denote the dimensions of the corresponding matrix algebras. The labels of the links denote the partial embeddings (not written when equal to unity)

At each finite level labelled by the integer n, let A_{θ_n} be the algebra of the noncommutative two-torus with rational deformation parameter $\theta_n = p_n/q_n$ given in (2.18), and generators $U_a^{(n)}$, $a = 1, 2$ obeying the relation

$$U_1^{(n)} U_2^{(n)} = \mathrm{e}^{2\pi i p_n/q_n}\, U_2^{(n)} U_1^{(n)}. \tag{2.27}$$

From (2.5) and (2.6) it follows that there exists a surjective algebra homomorphism

$$\pi : A_{\theta_n} \to \mathbb{M}_{q_n}(\mathbb{C}), \quad \pi\left(U_a^{(n)}\right) \equiv \widetilde{U}_a^{(n)}, \quad a = 1, 2 \tag{2.28}$$

and for $\widetilde{U}_1^{(n)}$ and $\widetilde{U}_2^{(n)}$ we may take the $q_n \times q_n$ clock and cyclic shift matrices, respectively,

$$\left[\widetilde{U}_1^{(n)}\right]_{kj} = \mathrm{e}^{2\pi i (j-1) p_n/q_n}\, \delta_{kj}, \quad \left[\widetilde{U}_2^{(n)}\right]_{kj} = \delta_{k,j-1}, \quad k, j = 1, \ldots, q_n \ (\mathrm{mod}\, q_n), \tag{2.29}$$

which also obey a relation like (2.27),

$$\widetilde{U}_1^{(n)} \widetilde{U}_2^{(n)} = \mathrm{e}^{2\pi i p_n/q_n}\, \widetilde{U}_2^{(n)} \widetilde{U}_1^{(n)}. \tag{2.30}$$

Thus, within each finite dimensional matrix algebra A_n there is the subalgebra $\pi(A_{\theta_n}) \oplus \pi(A_{\theta_{n-1}})$ which is represented by clock and shift matrices. The main result of Ref. [15] is the statement that the algebra $\pi(A_{\theta_n}) \oplus \pi(A_{\theta_{n-1}})$ can be taken to be a finite dimensional approximation of the algebra A_θ of the noncommutative torus in the following sense. First of all, notice that $\rho_n(\widetilde{U}_a^{(n-1)} \oplus \widetilde{U}_a^{(n-2)}) \neq \widetilde{U}_a^{(n)} \oplus \widetilde{U}_a^{(n-1)}$. Then, we have

Proposition 1 (Pimsner–Voiculescu**).**

$$\lim_{n\to\infty} \left\| \rho_n \left(\widetilde{U}_a^{(n-1)} \oplus \widetilde{U}_a^{(n-2)} \right) - \widetilde{U}_a^{(n)} \oplus \widetilde{U}_a^{(n-1)} \right\|_{A_n} = 0, \quad a = 1, 2.$$

Proposition 1 can be proven similarly to Proposition 3 below, and will therefore be omitted. It implies that there exist unitary operators $U_a \in A_\infty$, $a = 1, 2$, which are not themselves coherent sequences, but which can be written as a limit of such a sequence with respect to the operator norm of A_∞. Because of (2.17), (2.18) and (2.30), the operators U_a so defined satisfy (2.1) and therefore generate the subalgebra $A_\theta \subset A_\infty$. Thus, there exists a unital injective $*$-morphism $\rho : A_\theta \to A_\infty$.[3] This also means that at sufficiently large level n in the AF-algebra A_∞, the generators of the algebra (2.30) may be well approximated by the images under the injection ρ_n of the corresponding matrices generating $A_{\theta_{n-1}}$. It is in this sense that the elements of the algebra A_θ may be approximated by sufficiently large finite dimensional matrices. In what follows we shall show how to use this approximation to describe aspects of field theories over the noncommutative torus $\mathbf{T}_\theta^2$.

An important consequence of these results is the fact that Morita equivalent noncommutative tori can be embedded in the same AF-algebra A_∞. From (2.25) and (2.26) we know that $\mathrm{K}_0(A_\infty) = \mathbb{Z} + \theta\mathbb{Z}$ as an ordered group. On the other hand, it is known

[3] The canonical representation of A_θ is on the Hilbert space $L^2(\mathbf{T}^2)$, which by Fourier expansion coincides with $\ell^2(\mathbb{Z}^2)$.

[23] that $\mathbb{Z}+\theta\mathbb{Z}$ and $\mathbb{Z}+\theta'\mathbb{Z}$ are order isomorphic if and only if there is an element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,\mathbb{Z})$ such that

$$\theta' = \frac{a\theta+b}{c\theta+d}. \tag{2.31}$$

From the point of view of continued fraction expansions, if $\theta = [c_0, c_1, c_2, \dots]$ and $\theta' = [c'_0, c'_1, c'_2, \dots]$, the relation (2.31) is the statement that the two expansions have the same tails, i.e. that $c_n = c'_{n+m}$ for some integer m and for n sufficiently large [22]. But (2.31) is just the Morita equivalence relation between A_θ and $A_{\theta'}$ [24]. Thus, on the one hand we rediscover the known fact that Morita equivalent tori have the same K_0 group,[4] but we can also infer that Morita equivalent algebras can be embedded in the same (up to isomorphism) AF-algebra A_∞. Morita equivalent algebras can be embedded in the same A_∞ because their sequences of embeddings are the same up to a finite number of terms. In Sect. 4 this will be the key property which allows the construction of projective modules within the same approximation, and the physical consequences will be that dual noncommutative Yang–Mills theories all lie within the same AF-algebra A_∞.

Let us now describe the infinite dimensional Hilbert space $\mathcal{H}_\infty$ on which A_∞ is represented as (bounded) operators. It is similarly defined by an inductive limit determined by the Bratteli diagram of Fig. 1. For any integer n, consider the finite dimensional Hilbert space

$$\mathcal{H}_n = \mathbb{C}^{q_n} \oplus \mathbb{C}^{q_{n-1}} \tag{2.32}$$

on which the algebra A_n in (2.21) naturally acts. Next, consider the embeddings $\mathcal{H}_{n-1} \overset{\tilde\rho_n}{\hookrightarrow} \mathcal{H}_n$ defined by

$$\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix} \overset{\tilde\rho_n}{\longmapsto} \begin{pmatrix} \left.\begin{matrix} \frac{\boldsymbol{v}}{\sqrt{1+c_n}} \\ \vdots \\ \frac{\boldsymbol{v}}{\sqrt{1+c_n}} \end{matrix}\right\} c_n \\ \boldsymbol{w} \\ \frac{\boldsymbol{v}}{\sqrt{1+c_n}} \end{pmatrix}, \tag{2.33}$$

where $\boldsymbol{v} = \sum_{j=1}^{q_n} v^j \boldsymbol{e}_j^{(n)} \in \mathbb{C}^{q_n}$ and $\boldsymbol{w} = \sum_{j=1}^{q_{n-1}} w^j \boldsymbol{e}_j^{(n-1)} \in \mathbb{C}^{q_{n-1}}$. Then

$$\mathcal{H}_\infty = \overline{\bigcup_{n=0}^{\infty} \mathcal{H}_n}. \tag{2.34}$$

The normalization factors $(1+c_n)^{-1/2}$ in (2.33) are inserted so that the linear transformations $\tilde\rho_n$ are isometries,

$$\Big\langle \tilde\rho_n(\boldsymbol{v}\oplus\boldsymbol{w}),\ \tilde\rho_n(\boldsymbol{v}'\oplus\boldsymbol{w}')\Big\rangle_{\mathcal{H}_n} = \Big\langle \boldsymbol{v}\oplus\boldsymbol{w},\ \boldsymbol{v}'\oplus\boldsymbol{w}'\Big\rangle_{\mathcal{H}_{n-1}}. \tag{2.35}$$

[4] It is a general fact that Morita equivalent algebras have the same K-theory.

This ensures that the vectors of $\mathcal{H}_\infty$, which are built from the coherent sequences of $\bigcup_n \mathcal{H}_n$, are indeed convergent. Note that the elements of a coherent sequence are related inductively at each level by $\boldsymbol{v}_n \oplus \boldsymbol{w}_n = \tilde{\rho}_n(\boldsymbol{v}_{n-1} \oplus \boldsymbol{w}_{n-1})$ for n sufficiently large, or

$$\boldsymbol{v}_n = \underbrace{\frac{\boldsymbol{v}_{n-1}}{\sqrt{1+c_n}} \oplus \cdots \oplus \frac{\boldsymbol{v}_{n-1}}{\sqrt{1+c_n}}}_{c_n \text{ times}} \oplus \boldsymbol{w}_{n-1}, \quad \boldsymbol{w}_n = \frac{\boldsymbol{v}_{n-1}}{\sqrt{1+c_n}}. \tag{2.36}$$

The inner product in $\mathcal{H}_\infty$ is given by

$$\Big\langle (\psi'_n)_{n\in\mathbb{Z}^+}\,,\ (\psi_m)_{m\in\mathbb{Z}^+} \Big\rangle = \lim_{n\to\infty} \Big\langle \psi'_n\,,\ \psi_n \Big\rangle_{\mathcal{H}_n}. \tag{2.37}$$

In the same spirit by which we think of elements of A_∞ as infinite matrices, we also visualize elements of $\mathcal{H}_\infty$ as square summable complex sequences and write $\mathcal{H}_\infty \cong \ell^2\mathbb{Z}^+$.

2.3. Embedding the noncommutative torus in an AF-algebra: Rational case. Everything we have said in the previous subsection is true for irrational θ, but in many instances one is still interested in the case of rational deformation parameters. Even though Morita equivalence implies that the algebra A_θ is then equivalent in a certain sense to the algebra of functions on the ordinary torus $\mathbf{T}^2$, the physical theories built on the two algebras can have different characteristics (analogously to the case of T-duality between different brane worldvolume field theories). Indeed, physical correlation functions should not have a discontinuous behaviour between rational and irrational deformation parameters. Furthermore, as shown in [25], the noncommutative Yang–Mills description is the physically significant one in the infrared regime as a local field theory of the light degrees of freedom, even though this theory is equivalent by duality to ordinary Yang–Mills theory.

When θ is rational one can repeat, to some extent, the constructions of the previous subsection, but one needs to excercise some care due to the occurence of continued fraction expansions which are not simple, i.e. some c_n's in the expansion vanish. In this case, although the second equality in (2.18) does not make sense if $c_n = 0$, one can nonetheless define the n-convergent θ_n by the first equality in (2.18), i.e. $\theta_n = p_n/q_n$, with p_n and q_n defined recursively by the formulae (2.20) (recall that $q_n > 0$ always). Thus, we let $\theta = p/q$ with p, q relatively prime. The *simple* continued fraction expansion of θ, which is unique, will terminate at some level n_0, so that

$$\theta = \frac{p}{q} = \Big[c_0, c_1, \ldots, c_{n_0}\Big]. \tag{2.38}$$

However, we may still approximate θ by an infinite but *not simple* continued fraction expansion in the following manner. First, above the level n_0, we take all even c's to vanish,

$$c_{n_0+2n} = 0, \quad n \geq 0. \tag{2.39}$$

Consequently, from (2.20) we get

$$p_{n_0+2n} = p, \quad q_{n_0+2n} = q; \quad n \geq 0 \tag{2.40}$$

so that

$$\theta_{n_0+2n} = \frac{p}{q}, \quad n \geq 0. \tag{2.41}$$

As for the odd c's (above the level n_0), we shall not specify c_{n_0+1} at the moment, while we take

$$c_{n_0+2n+1} = 1, \quad n > 0. \tag{2.42}$$

From (2.20) we get

$$p_{n_0+2n+1} = np + p_{n_0+1}, \quad q_{n_0+2n+1} = nq + q_{n_0+1}; \quad n \geq 0 \tag{2.43}$$

so that

$$\theta_{n_0+2n+1} = \frac{np + p_{n_0+1}}{nq + q_{n_0+1}} \overset{n\to\infty}{\longrightarrow} \frac{p}{q}. \tag{2.44}$$

Thus, we can write the rational number p/q as the infinite but not simple continued fraction expansion

$$\frac{p}{q} = \Big[c_0, c_1, \dots, c_{n_0}, c_{n_0+1}, 0, 1, 0, 1, \dots\Big]. \tag{2.45}$$

If necessary, we shall use the arbitrariness in c_{n_0+1} to fix p_{n_0+1} and q_{n_0+1} in such a way that p_{n_0+2n+1} and q_{n_0+2n+1} are relatively prime integers. In this way we obtain infinite, strictly increasing sequences of relatively prime integers q_{n_0+2n+1} and $|p_{n_0+2n+1}|$, and the constructions and proofs of the previous subsection can be adapted to the present situation.

We are now ready to construct the AF-algebra A_∞ in which to embed the noncommutative torus with rational deformation parameter. Note that, generally, the isomorphism class of an AF-algebra is completely characterized by the infinite tail of its Bratteli diagram, which for the present case is depicted in Fig. 2a. A comparison with Fig. 1 for the irrational case shows that the algebra for rational θ is of the same kind, with the additional rule that for vanishing c's in the fractional expansion there is no link in the Bratteli diagram. From Fig. 2a we see that by going from an odd level to the next even one, one simply exchanges the factors in the decomposition, and thus it is better to 'glue' an odd level to the next even one. This produces the Bratteli diagram in Fig. 2b, which we stress describes the very same AF-algebra A_∞. There we have defined

$$\tilde{q}_n = q_{n_0+2n+1}, \quad n \geq 0. \tag{2.46}$$

The finite dimensional algebras at level n are then

$$B_n = \mathbb{M}_{\tilde{q}_n}(\mathbb{C}) \oplus \mathbb{M}_q(\mathbb{C}) \tag{2.47}$$

with embeddings $B_{n-1} \overset{\rho_n}{\hookrightarrow} B_n$ given by

$$\begin{pmatrix} \mathcal{M} & \\ & \mathcal{N} \end{pmatrix} \overset{\rho_n}{\longmapsto} \begin{pmatrix} \mathcal{M} & & \\ & \mathcal{N} & \\ & & \mathcal{N} \end{pmatrix}, \tag{2.48}$$

where $\mathcal{M}$ and $\mathcal{N}$ are $\tilde{q}_{n-1} \times \tilde{q}_{n-1}$ and $q \times q$ matrices, respectively. The norm closure of the inductive limit (2.47,2.48) is the desired AF-algebra A_∞. Note that, aside from

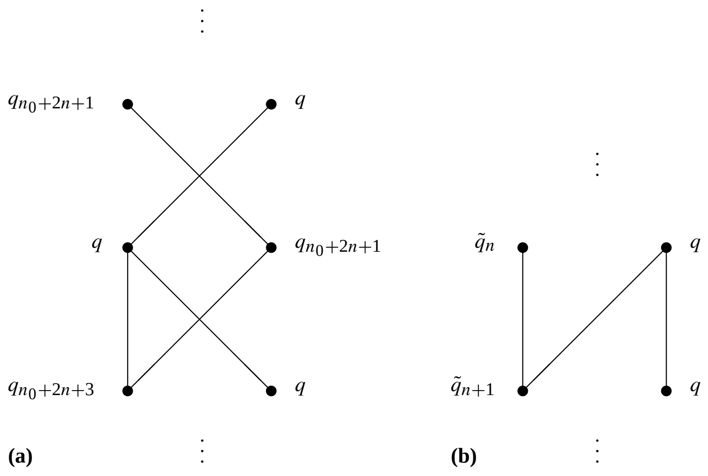

Fig. 2a,b. Equivalent Brattelli diagrams for the algebra A_∞ in the case of rational θ. The labels of the vertices denote the dimensions of the corresponding matrix algebras. All partial embeddings are equal to unity

the fact that it contributes to the increase of dimension in the first factor of B_n, the constant part $\mathbb{M}_q(\mathbb{C})$ is required at each level for K-theoretic reasons. The positive maps $\varphi_n : \mathbb{Z}^2 \to \mathbb{Z}^2$ associated with the embeddings (2.48) are now given by

$$\begin{pmatrix} \tilde{q}_n \\ q \end{pmatrix} = \varphi_n \begin{pmatrix} \tilde{q}_{n-1} \\ q \end{pmatrix}, \quad \varphi_n \equiv \varphi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.49}$$

As a consequence one finds

$$\mathrm{K}_0(A_\infty) = \mathbb{Z} \oplus \mathbb{Z} \tag{2.50}$$

with the cone of non-negative elements, which defines the ordering, given by

$$\begin{aligned} \mathrm{K}_0^+(A_\infty) &= \bigcup_{r=1}^{\infty} \varphi^{-r} \left(\mathbb{Z}^+ \oplus \mathbb{Z}^+\right) \\ &= \left\{(a,b) \in \mathbb{Z}^2 \;\middle|\; b > 0\right\} \cup \left\{(a,0) \in \mathbb{Z}^2 \;\middle|\; a \geq 0\right\}. \end{aligned} \tag{2.51}$$

In analogy with (2.46) we also define

$$\tilde{p}_n = p_{n_0+2n+1}, \quad n \geq 0 \tag{2.52}$$

and

$$\tilde{\theta}_n = \theta_{n_0+2n+1} = \frac{p_{n_0+2n+1}}{q_{n_0+2n+1}}, \quad n \geq 0. \tag{2.53}$$

Then, exactly as it happens for the irrational situation, within each finite dimensional matrix algebra B_n there is the subalgebra $\pi(A_{\tilde{\theta}_n}) \oplus \pi(A_{p/q})$ with $A_{\tilde{\theta}_n}$ and $A_{p/q}$ rational noncommutative tori and π the representation in finite dimensional matrices as given in (2.5), (2.6) and (2.28), (2.29), i.e., in terms of clock and shift matrices. In contrast to the irrational case, however, it now follows from the form of the second factor in the finite dimensional algebras that $\rho_n\big(\mathbf{0}_{\tilde{q}_{n-1}} \oplus \pi(A_{p/q})\big) = \mathbf{0}_{\tilde{q}_n} \oplus \pi(A_{p/q})$, while it is still true that $\rho_n\big(\pi(A_{\tilde{\theta}_{n-1}}) \oplus \mathbf{0}_q\big) \neq \pi\big(A_{\tilde{\theta}_{n-1}}\big) \oplus \mathbf{0}_q$. Consequently we have an analogue of Proposition 1 and the statement that the algebra $\pi(A_{\tilde{\theta}_n}) \oplus \pi(A_{p/q})$ can be taken to be a finite dimensional approximation of the algebra A_θ of the noncommutative torus with rational deformation parameter $\theta = p/q$. Finally, the infinite dimensional Hilbert space $\mathcal{H}_\infty$ on which A_∞ is represented is given at level n by the finite dimensional vector space

$$\mathcal{H}_n = \mathbb{C}^{\tilde{q}_n} \oplus \mathbb{C}^q \tag{2.54}$$

on which the algebra B_n in (2.47) naturally acts. The embeddings $\mathcal{H}_{n-1} \overset{\tilde{\rho}_n}{\hookrightarrow} \mathcal{H}_n$ can be read off from the Bratteli diagram in Fig. 2b and are given by

$$\tilde{\rho}_n(\boldsymbol{v}_{n-1} \oplus \boldsymbol{w}) = \boldsymbol{v}_n \oplus \boldsymbol{w}, \quad \boldsymbol{v}_n = \frac{1}{\sqrt{2}}\big(\boldsymbol{v}_{n-1} \oplus \boldsymbol{w}\big). \tag{2.55}$$

3. Approximating Correlation Functions

Consider an operator $\mathcal{G} \in A_\theta$ and states $\psi', \psi \in \mathcal{H}_\infty$. The element $\mathcal{G}$ is a particular combination of the generators U_a, $a = 1, 2$, of the noncommutative torus and the vectors ψ', ψ may be represented by particular coherent sequences $\{\psi'_n\}_{n\in\mathbb{Z}^+}$, $\{\psi_m\}_{m\in\mathbb{Z}^+}$ with $\psi'_n, \psi_n \in \mathcal{H}_n$. We are interested in evaluating the correlation function

$$C(\theta) = \langle \psi' \,,\, \mathcal{G}\psi \rangle, \tag{3.1}$$

where, for simplicity, we indicate only the dependence of the correlator on the deformation parameter of the algebra. According to Proposition 1 (and its counterpart for the rational case), there is a corresponding sequence of operators $\mathcal{G}_n \in \pi(A_{\theta_n}) \oplus \pi(A_{\theta_{n-1}})$, obtained by replacing the U_a's by $\widetilde{U}_a^{(n)} \oplus \widetilde{U}_a^{(n-1)}$ everywhere, which approximate $\mathcal{G}$ in the sense that $\lim_n \|\mathcal{G}_n - \mathcal{G}\| = 0$. Using this sequence we can also consider the correlation functions

$$C_n(\theta_n) = \langle \psi'_n \,,\, \mathcal{G}_n\psi_n \rangle_{\mathcal{H}_n}. \tag{3.2}$$

We wish to show that the correlators (3.2) for sufficiently large n give a "good" approximation to the correlation function (3.1), i.e., $C(\theta) = \lim_n C_n(\theta_n)$. This will be true if, as one moves from one level to the next in the coherent sequence, the corresponding expectation values of the operator $\mathcal{G}_{n+1}$ are approximated by the functions (3.2). This property will follow immediately from the following

Proposition 2. *Given any two sequences of vectors* $\psi'_{n-1}, \psi_{n-1} \in \mathcal{H}_{n-1}$, *define*

$$\mathcal{U}_a^{(n)} \equiv \Big\langle \psi'_{n-1},\, \Big(\widetilde{U}_a^{(n-1)} \oplus \widetilde{U}_a^{(n-2)}\Big)\,\psi_{n-1}\Big\rangle_{\mathcal{H}_{n-1}} - \Big\langle \tilde{\rho}_n(\psi'_{n-1})\,,\, \Big(\widetilde{U}_a^{(n)} \oplus \widetilde{U}_a^{(n-1)}\Big) \circ \tilde{\rho}_n(\psi_{n-1})\Big\rangle_{\mathcal{H}_n} \tag{3.3}$$

for $a = 1, 2$. *Then*

$$\lim_{n\to\infty} \mathcal{U}_a^{(n)} = 0.$$

Proof. We will give the proof for the case of irrational θ. The proof for the rational case is a straightforward modification of the normalizations of the immersions. Let $\psi_{n-1} = \boldsymbol{v}_{n-1} \oplus \boldsymbol{w}_{n-1}$ and $\psi'_{n-1} = \boldsymbol{v}'_{n-1} \oplus \boldsymbol{w}'_{n-1}$, with $\boldsymbol{v}_{n-1}, \boldsymbol{v}'_{n-1} \in \mathbb{C}^{q_{n-1}}$ and $\boldsymbol{w}_{n-1}, \boldsymbol{w}'_{n-1} \in \mathbb{C}^{q_{n-2}}$. The quantity (3.3) for $a = 1$ can be calculated to be

$$\begin{aligned} \mathcal{U}_1^{(n)} = & \sum_{j=1}^{q_{n-2}} \overline{w}_{n-1}^{j} \, w_{n-1}^{\prime j} \left(\mathrm{e}^{2\pi i \theta_{n-2}(j-1)} - \mathrm{e}^{2\pi i \theta_n (j-1+c_n q_{n-1})} \right) \\ & + \frac{1}{1+c_n} \sum_{k=0}^{c_n-1} \sum_{j=1}^{q_{n-1}} \overline{v}_{n-1}^{j} \, v_{n-1}^{\prime j} \left(\mathrm{e}^{2\pi i \theta_{n-1}(j-1)} - \mathrm{e}^{2\pi i \theta_n (j-1+k q_{n-1})} \right) . \end{aligned} \tag{3.4}$$

In the first sum in (3.4), we add and subtract $\mathrm{e}^{2\pi i \theta_{n-1}(j-1)}$ to each of the differences of exponentials there. From (2.17) it follows that the differences

$$\left| \mathrm{e}^{2\pi i \theta_{n-2}(j-1)} - \mathrm{e}^{2\pi i \theta_{n-1}(j-1)} \right| \tag{3.5}$$

each vanish in the limit $n \to \infty$. For the remaining differences

$$\left| \mathrm{e}^{2\pi i \theta_{n-1}(j-1)} - \mathrm{e}^{2\pi i \theta_n (j-1+c_n q_{n-1})} \right|, \tag{3.6}$$

we use the inequality [15]

$$\begin{aligned} \left| \mathrm{e}^{2\pi i \theta_{n-1} l} - \mathrm{e}^{2\pi i \theta_n (l+m q_{n-1})} \right| &= \left| \mathrm{e}^{2\pi i \theta_{n-1}(l+m q_{n-1})} - \mathrm{e}^{2\pi i \theta_n (l+m q_{n-1})} \right| \\ &\le 2\pi q_n \left| \theta_{n-1} - \theta_n \right| = \frac{2\pi}{q_{n-1}} \end{aligned} \tag{3.7}$$

which holds for every pair of integers l, m with $|l+m q_{n-1}| \le q_n$. From (2.20) it therefore follows that

$$\begin{aligned} & \left| \sum_{j=1}^{q_{n-2}} \overline{w}_{n-1}^{j} \, w_{n-1}^{\prime j} \left(\mathrm{e}^{2\pi i \theta_{n-2}(j-1)} - \mathrm{e}^{2\pi i \theta_n (j-1+c_n q_{n-1})} \right) \right| \\ & \qquad \le \left(\varepsilon_n + \frac{2\pi}{q_{n-1}} \right) \left| \left\langle \boldsymbol{w}_{n-1} \, , \, \boldsymbol{w}'_{n-1} \right\rangle_{\mathbb{C}^{q_{n-2}}} \right| , \end{aligned} \tag{3.8}$$

where $\varepsilon_n \to 0$ and we have assumed that n is sufficiently large. Because the vectors ψ_{n-1} and ψ'_{n-1} are Cauchy sequences in $\mathcal{H}_\infty$, the sequence of inner products in (3.8) converges. Since $q_n \to \infty$, this shows that the first sum in (3.4) vanishes as $n \to \infty$. In a similar way one proves that the second sum in (3.4) vanishes as $n \to \infty$.

For $a = 2$ the expression (3.3) can be written as

$$\mathcal{U}_2^{(n)} = \frac{\overline{v}_{n-1}^{q_{n-1}} \, v_{n-1}^{\prime 1}}{1+c_n} + \overline{w}_{n-1}^{q_{n-2}} \, w_{n-1}^{\prime 1} - \frac{\overline{v}_{n-1}^{q_{n-1}} \, w_{n-1}^{\prime 1} + \overline{w}_{n-1}^{q_{n-2}} \, v_{n-1}^{\prime 1}}{\sqrt{1+c_n}} . \tag{3.9}$$

Using Eq. (2.36) we may deduce how the $\mathcal{U}_2^{(n)}$ change with n, and we find

$$\begin{aligned}\mathcal{U}_2^{(n+1)} = & \frac{\overline{w}^{q_{n-2}}_{n-1}\, v'^{1}_{n-1}}{(1+c_{n+1})\sqrt{1+c_n}} + \frac{\overline{v}^{q_{n-1}}_{n-1}\, v'^{1}_{n-1}}{1+c_n} \\ & - \frac{1}{\sqrt{(1+c_n)(1+c_{n+1})}} \left(\overline{w}^{q_{n-2}}_{n-1}\, v'^{1}_{n-1} + \frac{\overline{v}^{q_{n-1}}_{n-1}\, v'^{1}_{n-1}}{\sqrt{1+c_n}} \right) .\end{aligned} \tag{3.10}$$

By using (3.9), (3.10) and an induction argument, we find that in general $\mathcal{U}_2^{(n+m)}$ can be bounded by the product of a convergent constant M_m, determined by the uniform bounds on the vectors ψ_{n-1} and ψ'_{n-1}, and a product of normalization factors $(1+c_n)^{-1/2}$. Since each $c_n \geq 1$, we then find

$$\left|\mathcal{U}_2^{(n+m)}\right| \leq M_m \left(\frac{1}{\sqrt{2}}\right)^m \tag{3.11}$$

which establishes the Proposition for $a = 2$. □

Proposition 2 can be generalized straightforwardly to arbitrary powers of the U_a's, and also to products $U_1 U_2$ by inserting a complete set of states of $\mathcal{H}_\infty$ in between U_1 and U_2. It represents the appropriate limiting procedure that one could use in a numerical simulation of the correlation functions. Namely, one starts with sufficiently large vectors and matrices which approximate a correlation function (3.1) and then iterates the vectors to the next level according to the embedding (2.33) (or (2.55) for the rational case). From this procedure one may in fact estimate the rate of convergence of the approximation to the desired correlator. As a simple example, we have checked numerically the convergence of the quantities

$$\begin{aligned}\Big\langle \tilde\rho_{n+m} \circ \tilde\rho_{n+m-1} \circ \cdots \circ \tilde\rho_n(\psi'), & \left(\widetilde{U}_a^{(n+m)} \oplus \widetilde{U}_a^{(n+m-1)} \right) \\ & \circ \tilde\rho_{n+m} \circ \tilde\rho_{n+m-1} \circ \cdots \circ \tilde\rho_n(\psi) \Big\rangle_{\mathcal{H}_{n+m}}\end{aligned} \tag{3.12}$$

for various cases. For the deformation parameter we have taken the Golden Ratio $\theta = \frac{\sqrt{5}+1}{2}$ which is characterized by $c_n = 1$, $\forall n \geq 0$, and which is known to be the slowest converging continued fraction. In this case $p_n = q_{n-1}$ is the n-th element of the Fibonacci sequence. Nevertheless, the convergence of the θ_n to θ is quite rapid: for $n = 15$ the accuracy is of one part in 10^6 and the matrices are of size 610×610. Starting with various choices of ψ', ψ and n, the expression (3.12) converges to definite values quite fast in m, with the difference between successive evaluations steadily decreasing. For example, for random vectors ψ' and ψ with a starting value $n = 5$ and for $m = 13$ immersions, the difference between successive evaluations is less than a part in 10^3 at the end of the iterations. For other irrational θ's the convergence will be faster, and so will be the growth in dimension of the matrices.

4. Approximating Geometries

Thus far the approximating schemes we have discussed have been at the level of C^*-algebras. In the context of noncommutative geometry, this means that all of our equivalences hold only at the level of topology (this is actually the geometrical meaning of

Morita equivalence). The algebra A_θ on its own does not specify the geometry of the underlying noncommutative space, and the latter is determined by the specification of a K-cycle [10,26]. The algebra $A_{M/N}$ is essentially just a matrix algebra, and for it there exists choices of K-cycles corresponding to the deformed torus, the fuzzy two-sphere, and even the fuzzy three-sphere [13]. In this section we will describe how to obtain the K-cycle appropriate to the noncommutative torus $\mathbf{T}^2_\theta$ from the embedding of A_θ into the AF-algebra A_∞. In a more physical language, this will tell us how to approximate derivative terms for field theories on the noncommutative torus and also how to approximate gauge theories, as in [9]. As far as large N Matrix theory is concerned, this choice of K-cycle will be just one possible D0-brane parameter space geometry in the noncommutative spacetime.

On $\mathbf{T}^2_\theta$, there are natural linear derivations δ_a defined by

$$\delta_a(U_b) = 2\pi i\,\delta_{ab}\,U_b, \quad a,b = 1,2. \tag{4.1}$$

These derivations can be used to construct the canonical Dirac operator on $\mathbf{T}^2_\theta$, and hence the K-cycle appropriate to the (noncommutative) Riemannian geometry of the two-torus. With the canonical derivations (4.1), a connection ∇_a on a vector bundle $\mathcal{H}$ over the noncommutative torus may be defined as a Hermitian operator acting on $\mathcal{H}$ and satisfying the property

$$[\nabla_a, U_b] = 2\pi\,\delta_{ab}\,U_b, \quad a,b = 1,2. \tag{4.2}$$

Here the bundle $\mathcal{H}$ is taken to be a finitely-generated, left projective module over the noncommutative torus and (4.2) is a statement about operators acting on the left on $\mathcal{H}$. Indeed, it is nothing but the usual Leibniz rule.

In general, it is not possible to approximate the defining property (4.2) by finite dimensional matrices. It is, however, straightforward to construct an exponentiated version of this constraint in each algebra A_n. For this, it is convenient to use a different representation for the generators of the algebra (2.30), namely

$$\begin{aligned} \left[\widetilde{U}_1^{(n)}\right]_{kj} &= \mathrm{e}^{2\pi i(j-1)/q_n}\,\delta_{kj}, \\ \left[\widetilde{U}_2^{(n)}\right]_{kj} &= \delta_{k,j-p_n+1} \quad k,j = 1,\dots,q_n \ (\operatorname{mod} q_n). \end{aligned} \tag{4.3}$$

We seek unitary matrices $\mathrm{e}^{i\nabla_a^{(n)}} \in A_n$, $(\nabla_a^{(n)})^\dagger = \nabla_a^{(n)}$,[5] which conjugate elements of $\pi(A_{\theta_n})$ in the sense

$$\mathrm{e}^{-i\nabla_a^{(n)}}\,\widetilde{U}_b^{(n)}\,\mathrm{e}^{i\nabla_a^{(n)}} = \mathrm{e}^{2\pi i\delta_{ab}r_a^{(n)}/q_n}\,\widetilde{U}_b^{(n)}, \quad a,b = 1,2, \tag{4.4}$$

where $r_a^{(n)}$ are sequences of integers such that

$$\lim_{n\to\infty}\frac{r_a^{(n)}}{q_n} = R_a, \quad a = 1,2 \tag{4.5}$$

[5] The construction given below, as well those of [15] and in the preceeding sections of this paper, are strictly speaking only true in the continuous category, i.e. at the level of the Lie group of unitary matrices. Once we have the required approximation at hand, however, we may pass to the corresponding Lie algebra of Hermitian matrices and hence to the smooth category wherein the connections lie.

are fixed, finite real numbers whose interpretation will be given below. A set of operators obeying the conditions (4.4) is given by

$$\begin{aligned}\left[\mathrm{e}^{i\nabla_1^{(n)}}\right]_{kj} &= \delta_{k-r_1^{(n)}+1,\,j}\,,\\ \left[\mathrm{e}^{i\nabla_2^{(n)}}\right]_{kj} &= \mathrm{e}^{2\pi i(j-1)r_2^{(n)}/p_nq_n}\,\delta_{kj}\,,\qquad k,\,j=1,\ldots,q_n(\mathrm{mod}\,q_n).\end{aligned}\tag{4.6}$$

Note that $\mathrm{e}^{i\nabla_a^{(n)}}\notin\pi(A_{\theta_n})$, and that the matrices (4.6) obey the commutation relation

$$\mathrm{e}^{i\nabla_1^{(n)}}\,\mathrm{e}^{i\nabla_2^{(n)}}=\mathrm{e}^{-2\pi i r_1^{(n)}r_2^{(n)}/p_nq_n}\,\mathrm{e}^{i\nabla_2^{(n)}}\,\mathrm{e}^{i\nabla_1^{(n)}}.\tag{4.7}$$

We are interested in the behaviour of these matrices as $n\to\infty$.

Proposition 3.

$$\lim_{n\to\infty}\left\|\rho_n\left(\mathrm{e}^{i\nabla_a^{(n-1)}}\oplus\mathrm{e}^{i\nabla_a^{(n-2)}}\right)-\mathrm{e}^{i\nabla_a^{(n)}}\oplus\mathrm{e}^{i\nabla_a^{(n-1)}}\right\|_{A_n}=0,\quad a=1,2.$$

Proof. Again we will explicitly demonstrate this in the case of irrational θ, the rational case being a straightforward modification. For $a=1$ the eigenvalues of the matrix

$$\underbrace{\mathrm{e}^{i\nabla_a^{(n-1)}}\oplus\cdots\oplus\mathrm{e}^{i\nabla_a^{(n-1)}}}_{c_n\ \text{times}}\oplus\mathrm{e}^{i\nabla_a^{(n-2)}}\oplus\mathrm{e}^{i\nabla_a^{(n-1)}}-\mathrm{e}^{i\nabla_a^{(n)}}\oplus\mathrm{e}^{i\nabla_a^{(n-1)}}\tag{4.8}$$

are readily found to be all equal to 0 (for any n). For $a=2$, the eigenvalues of (4.8) are of the generic form $\mathrm{e}^{ic_j^{(n-1)}/p_{n-1}q_{n-1}}-\mathrm{e}^{id_j^{(n)}/p_nq_n}$, where $c_j^{(n-1)}/p_{n-1}q_{n-1}\to 0$ and $d_j^{(n)}/p_nq_n\to 0$ as $n\to\infty$. □

Proposition 3 implies that the operators $\mathrm{e}^{i\nabla_a^{(n)}}\oplus\mathrm{e}^{i\nabla_a^{(n-1)}}\in A_n$ are norm convergent to unitary operators $\mathrm{e}^{i\nabla_a}\in A_\infty-A_\theta$. It follows from (4.4) and (4.5) that these operators conjugate elements of the algebra A_θ according to

$$\mathrm{e}^{-i\nabla_a}\,U_b\,\mathrm{e}^{i\nabla_a}=\mathrm{e}^{2\pi i R_a\delta_{ab}}\,U_b,\quad a,b=1,2.\tag{4.9}$$

Iterating (4.9) and continuing to $s\in\mathbb{R}$, this property is seen to be the $s=1$ limit of the equation

$$\mathrm{e}^{-is\nabla_a}\,U_b\,\mathrm{e}^{is\nabla_a}=\mathrm{e}^{2\pi isR_a\delta_{ab}}\,U_b.\tag{4.10}$$

Differentiating (4.10) with respect to s and then setting $s=0$ yields

$$[\nabla_a,U_b]=2\pi R_a\,\delta_{ab}\,U_b.\tag{4.11}$$

From this commutator we infer that the operators ∇_a satisfy the appropriate Leibniz rule and therefore define a connection on a bundle over the noncommutative torus $\mathbf{T}^2_\theta$. The matrices (4.6) thereby give a finite dimensional approximation, in the spirit of the present paper, to the connection ∇_a. From (4.11) we see that the numbers R_a defined by (4.5) represent the lengths of the two sides of $\mathbf{T}^2$. Moreover, from (4.7) we find that the connection ∇_a has constant curvature

$$\left[\nabla_1\,,\ \nabla_2\right]=\frac{2\pi i R_1R_2}{\theta}.\tag{4.12}$$

The objects presented here thereby define connections of the modules $\mathcal{H}_{0,1}$ over the noncommutative torus which have rank $|p - q\theta| = \theta$ and topological charge $q = 1$ [11]. Gauge fields may be introduced in the usual way now by constructing functions of elements in the commutants of the algebras generated by $U_a^{(n)}$ and U_a. The more general class of constant curvature modules $\mathcal{H}_{p,q}$ [11] can likewise be constructed using the tensor product decomposition described in [9]. We will omit the details of this somewhat tedious generalization. Notice that at the finite dimensional level, all of the operators we have defined live in the same algebra A_∞. In the inductive limit however, while the $U_a^{(n)}$ go to the algebra of the noncommutative torus, the unitary operators giving the connection ∇_a go to a Morita equivalent one. Thus in the large N limit here we reproduce the known fact [11] that the endomorphism algebra of $\mathcal{H}_{p,q}$ is a noncommutative torus which is Morita equivalent to the original one. The reason for this correct reproduction of gauge theories in the limit is K-theoretic and was discussed in Sect. 2.

5. Conclusions

The constructions presented in this paper show that it is indeed possible to represent both geometrical and physical quantities defined over the noncommutative torus as a certain limit of finite dimensional matrices. These results give a systematic and definitive way to realize the spectral geometry, and also the noncommutative gauge theory, of $\mathbf{T}^2_\theta$ for any $\theta \in \mathbb{R}$ by an infinite tower of finite dimensional matrix geometries. It should be stressed though that the types of large N limits described in this paper are somewhat different in spirit than those used for brane constructions from matrix models [2,3,5], which are rooted in the fuzzy space approximations to function algebras [13]. The present matrix approximations are more suited to the definition of noncommutative Yang–Mills theory in terms of Type IIB superstrings in D-brane backgrounds [8]. It would be interesting to carry out the constructions of string theoretical degrees of freedom in terms of the above decompositions of the noncommutative torus into finite dimensional matrices, and thus test the correspondence between noncommutative gauge theoretic predictions with those of the matrix models.

The constructions of this paper also shed some light on the precise meaning of Morita equivalence in such physical models. Although Morita equivalence does imply a certain duality between (noncommutative) Yang–Mills theories, within the matrix approximations there is essentially no distinction between rational and irrational deformation parameters and hence no reason for a model with rational θ to be regarded as completely equivalent to an ordinary (commutative) gauge theory. This is in agreement with the recent hierarchical classification of noncommutative Yang–Mills theories given in [25]. It should always be understood that Morita equivalence is a duality between C^*-algebras, and as such it is topological. The equivalence at the level of geometry typically goes away upon the introduction of appropriate K-cycles (as is the usual case for T-duality equivalences as well). On the other hand, we have shown that dual Yang–Mills theories all originate from the same AF-algebra A_∞.

We close with some remarks about how these results may be generalized to higher dimensional noncommutative tori and hence to more physically relevant noncommutative Yang–Mills theories. The algebra of functions on a d-dimensional noncommutative torus $\mathbf{T}^d_\theta$ is generated by d unitary operators satisfying the relations

$$U_a U_b = \mathrm{e}^{2\pi i \theta_{ab}}\, U_b U_a, \quad a, b = 1, \ldots, d, \tag{5.1}$$

where $\theta = [\theta_{ab}]$ is an antisymmetric, real-valued $d \times d$ matrix. It is always possible to rotate θ into a canonical skew-diagonal form with skew-eigenvalues ϑ_a,

$$\theta = \begin{pmatrix} 0 & \vartheta_1 & & & & \\ -\vartheta_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & \vartheta_r & \\ & & & -\vartheta_r & 0 & \\ & & & & & \mathbf{0}_{d-2r} \end{pmatrix}, \tag{5.2}$$

where $2r$ is the rank of θ. Thus one may embed the algebra of a higher dimensional noncommutative torus into a d-fold tensor product of algebras corresponding to r noncommutative two-tori $\mathbf{T}^2_{\vartheta_a}$ and an ordinary $(d-2r)$-torus $\mathbf{T}^{d-2r}$. This embedding preserves the appropriate K-theory groups

$$\mathrm{K}_0(\mathbf{T}^d) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{2^{d-1}\ \text{times}}. \tag{5.3}$$

However, the issue of generalizing the constructions of the present paper to higher dimensions in this manner is still a delicate issue. It turns out [27] that for almost all noncommutative tori (precisely, for a set of deformation parameters of Lebesgue measure 1) one may can construct an AF algebra in which to embed the algebra of functions on $\mathbf{T}^d_\theta$.

Acknowledgements. We thank L. Dabrowski, G. Elliott, R. Nest, J. Madore, J. Nishimura, M. Rieffel, M. Sheikh-Jabbari, A. Sitarz and J. Várilly for interesting discussions. This work was supported in part by the Danish Natural Science Research Council.

References

1. Witten, E.: Nucl. Phys. B **460**, 335 (1996)[hep-th/9510135]
2. Banks, T., Fischler, W., Shenker, S.H. and Susskind, L.: Phys. Rev. D **55**, 5112 (1997) [hep-th/9610043]
3. Banks, T., Seiberg, N. and Shenker, S.H.: Nucl. Phys. B **490**, 91 (1997) [hep-th/9612157]
4. Connes, A., Douglas, M.R. and Schwarz, A.: J. High Energy Phys. **9802**, 003 (1998)[hep-th/9711162]
5. Ishibashi, N., Kawai, H., Kitazawa, Y. and Tsuchiya, A.: Nucl. Phys. B **498**, 467 (1997) [hep-th/9612115]
6. Seiberg, N. and Witten, E.: J. High Energy Phys. **9909**, 032 (1999) [hep-th/9908142]
7. Rajeev, S.G.: Phys. Rev. D **42**, 2779 (1990); **44**, 1836 (1991);
   Lee, C.-W.H. and Rajeev, S.G.: Nucl. Phys.B **529**, 656 (1998) [hep-th/9712090];
   Lizzi, F. and Szabo, R.J.: Chaos Solitons Fractals **10**, 445 (1999) [hep-th/9712206]; J. High Energy Phys. Proc. corfu98/073 [hep-th/9904064]
8. Li, M.: Nucl. Phys. B **499**, 149 (1997) [hep-th/9612222];
   Aoki, H., Ishibashi, N., Iso, S., Kawai, H., Kitazawa, Y. and Tada, T.: Nucl. Phys. B **565**, 176 (2000) [hep-th/9908141];
   Ishibashi, N., Iso, S., Kawai, H. and Kitazawa, Y.: Nucl. Phys. B **573**, 573 (2000) [hep-th/9910004];
   Bars, I. and Minic, D.: Phys. Rev. D **62**, 105018 (2000) [hep-th/9910091]
9. Ambjørn, J., Makeenko, Y.M., Nishimura, J. and Szabo R.J.: J. High Energy Phys. **9911**, 029 (1999) [hep-th/9911041]
10. Connes, A.: *Noncommutative Geometry.* Academic Press, 1994
11. Connes, A.: C.R. Acad. Sci. Paris Sér. A **290**, 599 (1980);
    Connes, A. and Rieffel, M.A.: Contemp. Math. **62**, 237 (1987)
12. Wegge-Olsen, N.E.: *K-Theory and C^*-Algebras.* Oxford: Oxford Science Publications, 1993
13. de Wit, B., Hoppe, J. and Nicolai, H.: Nucl. Phys. B **305**[FS23], 545 (1988);
    Madore, J.: *An Introduction to Noncommutative Geometry and its Physical Applications*, Second Edition. Cambridge: Cambridge University Press, 1999
14. Rieffel, M.A.: Questions on Quantization. In: Proc. Int. Conf. *Operator Algebras and Operator Theory*, July 4–9 1997, Shanghai [quant-ph/9712009]

15. Pimsner, M. and Voiculescu, D.: J. Oper. Theory **4**, 201 (1980)
16. Schwarz, A.: Nucl. Phys. B **534**, 720 (1998) [hep-th/9805034];
Landi, G., Lizzi, F. and Szabo, R.J.: Commun. Math. Phys. **206**, 603 (1999) [hep-th/9806099];
Brace, D., Morariu, B. and Zumino, B.: Nucl. Phys. B **545**, 192 (1999) [hep-th/9810099];
Pioline, B. and Schwarz, A.: J. High. Energy Phys. **9908**, 021 (1999) [hep-th/9908019]
17. Weyl, H.: *The Theory of Groups and Quantum Mechanics*. Dover, 1931
18. Fairlie, D.B., Fletcher, P. and Zachos, C.K.: Phys. Lett. B **218**, 203 (1989); J. Math. Phys. **31**, 1088 (1990);
Fairlie, D.B. and Zachos, C.K.: Phys. Lett. B **224**, 101 (1989)
19. Sheikh-Jabbari, M.M.: J. High Energy Phys. **9906**, 015 (1999) [hep-th/9903107]
20. Bratteli, O.: Trans. Am. Math. Soc. **171**, 195 (1972);
Effros, E.G.: CBMS Reg. Conf. Ser. Math., no. 46 (Am. Math. Soc., 1981)
21. Elliott, G.A.: J. Algebra **38**, 29 (1976)
22. Hardy, G.H. and Wright, E.M.: *An Introduction to the Theory of Numbers*. Oxford, 1954
23. Effros, E.G. and Shen, C.L.: Indiana J. Math. **29**, 191 (1980)
24. Rieffel, M.A.: Pacific J. Math. **93**, 415 (1981); Contemp. Math. **105**, 191 (1990)
25. Hashimoto, A. and Itzhaki, N.: J. High Energy Phys. **9912**, 007 (1999) [hep-th/9911057]
26. Landi, G.: *An Introduction to Noncommutative Spaces and their Geometries*. Springer-Verlag, 1997;
Várilly, J.C.: An Introduction to Noncommutative Geometry, Lectures at the EMS Summer School on *Noncommutative Geometry and Applications*, September 1997, Portugal [physics/9709045];
Fröhlich, J., Grandjean, O. and Recknagel, A.: In: *Quantum Symmetries*, Connes, A., Gawędzki, K. and Zinn-Justin, J. (eds.). Les Houches Session **64**. Amsterdam: Elsevier, p. 221 [hep-th/9706132]
27. Boca, F.P.: J. Reine Angew. Math. **492**, 179 (1997)

Communicated by R. H. Dijkgraaf