Graph Algebras

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Abstract. These notes are the written version of the author's lectures at the CBMS-NSF conference on *Graph Algebras: Operator Algebras We Can See*, held at the University of Iowa from 30 May to 4 June 2004. They discuss the structure theory of the Cuntz-Krieger algebras of directed graphs, together with some applications and some recent generalisations of that theory.

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Preface

These notes were mainly written while I was preparing my lectures for the CBMS-NSF conference on *Graph Algebras: Operator Algebras We Can See*, which was held at the University of Iowa from 30 May to 4 June 2004. The ten chapters roughly correspond to the ten lectures, though for logical reasons they appear here in a slightly different order.

I am very grateful to those who organised the conference, those who came to the conference, and those who have helped me try to get the glitches out of these notes. In particular, I thank:

- Paul Muhly, David Pask and Mark Tomforde, who did a marvellous job of organising the conference: the informal atmosphere was great, and everything ran smoothly. (Well, everything which didn't go through Chicago airport.) Paul especially did an enormous amount of work on the details, always in his inimitably cheerful way. Thanks, Paul.
- Nathan Brownlowe, Tyrone Crisp, James Foster, Daniel Gow and Rishni Ratnam, who helped me organise lecture notes for an honours course I gave in 2002, which were my starting point for these notes.
- Astrid an Huef, Marcelo Laca, Paul Muhly, Aidan Sims, Mark Tomforde and Trent Yeend, who provided me with (sometimes embarrassingly long) lists of corrections on parts of the notes. The remaining mistakes are not Dana Williams' fault[1].

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[1] See [**114**, page xiv].

Introduction

A directed graph is a combinatorial object consisting of vertices and oriented edges joining pairs of vertices. We can represent such a graph by operators on a Hilbert space $\mathcal{H}$: the vertices are represented by mutually orthogonal closed subspaces, or more precisely the projections onto these subspaces, and the edges by operators between the appropriate subspaces. The graph algebra is, loosely speaking, the C^*-subalgebra of $B(\mathcal{H})$ generated by these operators.

When the graph is finite and highly connected, the graph algebras coincide with a family of C^*-algebras first studied by Cuntz and Krieger in 1980 [**16**]. The Cuntz-Krieger algebras were quickly recognised to be a rich supply of examples for operator algebraists, and also cropped up in some unexpected situations [**87, 131**]. In the past ten years there has been a great deal of interest in graph algebras associated to infinite graphs, and these have arisen in new contexts: in non-abelian duality [**83, 24**], as deformations of commutative algebras [**56, 57, 58**], in non-commutative geometry [**12**], and as models for the classification of simple C^*-algebras [**136**].

Graph algebras have an attractive structure theory in which algebraic properties of the algebra are related to combinatorial properties of paths in the directed graph. The fundamental theorems of the subject are analogues of those proved by Cuntz and Krieger, and include a uniqueness theorem and a description of the ideals in graph algebras. But we now know much more: just about any C^*-algebraic property a graph algebra might have can be determined by looking at the underlying graph.

Our goals here are to describe the structure theory of graph algebras, and to discuss two particularly promising extensions of that theory involving the topological graphs of Katsura [**73**] and the higher-rank graphs of Kumjian and Pask [**81**]. We provide full proofs of the fundamental theorems, and also when we think some insight can be gained by proving a special case of a published result or by taking an alternative route to it. Otherwise we concentrate on describing the main ideas and giving references to the literature.

Outline. The core material is in the first four chapters, where we discuss the uniqueness theorems and the ideal structure. These theorems were first proved for infinite graphs by realising the graph algebra as the C^*-algebra of a locally compact groupoid, and applying results of Renault [**83, 82**]. There are now several other approaches to this material; the elementary methods we use here are based on the original arguments of Cuntz and Krieger, but incorporate several simplifications which have been made over the years. These techniques work best for the row-finite graphs in which each vertex receives just finitely many edges; in Chapter 5 we describe a method of Drinen and Tomforde for reducing problems to the row-finite case.

In Chapter 6, we describe how graph algebras provided important insight into some problems in non-abelian duality for crossed products of C^*-algebras, and then in Chapter 7 we calculate the K-theory of graph algebras. In Chapter 8, we give an introduction to correspondences and Cuntz-Pimsner algebras, using graph algebras as motivation for the various constructions. This is important material for researchers in many areas: many interesting C^*-algebras are by definition the Cuntz-Pimsner algebra of some particular correspondence (see [**68, 73, 92**], for example), and the general theory seems to be a powerful tool. This is certainly the case for the C^*-algebras of topological graphs, which are the subject of Chapter 9. In the last chapter, we discuss higher-rank graphs.

Chapters 5, 6, 7 and 10 are essentially independent of each other, and can be read in any order after the first four chapters. Chapter 9, on the other hand, requires familiarity with Chapter 8.

Conventions. We use the standard conventions of our subject. Thus, for example, homomorphisms between C^*-algebras are always $*$-preserving, and representations of C^*-algebras are homomorphisms into the algebra $B(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$; ideals are always assumed to be closed and two-sided. As a more personal convention, we always denote the inner product of two elements h, k of a Hilbert space by $(h \,|\, k)$, and reserve the notation $\langle x, y \rangle$ for C^*-algebra-valued inner products, which for us are always conjugate-linear in the first variable (see Chapter 8).

There is, unfortunately, no standard set of conventions for graph algebras: one can choose to have the partial isometries representing edges going either in the same direction as the edge or in the opposite direction. Both conventions are used in the literature, and both have their advantages and disadvantages. We have opted to have the partial isometries going in the same direction as the edges. The main disadvantage of this is that we have to adopt a rather unnatural notation for paths in graphs (see Remark 1.13). However, when edges represent morphisms in a category, as they do in higher-rank graphs, for example, this convention means that composition of morphisms is compatible with multiplication of operators in $B(\mathcal{H})$. After several years of having bright students who talk about higher-rank graphs all the time, I have found it easier to adopt their conventions. (There was no chance they were going to change...) Nevertheless, I sympathise with those to whom changing seems an unnecessary nuisance, and I apologise to them.

Background. While writing these notes, I have been addressing a reader who has taken a first course in C^*-algebras, covering the Gelfand-Naimark Theorems, the continuous functional calculus, and positivity. This material is covered in some form or other in most of the standard books on the subject, and the first 3 chapters of [**93**], for example, contain everything we need.

For those with slightly different backgrounds, it might be helpful to mention some non-trivial facts about C^*-algebras which we use frequently.

(1) Every homomorphism between C^*-algebras is norm-decreasing, and every injective homomorphism is norm-preserving. In particular, every automorphism and every isomorphism of C^*-algebras preserves the norm.

(2) The range $\phi(A)$ of every homomorphism $\phi : A \to B$ between C^*-algebras is closed, and is therefore a C^*-subalgebra of B.

(3) Every C^*-algebra has a faithful representation as a C^*-algebra of bounded operators on Hilbert space. This is often used when we want to apply results about representations of C^*-algebras to more general homomorphisms.

Points (1) and (2) give the theory of C^*-algebras a rather algebraic flavour. However, it is essential in (1) and (2) that the algebras are complete and that the homomorphisms are everywhere defined. We often work with maps defined on dense $*$-subalgebras of C^*-algebras, and then we have to establish norm estimates to be sure that the maps extend to the C^*-algebras.

We have included an appendix in which we discuss some material which might not be covered in a first course, and which might be hard to locate in the literature. First we show how geometric properties of projections and partial isometries on a Hilbert space $\mathcal{H}$ can be encoded using the $*$-algebraic structure of $B(\mathcal{H})$. Then we look at some standard tricks which allow us to identify C^*-algebras as matrix algebras or direct sums and direct limits of such algebras.

CHAPTER 1

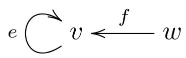
Directed graphs and Cuntz-Krieger families

A *directed graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and functions $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 are called *edges*. For each edge e, $s(e)$ is the *source* of e and $r(e)$ the *range* of e; if $s(e) = v$ and $r(e) = w$, then we also say that v *emits* e and that w *receives* e, or that e is an edge from v to w. All the graphs in these notes are directed, so we sometimes get lazy and call them graphs. If there is more than one graph around, we might write r_E and s_E to emphasise that we are talking about the range and source maps for E.

We usually draw a graph by placing the vertices in a plane, and drawing a directed line from $s(e)$ to $r(e)$ for each edge $e \in E^1$. If necessary, we label the edge by its name.

Example 1.1. If $E^0 = \{v, w\}$, $E^1 = \{e, f\}$, $r(e) = s(e) = v$, $s(f) = w$ and $r(f) = v$, then we could draw

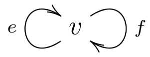
(1.1)



An edge which begins and ends at the same vertex v, like the edge e in Example 1.1, is called a *loop based at* v[1]. A vertex which does not receive any edges, like the vertex w in Example 1.1, is called a *source*. (Using the word "source" in two ways doesn't seem to cause confusion.) A vertex which emits no edges is called a *sink*.

Conversely, every drawing like (1.1) determines a graph.

Example 1.2. The drawing



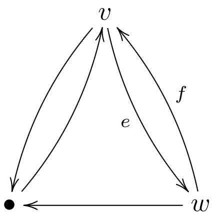
represents a graph E in which $E^0 = \{v\}$, $E^1 = \{e, f\}$ and e and f are both loops based at v. Notice that we are allowing multiple edges between the same pair of vertices; graph theorists often don't allow this.

Drawings are a useful aid when trying to follow arguments about graphs. However, there are many ways to draw the same graph, so it is important to remember that two directed graphs E and F are the same (formally, *isomorphic*) if and only if there are bijections $\phi^0 : E^0 \to F^0$ and $\phi^1 : E^1 \to F^1$ such that $r_F \circ \phi^1 = \phi^0 \circ r_E$ and $s_F \circ \phi^1 = \phi^0 \circ s_E$.

When it doesn't matter what an edge is called, we don't bother to label it in a drawing; when it doesn't matter what a vertex is called, we denote it by a $\bullet$.

[1]This is standard graph-theory terminology. Unfortunately the word "loop" is used in the graph-algebra literature to mean a closed path.

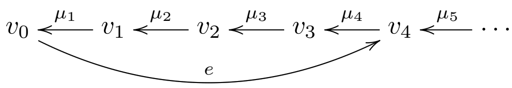
EXAMPLE 1.3. The drawing



represents a graph E with three vertices, two of which are called v and w, and five edges, two of which are called e and f. We do this to simplify notation when we are only going to refer to e and f.

These first three examples are all *finite graphs* in which both E^0 and E^1 are finite sets. In general, we allow either or both to be infinite.

EXAMPLE 1.4. The drawing



represents a graph E in which $E^0 = \{v_n : n \geq 0\}$ is infinite, and E^1 is the union of a singleton set $\{e\}$ and an infinite set $\{\mu_i : i \geq 1\}$.

EXAMPLE 1.5. The drawing

$$\cdots \longrightarrow v_3 \longrightarrow v_2 \longrightarrow v_1 \xrightarrow{(\infty)} v_0 \longrightarrow v_{-1} \longrightarrow v_{-2} \longrightarrow \cdots$$

represents a graph E with $E^0 = \{v_i : i \in \mathbb{Z}\}$, one edge from v_{i+1} to v_i for each $i \neq 0$, and infinitely many edges from v_1 to v_0.

For reasons which we will discuss in Chapter 5, graphs in which some vertices receive infinitely many edges pose extra problems for us. So we shall consider mainly the *row-finite graphs* in which each vertex receives at most finitely many edges, that is, in which $r^{-1}(v)$ is a finite set for every $v \in E^0$. The graph in Example 1.4 is row-finite, but that in Example 1.5 is not because v_0 receives infinitely many edges.

REMARK 1.6. The word "row-finite" refers to the corresponding property of the *vertex matrix* A_E of the graph E, which is the $E^0 \times E^0$ matrix defined by

$$A_E(v, w) = \#\{e \in E^1 : r(e) = v,\ s(e) = w\}.$$

(A_E is sometimes called the *adjacency matrix* of E.) The graph E is row-finite if and only if each row $\{A_E(v, w) : w \in E^0\}$ of A_E has finite sum.

We now seek to represent a directed graph by operators on Hilbert space: the vertices will be represented by orthogonal projections and the edges by partial isometries. Formally, let E be a row-finite directed graph and $\mathcal{H}$ a Hilbert space. A *Cuntz-Krieger E-family* $\{S, P\}$ *on* $\mathcal{H}$ consists of a set $\{P_v : v \in E^0\}$ of mutually orthogonal projections on $\mathcal{H}$ and a set $\{S_e : e \in E^1\}$ of partial isometries on $\mathcal{H}$, such that

(CK1) $S_e^* S_e = P_{s(e)}$ for all $e \in E^1$; and

(CK2) $P_v = \sum_{\{e \in E^1 : r(e) = v\}} S_e S_e^*$ whenever v is not a source.

Conditions (CK1) and (CK2) are called the *Cuntz-Krieger relations*, and Condition (CK2) in particular is often called the *Cuntz-Krieger relation at* v.

Saying that the projections P_v are mutually orthogonal means that the ranges $P_v\mathcal{H}$ are mutually orthogonal subspaces of $\mathcal{H}$. The relation (CK1) says that S_e is a partial isometry with initial space $P_{s(e)}\mathcal{H}$ (Proposition A.4); relation (CK2) implies that the range projection $S_eS_e^*$ of S_e is dominated by $P_{r(e)}$, and hence that $S_e\mathcal{H} \subset P_{r(e)}\mathcal{H}$ (Proposition A.1). Thus S_e is an isometry of $P_{s(e)}\mathcal{H}$ onto a closed subspace of $P_{r(e)}\mathcal{H}$; expressing this algebraically gives the relation

$$S_e = P_{r(e)}S_e = S_eP_{s(e)}, \tag{1.2}$$

which is used all the time in manipulations with Cuntz-Krieger families. Relation (CK2) also implies that the partial isometries S_e associated to the edges e with $r(e) = v$ have mutually orthogonal ranges (see Corollary A.3) with span $P_v\mathcal{H}$, so

$$P_v\mathcal{H} = \textstyle\bigoplus_{\{e\in E^1\,:\,r(e)=v\}} S_e\mathcal{H},$$

in the sense that the map $(h_e) \mapsto \sum_e h_e$ is an isomorphism of the direct sum onto $P_v\mathcal{H}$.

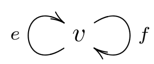
Remark 1.7. Since the initial and range projections of the S_e are all contained in $\mathcal{H}_e := \overline{\operatorname{span}}\big\{\bigcup_{v\in E^0} P_v\mathcal{H}\big\}$, we may as well assume that $\mathcal{H} = \mathcal{H}_e$, in which case we say the family is *non-degenerate*. If $\{S, P\}$ is a non-degenerate Cuntz-Krieger E-family, then the mutual orthogonality of the P_v implies that $\mathcal{H} = \bigoplus_{v\in E^0} P_v\mathcal{H}$, in the sense that the obvious map $(h_v) \mapsto \sum_v h_v$ of

$$\textstyle\bigoplus_{v\in E^0} P_v\mathcal{H} := \big\{(h_v) \in \prod_{v\in E^0} P_v\mathcal{H} : \sum_v \|h_v\|^2 < \infty\big\}$$

into $\mathcal{H}$ is an isomorphism (that $\sum_v \|h_v\|^2 < \infty$ implies that the sum $\sum_v h_v$ converges in norm in $\mathcal{H}$).

Remark 1.8. Since the orthogonal projections on closed subspaces of $\mathcal{H}$ are the bounded operators P satisfying $P^2 = P = P^*$ and the partial isometries are the bounded operators S satisfying $S = SS^*S$, we can talk about projections and partial isometries in any C^*-algebra B (see Appendix A.1). A Cuntz-Krieger E-family in B then consists of projections $\{P_v \in B : v \in E^0\}$ satisfying $P_vP_w = 0$ for $v \neq w$ (so that $\{P_v\}$ is a mutually orthogonal family of projections) and partial isometries $\{S_e \in B : e \in E^1\}$ satisfying (CK1) and (CK2).

Example 1.9. Consider the directed graph:



We have $S_e^*S_e = P_v = S_f^*S_f$, $P_v = S_eS_e^* + S_fS_f^*$. Take $\mathcal{H} = \ell^2(\mathbb{N}) = \overline{\operatorname{span}}\{e_n : n \geq 0\}$, P_v to be the identity operator 1, $S_e(e_n) = e_{2n}$ and $S_f(e_n) = e_{2n+1}$. Then $\{S, P\}$ is a Cuntz-Krieger family for this graph.

In any Cuntz-Krieger family $\{T, Q\}$ for this graph with Q_v non-zero, $Q_v\mathcal{H}$ must be infinite-dimensional. To see this, note that T_e is an isometry of $Q_v\mathcal{H}$ onto $T_e\mathcal{H}$, so $\dim Q_v\mathcal{H} = \dim T_e\mathcal{H}$. Similarly, $\dim Q_v\mathcal{H} = \dim T_f\mathcal{H}$. Thus $Q_v\mathcal{H} = T_e\mathcal{H} \oplus T_f\mathcal{H}$ implies

$$\dim Q_v\mathcal{H} = \dim T_e\mathcal{H} + \dim T_f\mathcal{H} = 2\dim Q_v\mathcal{H},$$

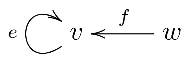
so $\dim Q_v\mathcal{H}$ can only be 0 or ∞.

In general there is no problem finding Cuntz-Krieger E-families with every P_v and every S_e non-zero: take $\mathcal{H}_v$ to be a separable infinite-dimensional Hilbert space for each $v \in E^0$, set $\mathcal{H} = \bigoplus_v \mathcal{H}_v$, take P_v to be the projection of $\mathcal{H}$ on $\mathcal{H}_v$, decompose $\mathcal{H}_v$ as a direct sum $\mathcal{H}_v = \bigoplus_{r(e)=v} \mathcal{H}_{v,e}$ of infinite-dimensional subspaces, and take S_e to be a unitary isomorphism of $\mathcal{H}_{s(e)}$ onto $\mathcal{H}_{r(e),e}$, viewed as a partial isometry on $\mathcal{H}$ with initial space $\mathcal{H}_{s(e)}$.

Example 1.9 shows why we need to take the spaces $\mathcal{H}_v$ to be infinite-dimensional. This is not always necessary, though:

EXAMPLE 1.10. Consider the graph E which consists of a single vertex v and a single loop e based at v. Then the Cuntz-Krieger relations say that $S_e^* S_e = P_v = S_e S_e^*$, so that S_e is a unitary operator on $P_v\mathcal{H}$ (and is 0 on $(P_v\mathcal{H})^\perp$). There is no other restriction on S_e: if U is a unitary operator on $\mathcal{H}$, then we can take $P_v = 1$ and $S_e = U$. So there is no restriction on $\dim \mathcal{H}$; we could even take $\mathcal{H} = \mathbb{C}$ and U to be multiplication by $e^{i\theta}$, for example.

EXAMPLE 1.11. For the graph



we can define a Cuntz-Krieger family on $\mathcal{H} = \ell^2$ by

$$P_v(x_0, x_1, x_2, \cdots) = (0, x_1, x_2, \cdots), \quad P_w(x_0, x_1, x_2, \cdots) = (x_0, 0, 0, \cdots),$$
$$S_f(x_0, x_1, x_2, \cdots) = (0, x_0, 0, \cdots), \quad \text{and} \quad S_e(x_0, x_1, x_2, \cdots) = (0, 0, x_1, x_2, \cdots).$$

It is important here that $P_v\mathcal{H}$ is infinite-dimensional: in any Cuntz-Krieger family for this graph, the Cuntz-Krieger relation at v implies that

$$\dim(P_v\mathcal{H}) = \dim(S_f\mathcal{H}) + \dim(S_e\mathcal{H}) = \dim(P_w\mathcal{H}) + \dim(P_v\mathcal{H}),$$

so if P_w and P_v are both non-zero, $P_v\mathcal{H}$ must be infinite-dimensional. It is worth observing now that the crucial factor in this argument is the presence of the edge f entering the loop.

We will be interested in the C^*-algebras $C^*(S, P)$ generated by Cuntz-Krieger families $\{S, P\}$, so we now investigate the $*$-algebraic consequences of the Cuntz-Krieger relations.

PROPOSITION 1.12. *Suppose that E is a row-finite graph and $\{S, P\}$ is a Cuntz-Krieger E-family in a C^*-algebra B. Then*

(a) *the projections $\{S_e S_e^* : e \in E^1\}$ are mutually orthogonal;*
(b) $S_e^* S_f \neq 0 \Longrightarrow e = f$;
(c) $S_e S_f \neq 0 \Longrightarrow s(e) = r(f)$;
(d) $S_e S_f^* \neq 0 \Longrightarrow s(e) = s(f)$.

PROOF. For part (a), suppose first that $r(e) = r(f)$. Then the Cuntz-Krieger relation at $r(e)$ implies that $P_{r(e)}$ is the sum of $S_e S_e^*$, $S_f S_f^*$ and other projections, which because $P_{r(e)}$ is a projection implies that $S_e S_e^*$ and $S_f S_f^*$ are mutually orthogonal (see Corollary A.3). On the other hand, if $r(e) \neq r(f)$, then (1.2) implies that

$$(S_e S_e^*)(S_f S_f^*) = (S_e S_e^* P_{r(e)})(P_{r(f)} S_f S_f^*) = (S_e S_e^*) 0 (S_f S_f^*) = 0.$$

Part (b) follows from (a), since $S_e^*S_f = S_e^*(S_eS_e^*)(S_fS_f^*)S_f = 0$ when $e \neq f$. For (c), we just note that part (a) implies that $S_eS_f = (S_eP_{s(e)})(P_{r(f)}S_f)$ vanishes unless $s(e) = r(f)$, and a similar argument gives (d). $\square$

Part (c) of Proposition 1.12 is particularly crucial: it says that S_eS_f is zero unless the pair ef is a path of length 2 in the graph E. More generally, a *path of length* n in a directed graph E is a sequence $\mu = \mu_1\mu_2\cdots\mu_n$ of edges in E such that $s(\mu_i) = r(\mu_{i+1})$ for $1 \le i \le n-1$. We write $|\mu| := n$ for the length of μ, and regard vertices as paths of length 0; we denote by E^n the set of paths of length n, and write $E^* := \bigcup_{n\ge 0} E^n$. (Now our notation for the sets of vertices and edges should make more sense.) We extend the range and source maps to E^* by setting $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_{|\mu|})$ for $|\mu| > 1$, and $r(v) = v = s(v)$ for $v \in E^0$. If μ and ν are paths with $s(\mu) = r(\nu)$, we write $\mu\nu$ for the path $\mu_1\cdots\mu_{|\mu|}\nu_1\cdots\nu_{|\nu|}$.

Remark 1.13. It may seem slightly odd to define the source of $\mu = \mu_1\mu_2\cdots\mu_n$ to be $s(\mu_n)$ rather than $s(\mu_1)$. However, this is forced on us by Proposition 1.12(c): the conventions of composition, which is the multiplication in $B(\mathcal{H})$, say that in the product RT we perform T first. This means that if we want juxtaposition of edges in a path μ to be consistent with juxtaposition of the corresponding partial isometries S_{μ_i} in $B(\mathcal{H})$, then we need to traverse μ_{i+1} before μ_i.

For $\mu \in \prod_{i=1}^n E^1$, we define $S_\mu := S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n}$, and for $v \in E^0$, we define $S_v := P_v$. Proposition 1.12(c) says that $S_\mu = 0$ unless μ is a path; if μ is a path, then

$$
\begin{aligned}
S_\mu^*S_\mu &= (S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n})^*S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n} \\
&= S_{\mu_n}^*\cdots S_{\mu_2}^*(S_{\mu_1}^*S_{\mu_1})S_{\mu_2}\cdots S_{\mu_n} \\
&= S_{\mu_n}^*\cdots S_{\mu_2}^*P_{s(\mu_1)}S_{\mu_2}\cdots S_{\mu_n} \\
&= S_{\mu_n}^*\cdots S_{\mu_2}^*P_{r(\mu_2)}S_{\mu_2}\cdots S_{\mu_n} \\
&= S_{\mu_n}^*\cdots S_{\mu_3}(S_{\mu_2}^*S_{\mu_2})S_{\mu_3}^*\cdots S_{\mu_n} \\
&\ \vdots \\
&= P_{s(\mu_n)} = P_{s(\mu)}.
\end{aligned}
\tag{1.3}
$$

Thus for $\mu \in E^*$, S_μ is a partial isometry with initial projection $P_{s(\mu)}$, and since $P_{r(\mu)}S_\mu S_\mu^* = S_\mu S_\mu^*$, the range of S_μ is a subspace of $P_{r(\mu)}\mathcal{H}$.

Proposition 1.12 extends to the partial isometries S_μ as follows:

Corollary 1.14. *Suppose that E is a row-finite graph and $\{S, P\}$ is a Cuntz-Krieger E-family in a C^*-algebra B. Let $\mu, \nu \in E^*$. Then*

(a) *if $|\mu| = |\nu|$ and $\mu \neq \nu$, then $(S_\mu S_\mu^*)(S_\nu S_\nu^*) = 0$;*

(b) $S_\mu^*S_\nu = \begin{cases} S_{\mu'}^* & \text{if } \mu = \nu\mu' \text{ for some } \mu' \in E^* \\ S_{\nu'} & \text{if } \nu = \mu\nu' \text{ for some } \nu' \in E^* \\ 0 & \text{otherwise;} \end{cases}$

(c) *if $S_\mu S_\nu \neq 0$, then $\mu\nu$ is a path in E and $S_\mu S_\nu = S_{\mu\nu}$;*

(d) *if $S_\mu S_\nu^* \neq 0$, then $s(\mu) = s(\nu)$.*

PROOF. For (a), let i be the smallest integer such that $\mu_i \neq \nu_i$. Then, applying (1.3) to $\mu_1\mu_2\cdots\mu_{i-1}$ gives

$$\begin{aligned} S_\mu^* S_\nu &= (S_{\mu_1}S_{\mu_2}\cdots S_{\mu_n})^* S_{\nu_1}S_{\nu_2}\cdots S_{\nu_n} \\ &= S_{\mu_n}^*\cdots S_{\mu_i}^*(S_{\mu_{i-1}}^*\cdots S_{\mu_1}^*)(S_{\mu_1}S_{\mu_2}\cdots S_{\mu_{i-1}})S_{\nu_i}\cdots S_{\nu_n} \\ &= S_{\mu_n}^*\cdots S_{\mu_i}^* P_{s(\mu_{i-1})}S_{\nu_i}\cdots S_{\nu_n} \\ &= S_{\mu_n}^*\cdots S_{\mu_i}^* P_{r(\mu_i)}S_{\nu_i}\cdots S_{\nu_n} \\ &= S_{\mu_n}^*\cdots S_{\mu_i}^* S_{\nu_i}\cdots S_{\nu_n}, \end{aligned}$$

which vanishes by Proposition 1.12(b), giving (a).

For part (b), assume first that $n := |\mu| \leq |\nu|$, and factor $\nu = \alpha\nu'$ with $|\alpha| = n$. Then

$$S_\mu^* S_\nu = S_\mu^*(S_\alpha S_{\nu'}) = (S_\mu^* S_\alpha)S_{\nu'}.$$

If $\mu = \alpha$, then (1.3) implies that

$$S_\mu^* S_\nu = P_{s(\mu)}S_{\nu'} = P_{r(\nu')}S_{\nu'} = S_{\nu'}.$$

If $\mu \neq \alpha$, then part (a) implies that $S_\mu^* S_\nu = (S_\mu^* S_\alpha)S_{\nu'} = 0$. This gives (b) when $|\mu| \leq |\nu|$. When $|\mu| > |\nu|$, we can either run a similar argument factoring $\mu = \beta\mu'$, or take adjoints and apply what we have just proved.

Parts (c) and (d) follow from the corresponding parts of Proposition 1.12. □

COROLLARY 1.15. *Suppose that E is a row-finite graph and $\{S, P\}$ is a Cuntz-Krieger E-family in a C^*-algebra B. For μ, ν, α, $\beta \in E^*$, we have*

$$(1.4) \qquad (S_\mu S_\nu^*)(S_\alpha S_\beta^*) = \begin{cases} S_{\mu\alpha'}S_\beta^* & \text{if } \alpha = \nu\alpha' \\ S_\mu S_{\beta\nu'}^* & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise.} \end{cases}$$

In particular, it follows that every non-zero finite product of the partial isometries S_e and S_f^* has the form $S_\mu S_\nu^*$ for some $\mu, \nu \in E^*$ with $s(\mu) = s(\nu)$.

PROOF. The formula follows from part (b) of Corollary 1.14. To see the last statement, we suppose that W is a non-zero word — that is, a product of finitely many S_e and S_f^*. Any adjacent S_e's can be combined into a single term S_μ, and since W is non-zero, μ must be a path. Similarly, any adjacent S_f^*'s can be combined into an S_ν^* for some $\nu \in E^*$. Thus W is a product of terms of the form $S_\mu S_\nu^*$ for $\mu, \nu \in E^*$. (Since $E^0 \subset E^*$, we can write S_ν^*, for example, as $S_{s(\nu)}S_\nu^* = P_{s(\nu)}S_\nu^*$.) The formula (1.4) implies that we can combine this product into one term of the same form. □

COROLLARY 1.16. *If $\{S, P\}$ is a Cuntz-Krieger E-family for a row-finite graph E, then*

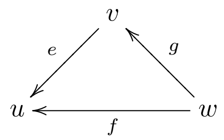
$$C^*(S, P) = \overline{\operatorname{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^*,\ s(\mu) = s(\nu)\}.$$

PROOF. The formula (1.4) implies that

$$\operatorname{span}\{S_\mu S_\nu^* : \mu, \nu \in E^*,\ s(\mu) = s(\nu)\}$$

is a subalgebra of $C^*(S, P)$, and since $(S_\mu S_\nu^*)^* = S_\nu S_\mu^*$, it is a $*$-subalgebra. Thus its closure is a C^*-subalgebra of $C^*(S, P)$, and since it contains the generators $S_e = S_e S_{s(e)}^*$ and $P_v = S_v S_v^*$, it is all of $C^*(S, P)$. □

EXAMPLE 1.17. Let $\{S, P\}$ be a Cuntz-Krieger family for the following directed graph E:



When $s(\mu) = s(\nu)$ we have $S_\mu S_\nu^* = S_\mu P_{s(\mu)} S_\nu^*$; unless $s(\mu) = w$, we can apply the Cuntz-Krieger relation at $s(\mu)$, and keep doing this until the paths begin at w. For example,

$$\begin{aligned} P_u &= S_e S_e^* + S_f S_f^* = S_e P_v S_e^* + S_f S_f^* \\ &= S_e \left(S_g S_g^* \right) S_e^* + S_f S_f^* \\ &= S_{eg} S_{eg}^* + S_f S_f^*. \end{aligned}$$

Thus

$$\begin{aligned} C^*(S, P) &= \overline{\operatorname{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^*,\, s(\mu) = s(\nu) = w\} \\ &= \operatorname{span}\{S_\mu S_\nu^* : \mu, \nu \in \{w, f, g, eg\}\}. \end{aligned}$$

Since w is a source, two paths μ, ν with $s(\mu) = w = s(\nu)$ cannot satisfy $\nu = \mu\nu'$ unless $\mu = \nu$. Hence

$$(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = \begin{cases} S_\mu S_\beta^* & \text{if } \alpha = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\{S_\mu S_\nu^* : \mu, \nu \in \{w, f, g, eg\}\}$ is a set of matrix units which spans $C^*(S, P)$. So we have by Proposition A.5 that if one is non-zero, they all are, and $C^*(S, P)$ is isomorphic to $M_4(\mathbb{C})$.

A path μ in a directed graph E is a *cycle*[2] if $|\mu| \geq 1$, $r(\mu) = s(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for $i \neq j$. In the graph-theory literature, people sometimes insist that $|\mu| > 1$, but this would make our statements much more complicated. The crucial feature of the graph in Example 1.17 is that there are no cycles.

PROPOSITION 1.18. *Suppose E is a finite directed graph with no cycles, and $w_1, \cdots, w_n$ are the sources in E. Then for every Cuntz-Krieger E-family $\{S, P\}$ in which each P_v is non-zero we have*

$$C^*(S, P) \cong \bigoplus_{i=1}^n M_{|s^{-1}(w_i)|}(\mathbb{C}),$$

where $s^{-1}(w_i) = \{\mu \in E^* : s(\mu) = w_i\}$.

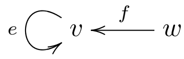
PROOF. As in Example 1.17, finitely many applications of the Cuntz-Krieger relations show that

$$C^*(S, P) = \operatorname{span}\{S_\mu S_\nu^* : s(\mu) = s(\nu) = w_i \text{ for some } i\}$$

and $A_i := \operatorname{span}\{S_\mu S_\nu^* : s(\mu) = s(\nu) = w_i\}$ is isomorphic to $M_{|s^{-1}(w_i)|}(\mathbb{C})$. When $\mu \in s^{-1}(w_i)$ and $\alpha \in s^{-1}(w_j)$ for some $j \neq i$, μ cannot extend α and vice versa. Thus $A_i A_j = 0$, and $C^*(S, P) \cong \bigoplus_{i=1}^n A_i$ by Proposition A.7. $\square$

[2]In the graph-algebra literature, cycles are called *simple loops*.

EXAMPLE 1.19. Consider a Cuntz-Krieger family $\{S, P\}$ for the following directed graph E:



The Cuntz-Krieger relations say that $S_e^*S_e = P_{s(e)} = P_v$, $S_f^*S_f = P_w$ and $P_v = S_eS_e^* + S_fS_f^*$. The element $P_v + P_w$ is an identity for $C^*(S, P)$. The element $S_e + S_f$ satisfies

$$(S_e + S_f)^*(S_e + S_f) = S_e^*S_e + S_f^*S_e + S_e^*S_f + S_f^*S_f = P_v + 0 + 0 + P_w,$$

and hence is an isometry in $C^*(S, P)$. Since

$$(S_e + S_f)(S_e + S_f)^* = S_eS_e^* + S_fS_e^* + S_eS_f^* + S_fS_f^* = P_v,$$

we can recover P_v, $P_w = (S_e + S_f)^*(S_e + S_f) - P_v$, $S_e = (S_e + S_f)P_v$ and $S_f = (S_e + S_f)P_w$ from the single element $S_e + S_f$. Thus $C^*(S, P)$ is generated by the isometry $S_e + S_f$. Conversely, if V is an isometry, then $P_w = 1 - VV^*$, $P_v = VV^*$, $S_e = VP_v$, $S_f = VP_w$ defines a Cuntz-Krieger E-family such that $C^*(S, P) = C^*(V)$.

Coburn's theorem [**93**, Theorem 3.5.18] says that all C^*-algebras generated by one non-unitary isometry are isomorphic, and in particular isomorphic to the *Toeplitz algebra* $\mathcal{T}$ generated by the unilateral shift. The isometry $S_e + S_f$ is non-unitary precisely when $P_w \neq 0$, so we deduce that all Cuntz-Krieger E-families with $P_w \neq 0$ generate C^*-algebras isomorphic to $\mathcal{T}$.

Proposition 1.18 and Example 1.19 suggest that, provided two Cuntz-Krieger families are non-trivial in the sense that appropriate vertex projections P_v are non-zero, the Cuntz-Krieger families generate isomorphic C^*-algebras. This is indeed a general phenomenon. To study it, we introduce a C^*-algebra which is universal for C^*-algebras generated by Cuntz-Krieger E-families, and analyse the representations of this C^*-algebra.

To build the universal C^*-algebra generated by a Cuntz-Krieger E-family, we mimic the behaviour of the spanning set $\{S_\mu S_\nu^*\}$. In the next proposition, the symbols $d_{\mu,\nu}$ are purely formal, all but finitely many coefficients $z_{\mu,\nu}$ in each sum are 0, and the vector space operations on the formal sums are defined by

$$a\big(\sum w_{\mu,\nu}d_{\mu,\nu}\big) + b\big(\sum z_{\mu,\nu}d_{\mu,\nu}\big) = \sum(aw_{\mu,\nu} + bz_{\mu,\nu})d_{\mu,\nu}.$$

The elements $d_{\alpha,\beta}$ obtained by setting $z_{\alpha,\beta} = 1$ and $z_{\mu,\nu} = 0$ otherwise then form a basis for V.

PROPOSITION 1.20. *Let E be a row-finite directed graph. Then the vector space V of formal linear combinations*

$$V = \big\{\sum z_{\mu,\nu}d_{\mu,\nu} : \mu, \nu \in E^*, s(\mu) = s(\nu)\big\}$$

is a $*$-algebra with $(d_{\mu,\nu})^* = d_{\nu,\mu}$ and

$$d_{\mu,\nu}d_{\alpha,\beta} = \begin{cases} d_{\mu\alpha',\beta} & \text{if } \alpha = \nu\alpha' \\ d_{\mu,\beta\nu'} & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise.} \end{cases}$$

To prove this, one has to check that the product is associative and compatible with the $*$-operation. This is tedious but routine.

For every Cuntz-Krieger family $\{S, P\}$ on $\mathcal{H}$, the operators $\{S_\mu S_\nu^*\}$ satisfy the relations imposed on the $\{d_{\mu,\nu}\}$, and hence there is a $*$-representation $\pi_{S,P}$ of V

on $\mathcal{H}$ such that $\pi_{S,P}(d_{\mu,\nu}) = S_\mu S_\nu^*$. Since the norm of a projection P satisfies $\|P\|^2 = \|P^*P\| = \|P\|$, every non-zero projection has norm 1, and thus for every non-zero partial isometry W, we have $\|W\|^2 = \|W^*W\| = 1$. Thus

$$\big\|\pi_{S,P}\big(\textstyle\sum z_{\mu,\nu} d_{\mu,\nu}\big)\big\| \leq \sum |z_{\mu,\nu}|\, \|S_\mu S_\nu^*\| \leq \sum |z_{\mu,\nu}|.$$

It follows that

$$\|a\|_1 := \sup\{\|\pi_{S,P}(a)\| : \{S,P\} \text{ is a Cuntz-Krieger } E\text{-family}\}$$

is finite for every v in V, and $\|\cdot\|_1$ is an algebra seminorm satisfying $\|a^*a\|_1 = \|a\|_1^2$. Let I be the $*$-ideal $\{u \in V : \|u\|_1 = 0\}$. Then $V_0 = V/I$ is a $*$-algebra, and the quotient norm $\|\cdot\|_0$ defined by $\|v+I\|_0 = \inf\{\|u+j\|_1 : j \in I\}$ is a C^*-norm, so the completion $\overline{V_0}$ is a C^*-algebra. Each $\pi_{S,P}$ is $\|\cdot\|_0$-continuous, and hence extends uniquely to a representation of $\overline{V_0}$.

We have now outlined the main steps in the proof of:

Proposition 1.21. *For any row-finite directed graph E, there is a C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{s,p\}$ such that for every Cuntz-Krieger E-family $\{T,Q\}$ in a C^*-algebra B, there is a homomorphism $\pi_{T,Q}$ of $C^*(E)$ into B satisfying $\pi_{T,Q}(s_e) = T_e$ for every $e \in E^1$ and $\pi_{T,Q}(p_v) = Q_v$ for every $v \in E^0$.*

Proof. Take $C^*(E) = \overline{V_0}$, and check that $s_e := d_{e,s(e)}$, $p_v := d_{v,v}$ form a Cuntz-Krieger E-family which generates V_0. To get $\pi_{T,Q}$, choose a faithful representation $\rho : B \to B(\mathcal{H})$, and take $\pi_{T,Q} = \rho^{-1} \circ \pi_{\rho(T),\rho(Q)}$. □

The C^*-algebra $C^*(E)$ is called the *C^*-algebra of the graph E* or the *Cuntz-Krieger algebra of E*, and is generically described as a *graph algebra*. In these notes, $\{s,p\}$ will always be the universal family which generates $C^*(E)$; in general, we will try to use lower-case letters for a Cuntz-Krieger family only when we think the family has a universal property.

Those whose native languages have definite and indefinite articles may have noticed that we have been making implicit uniqueness assertions about $C^*(E)$ and $\{s,p\}$. We take the view that the next corollary justifies this, and that it is okay to talk about “the” graph algebra $C^*(E)$ and “the” generating family $\{s,p\}$, provided we remember when it matters that they are only unique up to isomorphism in the following precise sense.

Corollary 1.22. *Suppose E is a row-finite directed graph, and C is a C^*-algebra generated by a Cuntz-Krieger E-family $\{w,r\}$ such that for every Cuntz-Krieger E-family $\{T,Q\}$ in a C^*-algebra B, there is a homomorphism $\rho_{T,Q}$ of C into B satisfying $\rho_{T,Q}(w_e) = T_e$ for every $e \in E^1$ and $\rho_{T,Q}(r_v) = Q_v$ for every $v \in E^0$. Then there is an isomorphism ϕ of $C^*(E)$ onto C such that $\phi(s_e) = w_e$ for every $e \in E^1$ and $\phi(p_v) = r_v$ for every $v \in E^0$.*

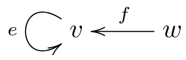
Proof. We take $\phi := \pi_{w,r}$. It is onto because the range of $\pi_{w,r}$ is a C^*-algebra containing $\{w_e, r_v\}$, hence is all of C. Since $\rho_{s,p} \circ \pi_{w,r}$ is the identity on $\{s,p\}$, it is the identity on all of $C^*(E)$. Thus

$$\phi(a) = 0 \Longrightarrow \pi_{w,r}(a) = 0 \Longrightarrow a = \rho_{s,p}(\pi_{w,r}(a)) = 0,$$

and ϕ is injective. □

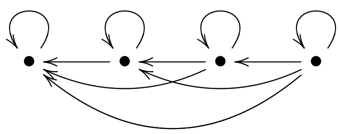
EXAMPLE 1.23. For the graph E which consists of a single loop at a single vertex v, Cuntz-Krieger E-families $\{S, P\}$ are determined by the single operator S_e, which is a unitary operator of $P_v\mathcal{H}$ onto $P_v\mathcal{H}$. The operator P_v is an identity for $C^*(S, P)$, and S_e is a unitary element of $C^*(S, P)$. So $(C^*(E), s_e)$ is universal for C^*-algebras generated by a unitary element: if U is a unitary element of a C^*-algebra B, then there is a homomorphism $\pi_U : C^*(E) \to B$ such that $\pi(s_e) = U$. We know from spectral theory that if $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and $\iota : \mathbb{T} \to \mathbb{C}$ is the function $\iota(z) = z$, then $(C(\mathbb{T}), \iota)$ has this same universal property. So Corollary 1.22 gives an isomorphism ϕ of $C^*(E)$ onto $C(\mathbb{T})$ such that $\phi(s_e) = \iota$.

EXAMPLE 1.24. In Example 1.19 we considered Cuntz-Krieger families $\{S, P\}$ for the following directed graph E:



We saw there that $S_e + S_f$ is an isometry which generates $C^*(S, P)$, and that every isometry on Hilbert space gives a Cuntz-Krieger E-family. Thus $(C^*(E), s_e + s_f)$ is the universal C^*-algebra (A, a) generated by an isometry a. In the representation-theoretic analysis of (A, a), Coburn's theorem becomes the assertion that, if π is a representation of A and $\pi(a)$ is non-unitary, then π is faithful on A (see [**1**], for example).

EXAMPLE 1.25. For a more exotic example, consider the following graph E:



Hong and Szymański prove in [**56**, Theorem 4.4] that $C^*(E)$ is isomorphic to the non-commutative sphere $C(S_q^7)$ of Vaksman and Soibelman, by checking that $C(S_q^7)$ has the universal property which characterises $C^*(E)$ and applying Corollary 1.22. In [**56**] and [**58**], they show that a broad range of non-commutative spheres, projective spaces and lens spaces are isomorphic to the C^*-algebras of suitable directed graphs.

REMARK 1.26. We have insisted that our graphs are countable, and hence all our graph algebras are separable. However, we have not really used this hypothesis, and one can talk about graph algebras of uncountable graphs. Katsura has recently shown that there is a graph E with uncountably many vertices whose C^*-algebra is prime but not primitive [**77**, Proposition 13.4]. Of course one can take this two ways: as evidence that uncountable graphs are interesting, or as evidence that they should be avoided.

CHAPTER 2

Uniqueness theorems for graph algebras

Since the C^*-algebra of a graph E has a universal property, we can prove that a C^*-algebra B is isomorphic to $C^*(E)$ by finding a Cuntz-Krieger E-family $\{T,Q\}$ which generates B and has the universal property (see Corollary 1.22). The first two big theorems of the subject say that it is often not necessary to check that $\{T,Q\}$ has the universal property. In this chapter we discuss these theorems and their implications, and in the next chapter we will prove them.

The first of these uniqueness theorems says that $C^*(E)$ is characterised by the existence of a special action of the circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, which is a compact topological group under multiplication. In general, an *action* of a locally compact group G on a C^*-algebra A is a homomorphism $s \mapsto \alpha_s$ of G into the group $\operatorname{Aut} A$ of automorphisms of A such that $s \mapsto \alpha_s(a)$ is continuous for each fixed $a \in A$. The particular action γ constructed in the next proposition is called the *gauge action* of $\mathbb{T}$ on $C^*(E)$.

Proposition 2.1. *Let E be a row-finite directed graph. Then there is an action γ of $\mathbb{T}$ on $C^*(E)$ such that $\gamma_z(s_e) = zs_e$ for every $e \in E^1$ and $\gamma_z(p_v) = p_v$ for every $v \in E^0$.*

Proof. Fix $z \in \mathbb{T}$. Then $\{zs, p\} = \{zs_e, p_v\}$ is a Cuntz-Krieger E-family which generates $C^*(E)$. If $\{T,Q\}$ is a Cuntz-Krieger E-family in a C^*-algebra B, then so is $\{\overline{z}T, Q\}$, and

$$\pi_{\overline{z}T,Q}(zs_e) = z\pi_{\overline{z}T,Q}(s_e) = z(\overline{z}T_e) = T_e.$$

Thus with $\rho_{T,Q} := \pi_{\overline{z}T,Q}$, the pair $(C^*(E), \{zs,p\})$ has the property described in Corollary 1.22, and hence there is an isomorphism $\gamma_z : C^*(E) \to C^*(E)$ such that $\gamma_z(s_e) = zs_e$ and $\gamma_z(p_v) = p_v$. For $w \in \mathbb{T}$, the isomorphisms $\gamma_z \circ \gamma_w$ and γ_{zw} agree on generators, and hence on all of $C^*(E)$. So γ is a homomorphism of $\mathbb{T}$ into $\operatorname{Aut} C^*(E)$.

To check continuity, fix $z \in \mathbb{T}$, $a \in C^*(E)$ and $\epsilon > 0$. Choose $c := \sum \lambda_{\mu,\nu} s_\mu s_\nu^*$ such that $\|a - c\| < \epsilon/3$. Notice that $\gamma_z(s_\mu) = z^{|\mu|}s_\mu$. Thus, since scalar multiplication is continuous, so is

$$w \mapsto \gamma_w(c) = \sum \lambda_{\mu,\nu} w^{|\mu|-|\nu|} s_\mu s_\nu^*,$$

and there exists $\delta > 0$ such that $|w - z| < \delta \implies \|\gamma_w(c) - \gamma_z(c)\| < \epsilon/3$. Since automorphisms of C^*-algebras preserve the norm, we have $\|\gamma_z(a-c)\| < \epsilon/3$. Thus for $|w - z| < \delta$ we have

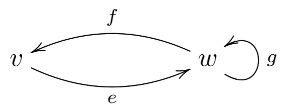
$$\|\gamma_w(a) - \gamma_z(a)\| \le \|\gamma_w(a-c)\| + \|\gamma_w(c) - \gamma_z(c)\| + \|\gamma_z(a-c)\| < 3(\epsilon/3) = \epsilon,$$

as required. □

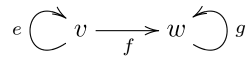
THEOREM 2.2 (The gauge-invariant uniqueness theorem). *Let E be a row-finite directed graph, and suppose that $\{T, Q\}$ is a Cuntz-Krieger E-family in a C^*-algebra B with each $Q_v \neq 0$. If there is a continuous action $\beta : \mathbb{T} \to \operatorname{Aut} B$ such that $\beta_z(T_e) = zT_e$ for every $e \in E^1$ and $\beta_z(Q_v) = Q_v$ for every $v \in E^0$, then $\pi_{T,Q}$ is an isomorphism of $C^*(E)$ onto $C^*(T, Q)$.*

Notice that the gauge-invariant uniqueness theorem has no hypotheses on the graph, and hence it is very useful for proving general statements about graph algebras. For a wide class of graphs, though, we can do even better. Recall that a *cycle* in a directed graph E is a path $\mu = \mu_1 \cdots \mu_n$ with $n \geq 1$, $s(\mu_n) = r(\mu_1)$ and $s(\mu_i) \neq s(\mu_j)$ for $i \neq j$. An edge e is an *entry* to the cycle μ if there exists i such that $r(e) = r(\mu_i)$ and $e \neq \mu_i$. Our second uniqueness theorem says that, provided every cycle has an entry, all non-trivial Cuntz-Krieger families generate isomorphic C^*-algebras. Notice that this hypothesis may be trivially satisfied if, for example, E has no cycles.

EXAMPLE 2.3. In the following directed graph E



ef, fe and g are cycles; fge and $efef$ are closed paths but are not cycles, because they visit w twice. Every cycle has an entry; for example, g is an entry to ef. However, in the graph F



the cycle e has no entry.

THEOREM 2.4 (The Cuntz-Krieger uniqueness theorem). *Suppose E is a row-finite directed graph in which every cycle has an entry, and $\{T, Q\}$ is a Cuntz-Krieger E-family in a C^*-algebra B such that $Q_v \neq 0$ for every $v \in E^0$. Then the homomorphism $\pi_{T,Q} : C^*(E) \to B$ is an isomorphism of $C^*(E)$ onto $C^*(T, Q)$.*

As the name suggests, this theorem is essentially due to Cuntz and Krieger [**16**] (see Remark 2.17 below). The name is often used to refer to the following consequence of Theorem 2.4, which looks more like their original uniqueness theorem.

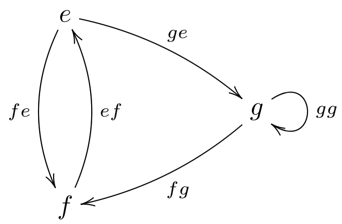
COROLLARY 2.5 (The Cuntz-Krieger uniqueness theorem). *Suppose E is a row-finite directed graph in which every cycle has an entry. If $\{S, P\}$ and $\{T, Q\}$ are two Cuntz-Krieger E-families on Hilbert space such that $P_v \neq 0$ and $Q_v \neq 0$ for all $v \in E^0$, then then there is an isomorphism ϕ of $C^*(S, P)$ onto $C^*(T, Q)$ such that $\phi(S_e) = T_e$ for every $e \in E^1$ and $\phi(P_v) = Q_v$ for every $v \in E^0$.*

PROOF. Theorem 2.4 implies that both $\pi_{S,P}$ and $\pi_{T,Q}$ are faithful representations of $C^*(E)$. Then $\phi := \pi_{T,Q} \circ \pi_{S,P}^{-1}$ has the required properties. $\square$

We will prove these uniqueness theorems in the next chapter, but first we want to illustrate how they are used. We begin with two applications of the gauge-invariant uniqueness theorem. As we will see in Remark 2.8, the first has historical interest.

COROLLARY 2.6. *Suppose E is a row-finite directed graph with no sources, and define the* dual graph $\widehat{E}$ *by* $\widehat{E}^0 = E^1$, $\widehat{E}^1 = E^2$, $r_{\widehat{E}}(ef) = e$ *and* $s_{\widehat{E}}(ef) = f$. *Then* $\widehat{E}$ *is row-finite and* $C^*(\widehat{E}) \cong C^*(E)$.

EXAMPLE 2.7. For the graph E of Example 2.3, the dual graph $\widehat{E}$ looks like



PROOF OF COROLLARY 2.6. For $e \in \widehat{E}^0 = E^1$,

$$\#{r_{\widehat{E}}}^{-1}(e) = \#\{ef \in E^2 = \widehat{E}^1\} = \#\{f : s_E(e) = r_E(f)\} = \#r_E^{-1}(s_E(e))$$

is finite because E is row-finite. Thus $\widehat{E}$ is row-finite.

Let $\{s, p\}$ be the universal Cuntz-Krieger family generating $C^*(E)$, and define $Q_e := s_e s_e^*$, $T_{fe} := s_f s_e s_e^*$. The projections Q_e are mutually orthogonal because the range projections in any Cuntz-Krieger family are mutually orthogonal. For $fe \in \widehat{E}^1$ we have

$$\begin{aligned} T_{fe}^* T_{fe} &= (s_f s_e s_e^*)^*(s_f s_e s_e^*) = s_e s_e^* s_f^* s_f s_e s_e^* \\ &= s_e s_e^* p_{s(f)} s_e s_e^* = s_e s_e^* p_{r(e)} s_e s_e^* \\ &= s_e s_e^* = Q_e = Q_{s(fe)}; \end{aligned}$$

since Q_e is a projection, this also implies that T_{fe} is a partial isometry. To verify the Cuntz-Krieger $\widehat{E}$-relation at $f \in \widehat{E}^0$, we compute

$$\begin{aligned} Q_f &= s_f s_f^* = s_f p_{s(f)} s_f^* \\ &= s_f \Big(\sum_{r(e)=s(f)} s_e s_e^* \Big) s_f^* = s_f \Big(\sum_{r(e)=s(f)} (s_e s_e^*)(s_e s_e^*) \Big) s_f^* \\ &= \sum_{r(e)=s(f)} s_f (s_e s_e^*)(s_e s_e^*) s_f^* = \sum_{r(fe)=f} T_{fe} T_{fe}^*. \end{aligned}$$

Thus $\{T, Q\}$ is a Cuntz-Krieger $\widehat{E}$-family, and the universal property of $C^*(\widehat{E})$ gives a homomorphism $\pi_{T,Q} : C^*(\widehat{E}) \to C^*(E)$ which carries the canonical generating Cuntz-Krieger family $\{t, q\}$ into $\{T, Q\}$. We can check on generators that the homomorphism $\pi_{T,Q}$ intertwines the gauge actions, and since all the s_e are non-zero, so are the projections Q_e. Thus the gauge-invariant uniqueness theorem implies that $\pi_{T,Q}$ is an isomorphism of $C^*(\widehat{E})$ onto $C^*(T, Q)$. Since the generators $p_v = \sum_{r(e)=v} Q_e$ and $s_f = \sum_{s(f)=r(e)} T_{fe}$ of $C^*(E)$ lie in the range of $\pi_{T,Q}$ and the range of $\pi_{T,Q}$ is a C^*-algebra, $\pi_{T,Q}$ is surjective. (It is at this last step that we need to know that E has no sources: if v is a source, we cannot recover p_v from the Q_e.) □

REMARK 2.8. In their pioneering paper [**16**], Cuntz and Krieger considered an $n \times n$ matrix $A = (a_{ij})$ with each entry a_{ij} either 0 or 1 and no zero rows or columns. Nowadays, we define the *Cuntz-Krieger algebra* $\mathcal{O}_A$ to be the universal

C^*-algebra generated by partial isometries s_i satisfying $s_i^* s_i = \sum_j a_{ij} s_j s_j^*$. (For the original definition, see Remark 2.17 below.) We can define a graph E_A by $E_A^0 = \{1, 2, \cdots, n\}$, $E_A^1 = \{ij : a_{ij} = 1\}$, $s(ij) = j$ and $r(ij) = i$, and recover A as the vertex matrix of E_A. The argument of Corollary 2.6 shows that $\mathcal{O}_A$ is isomorphic to $C^*(E_A)$.

The vertex matrix A_E of a finite directed graph E does not necessarily have entries in $\{0, 1\}$, but the *edge matrix* B_E defined by

$$B_E(e, f) = \begin{cases} 1 & \text{if } s(e) = r(f) \\ 0 & \text{otherwise} \end{cases}$$

does. The graph E_{B_E} is the dual graph $\widehat{E}$, so Corollary 2.6 implies that $C^*(E) \cong C^*(\widehat{E}) \cong \mathcal{O}_{B_E}$. Thus the Cuntz-Krieger algebras are the C^*-algebras of finite graphs with no sinks or sources. These connections were made by Enomoto and Watatani [**36**] very soon after [**16**] appeared.

Remark 2.9. In [**6**], Bates extended Corollary 2.6 to the higher-order duals $E_{m,n}$ in which $m < n$, $E_{m,n}^0 = E^m$, $E_{m,n}^1 = E^n$, $s(\mu) = \mu_{n-m+1} \cdots \mu_n$ and $r(\mu) = \mu_1 \cdots \mu_m$. The main result of [**6**] says that $C^*(E_{m,n})$ is naturally isomorphic to $C^*(E_{p,q})$ when $n - m = q - p$, and not in general otherwise. The graphs $E_{0,2}$ were used by Hong and Szymański in their description of the non-commutative real projective spaces (see [**56**, Proposition 1.1]).

Our second application of the gauge-invariant uniqueness theorem is a technique for handling problems involving graphs with sinks and sources. Suppose E is a row-finite directed graph. We form a new graph F by *adding a head*

$$v \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \cdots$$

to each source v, and *adding a tail*

$$\cdots \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow w$$

to each sink w. The idea is that for many purposes, replacing E by F does not change the graph algebra in a substantial way. To make this precise, we need some C^*-algebra background and a lemma.

If A is a C^*-algebra which does not have an identity, such as $C^*(E)$ when E^0 is infinite, there are several ways of embedding it in a larger C^*-algebra with an identity. The *multiplier algebra* $M(A)$ consists of pairs (L, R) of maps from A to itself which satisfy the relation $aL(b) = R(a)b$. The maps L and R in such a pair are automatically bounded and linear with the same operator norm, and $M(A)$ is a C^*-algebra with $\|(L, R)\| = \|L\| = \|R\|$, $(L_1, R_1)(L_2, R_2) = (L_1 \circ L_2, R_2 \circ R_1)$ and $(L, R)^* = (R^\sharp, L^\sharp)$, where $R^\sharp(a) = R(a^*)^*$ (see [**93**, Theorem 2.1.5], for example). For $a \in A$, the pair (L_a, R_a) defined by $L_a(b) = ab$ and $R_a(b) = ba$ is a multiplier of A, and the map $a \mapsto (L_a, R_a)$ is an isomorphism of A onto an ideal A in $M(A)$. In general we think of the maps L_m, R_m defining a multiplier $m \in M(A)$ as left and right multiplication by m, and write ma for $L_m(a)$ and am for $R_m(a)$. Motivating examples are $A = \mathcal{K}(\mathcal{H})$, when $T \mapsto (L_T, R_T)$ is an isomorphism of $B(\mathcal{H})$ onto $M(\mathcal{K}(\mathcal{H}))$, and $A = C_0(X)$ for X locally compact, when pointwise multiplication of functions gives an isomorphism of the C^*-algebra $C_b(X)$ of bounded continuous functions onto $M(C_0(X))$. (These are proved in [**114**, Proposition 2.55]; a different (equivalent) definition of $M(A)$ is used in [**114**], but the ideas of the proof work with both definitions.)

For us, multiplier algebras arise because we want to make sense of infinite sums of vertex projections in $C^*(E)$. An infinite sum $\sum_{n=1}^{\infty} p_n$ of mutually orthogonal projections cannot converge in norm, because each difference $\sum_{n=M+1}^{N} p_n$ between partial sums is a projection and hence has norm 1. However, we can safely think of the projection p_V in the following lemma as $\sum_{v\in V} p_v$.

LEMMA 2.10. *Let E be a row-finite graph, and let V be an infinite set of vertices in E. Then there is a projection p_V in $M(C^*(E))$ such that*

$$p_V s_\mu s_\nu^* = \begin{cases} s_\mu s_\nu^* & \text{if } r(\mu) \in V \\ 0 & \text{if } r(\mu) \notin V. \end{cases} \tag{2.1}$$

PROOF. Since the elements $s_\mu s_\nu^*$ span a dense subspace of $C^*(E)$ and left multiplication by a multiplier is linear and continuous, there is at most one multiplier p_V satisfying (2.1). So while we choose to list V as $V = \{v_1, v_2, \cdots\}$, the multiplier we define will not depend on this choice of listing.

For $N \in \mathbb{N}$, let $p_N = \sum_{n=1}^{N} p_{v_n}$. Then

$$p_N s_\mu s_\nu^* = \begin{cases} s_\mu s_\nu^* & \text{if } r(\mu) = v_n \text{ for some } n \le N \\ 0 & \text{otherwise,} \end{cases}$$

and hence for $a \in A_0 := \operatorname{span}\{s_\mu s_\nu^* : s(\mu) = s(\nu)\}$, the sequence $\{p_N a : N \in \mathbb{N}\}$ is eventually constant. We define $L, R : A_0 \to A_0$ by $L(a) = \lim_{N\to\infty} p_N a$ and $R(a) = \lim_{N\to\infty} a p_N$. Since each p_N is a projection and hence has norm 1, both L and R are continuous, and extend to linear maps of $C^*(E) = \overline{A_0}$ into itself; an $\epsilon/3$ argument shows that we still have $L(b) = \lim_{N\to\infty} p_N b$ for all $b \in C^*(E)$. Thus for $a, b \in C^*(E)$, we have

$$aL(b) = a\big(\lim_{N\to\infty} p_N b\big) = \lim_{N\to\infty} a(p_N b) = \lim_{N\to\infty}(a p_N)b = R(a)b,$$

and $p_V := (L, R)$ is a multiplier of $C^*(E)$. For each $s_\mu s_\nu^*$, $r(\mu) \in V$ if and only if $r(\mu) = v_n$ for some n, and hence if and only if $p_N s_\mu s_\nu^* = s_\mu s_\nu^*$ for large N. Thus p_V satisfies (2.1). Since $p_M p_N = p_M$ for $N \ge M$, we have

$$p_V(p_V b) = \lim_M p_M(\lim_N p_N b) = \lim_M(\lim_N p_M p_N b) = \lim_M p_M b = p_V b,$$

so $p_V^2 = p_V$. Since it is easy to check that $p_V^* = p_V$, it follows that p_V is a projection. □

If p is a projection in the multiplier algebra $M(A)$ of a C^*-algebra A, then the set $pAp := \{pap : a \in A\}$ is a C^*-subalgebra of A, which is called a *corner* of A. The corner is *full* if it is not contained in any proper ideal of A, or in other words if the set ApA spans a dense subspace of A. The motivating example is the projection $p = e_{11}$ in $A = M_n(\mathbb{C})$, when $e_{11}M_n(\mathbb{C})e_{11}$ is the algebra of matrices which are non-zero only in the top left-hand corner. As we shall see later, full corners inherit many important properties from the ambient C^*-algebra A.

We can now state our second application of the gauge-invariant uniqueness theorem.

COROLLARY 2.11. *Suppose that E is a row-finite directed graph and F is the directed graph obtained by adding a head to every source of E and a tail to every sink. Denote by $\{s, p\}$ and $\{t, q\}$ the canonical Cuntz-Krieger families generating $C^*(E)$ and $C^*(F)$, and let q_E be the projection in $M(C^*(F))$ obtained by applying Lemma 2.10 to the subset E^0 of F^0. Then $q_E C^*(F) q_E$ is a full corner in $C^*(F)$,* *and there is an isomorphism ϕ of $C^*(E)$ onto $q_E C^*(F) q_E$ such that $\phi(s_e) = t_e$ for $e \in E^1$ and $\phi(p_v) = q_v$ for $v \in E^0$.*

Proof. Because there is no Cuntz-Krieger relation at a source in E, and because all the edges entering a sink in E are in E^1, the elements $\{t_e : e \in E^1\}$ and $\{q_v : v \in E^0\}$ form a Cuntz-Krieger E-family in $C^*(F)$. The resulting homomorphism $\phi := \pi_{t,q} : C^*(E) \to C^*(F)$ intertwines the gauge actions, and hence the gauge-invariant uniqueness theorem implies that ϕ is injective.

For $\mu, \nu \in F^*$ with $s(\mu) = s(\nu)$, we have

$$q_E t_\mu t_\nu^* q_E = \begin{cases} t_\mu t_\nu^* & \text{if } r(\mu) \in E^0 \text{ and } r(\nu) \in E^0 \\ 0 & \text{otherwise.} \end{cases}$$

If $r(\mu), r(\nu) \in E^0$, then either $s(\mu) = s(\nu) \in E^0$ or $s(\mu) = s(\nu)$ lies on one of the heads we have added at a source w, in which case there is a final segment α of the head such that $\mu = \mu'\alpha$ and $\nu = \nu'\alpha$, and

$$t_\mu t_\nu^* = t_{\mu'} t_\alpha t_\alpha^* t_{\nu'}^* = t_{\mu'} p_w t_{\nu'}^* = t_{\mu'} t_{\nu'}^*.$$

Thus every non-zero element of the form $q_E t_\mu t_\nu^* q_E$ is equal to one of the form $t_{\mu'} t_{\nu'}^*$ with $r(\mu')$, $r(\nu')$ and $s(\mu') = s(\nu')$ all in E^0. Since $a \mapsto q_E a q_E$ is continuous and linear, we deduce that

$$\begin{aligned} q_E C^*(F) q_E &= \overline{\operatorname{span}}\{t_\mu t_\nu^* : r(\mu) \in E^0,\ r(\nu) \in E^0 \text{ and } s(\mu) = s(\nu) \in E^0\} \\ &= \pi_{t,q}(C^*(E)). \end{aligned}$$

To see that $q_E C^*(F) q_E$ is full, suppose first that x is a vertex on a head. Then there is a unique path α from x to a source w of E, and $t_\alpha = q_w t_\alpha$, so t_α and $q_x = t_\alpha^* t_\alpha$ belong to the ideal generated by $q_E C^*(F) q_E$. It follows that every $t_f = q_{r(f)} t_f$ associated to an edge f on a head is also in this ideal. A similar argument works on tails, so the entire generating family $\{t_f, q_v\}$ lies in the ideal generated by $q_E C^*(F) q_E$. Thus this ideal is all of $C^*(F)$, and q_E is full. □

Corners arise frequently in this subject, and we digress briefly to discuss another situation where they arise. We say that a C^*-algebra A is *approximately finite-dimensional*, or that A is an *AF-algebra*, if there is a sequence of finite-dimensional C^*-subalgebras A_n such that $A_n \subset A_{n+1}$ and $A = \overline{\bigcup_{n=1}^\infty A_n}$. We will prove a theorem of Drinen which says that every AF-algebra is isomorphic to a corner in a graph algebra. For clarity, we assume here that A is unital; the details in the non-unital case are in [**28**] and [**145**].

A *Bratteli diagram* is a locally finite directed graph E with no sources and just one sink v_0, in which E^0 is partitioned as a disjoint union $E^0 = \bigcup_{n=0}^\infty V_n$ of finite sets V_n such that $V_0 = \{v_0\}$, and such that for each edge e there exists $n \in \mathbb{N}$ such that $s(e) \in V_n$ and $r(e) \in V_{n-1}$. (Warning: we have had to reverse the usual convention to fit our definition of Cuntz-Krieger family.) For each $v \in E^0$, we denote by n_v the number of paths $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = v_0$. A Bratteli diagram E is a *Bratteli diagram of an AF-algebra A* if there is an increasing sequence of finite-dimensional unital C^*-subalgebras A_n isomorphic to $\bigoplus_{v \in V_n} M_{n_v}(\mathbb{C})$ such that $A = \overline{\bigcup_{n=1}^\infty A_n}$, and such that the inclusion of A_n in A_{n+1} maps $M_{n_v}(\mathbb{C})$ into $M_{n_w}(\mathbb{C})$ with multiplicity

$$A_E(v, w) := \#\{e \in E^1 : r(e) = v, s(e) = w\},$$

so that the dimensions of the $M_{n_v}(\mathbb{C})$ satisfy $n_w = \sum_{v\in V_n} A_E(v,w)n_v$. Bratteli investigated the relationship between the structure of AF-algebras and the associated Bratteli diagrams [**11**], and in particular proved that two unital AF-algebras with the same diagram are isomorphic (see [**18**, Proposition III.2.7]).

PROPOSITION 2.12. *Let A be a unital AF-algebra and let $(E, \{V_n\}, v_0)$ be a Bratteli diagram for A. Then there is an isomorphism of A onto $p_{v_0}C^*(E)p_{v_0}$, and this is a full corner of $C^*(E)$.*

PROOF. For each n and $v \in V_n$, we let

$$B_n(v) := \operatorname{span}\{s_\mu s_\nu^* : r(\mu) = r(\nu) = v_0, s(\mu) = s(\nu) = v\}.$$

The elements $s_\mu s_\nu^*$ are matrix units which span $B_n(v)$; since there are n_v paths μ with $s(\mu) = v$ and $r(\mu) = v_0$, it follows from Proposition A.5 that $B_n(v)$ is isomorphic to $M_{n_v}(\mathbb{C})$. If w is another vertex in V_n, then $B_n(v)B_n(w) = 0$, and hence it follows from Proposition A.7 that

$$B_n := \operatorname{span}\{s_\mu s_\nu^* : r(\mu) = r(\nu) = v_0, s(\mu) = s(\nu) \in V_n\}$$

is the C^*-algebraic direct sum $\bigoplus_{v\in V_n} B_n(v)$. The Cuntz-Krieger relations tell us how B_n embeds in B_{n+1}: if $r(\mu) = r(\nu) = v_0$ and $v = s(\mu) = s(\nu) \in V_n$, then

$$s_\mu s_\nu^* = s_\mu p_{s(\mu)} s_\nu^* = \sum_{r(e)=s(\mu)} s_\mu s_e s_e^* s_\nu^* = \sum_{r(e)=s(\mu)} s_{\mu e} s_{\nu e}^*$$

contains exactly $A_E(v,w)$ terms in which $s(\mu e) = w = s(\nu e)$. Thus $B := \overline{\bigcup_{n=1}^\infty B_n}$ is an AF-algebra with Bratteli diagram E.

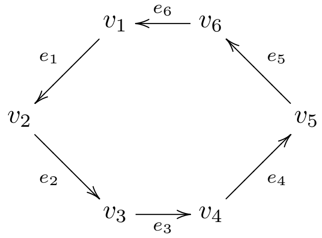
As in the proof of Corollary 2.11, compressing with p_{v_0} shows that

$$p_{v_0}C^*(E)p_{v_0} = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu,\nu \in E^*, s(\mu) = s(\nu) \text{ and } r(\mu) = r(\nu) = v_0\},$$

which is B. So the theorem of Bratteli we quoted above implies that A is isomorphic to $p_{v_0}C^*(E)p_{v_0}$. To see that the corner is full, let $v \in E^0$, and choose a path μ with $s(\mu) = v$ and $r(\mu) = v_0$. Then $s_\mu = p_{v_0}s_\mu$ belongs to the ideal generated by $p_{v_0}C^*(E)p_{v_0}$, and hence so does $p_v = s_\mu^* s_\mu$. Thus every vertex projection is in the ideal, and so is every $s_e = s_e p_{s(e)}$. □

REMARK 2.13. The graph algebra $C^*(E)$ of the Bratteli diagram E is itself AF, though it is not unital. Indeed, it is shown in [**82**, Theorem 2.4] that the C^*-algebra of a row-finite graph E is AF if and only if the graph E contains no cycles. This certainly applies to any Bratteli diagram, but it is easy to draw infinite graphs without cycles which are not Bratteli diagrams.

EXAMPLE 2.14. To see how there can fail to be uniqueness when there is no gauge action, consider the graph C_n which consists of a single cycle with n edges. Label the vertices as $\{v_1, v_2, \cdots, v_n\}$ and the edges as $\{e_1, e_2, \cdots, e_n\}$ in such a way that $s(e_i) = v_i$. For example, C_6 looks like



Let P_{v_i} be the $n \times n$ matrix unit e_{ii}, let S_{e_i} be the matrix unit $e_{(i+1)i}$ for $i < n$, and let $S_{e_n} = e_{1n}$. Then $\{S, P\}$ is a Cuntz-Krieger C_n-family which generates $M_n(\mathbb{C})$. However, $\pi_{S,P}$ is definitely not an isomorphism. The problem is that this family satisfies extra relations like $P_{v_1} = S_\mu := S_{e_n} \cdots S_{e_2} S_{e_1}$, whereas the Cuntz-Krieger relations demand only that S_μ has P_{v_1} as its initial and final projection, or in other words that S_μ is a unitary on the range of P_{v_1}. This suggests the fix: the C^*-algebra generated by the unitary could be any quotient of $C(\mathbb{T})$, so we need to insert a copy of $\mathbb{T}$.

Consider the C^*-algebra $C(\mathbb{T}, M_n(\mathbb{C}))$ with operations defined pointwise and norm defined in terms of the usual norm on $M_n(\mathbb{C})$ by $\|f\| := \sup_z \|f(z)\|$. We claim that if we set $P_{v_i}(z) = e_{ii}$, $S_{e_i}(z) = e_{(i+1)i}$ for $i < n$, and $S_{e_n}(z) = ze_{1n}$, then $\pi_{S,P}$ is an isomorphism of $C^*(C_n)$ onto $C(\mathbb{T}, M_n(\mathbb{C}))$. (Notice that the unitary S_μ is then the function $z \in C(\mathbb{T}) = e_{11}C(\mathbb{T}, M_n(\mathbb{C}))e_{11}$ which we know to be universal for unitary elements of C^*-algebras.)

It is easy to check by manipulating matrix units that $\{S, P\}$ is a Cuntz-Krieger family. Next, observe that since every e_{ij} can be factored as a product involving arbitrarily many copies of the product $e_{1n}e_{n(n-1)} \cdots e_{21}$, every function of the form $z \mapsto z^m e_{ij}$ for $m \geq 0$ is in the $*$-algebra generated by $\{S, P\}$; taking adjoints shows that this is also true for $m < 0$. Thus the range of $\pi_{S,P}$ contains every function of the form $z \mapsto \sum_{i,j=1}^n f_{ij}(z)e_{ij}$ in which each f_{ij} is a trigonometric polynomial. Since such functions are dense in $C(\mathbb{T}, M_n(\mathbb{C}))$, the range of $\pi_{S,P}$ must be all of $C(\mathbb{T}, M_n(\mathbb{C}))$. Next we need to find a candidate for the gauge action. For fixed $w \in \mathbb{T}$, let U_w be the diagonal unitary matrix $\sum_{j=1}^n w^j e_{jj}$, and define $\beta_w \in \operatorname{Aut} C(\mathbb{T}, M_n(\mathbb{C}))$ by

$$\beta_w(f)(z) = U_w f(w^n z) U_w^*.$$

The functions $P_{v_i} : z \mapsto e_{ii}$ are constant, and the diagonal matrix U_w commutes with e_{ii}, so $\beta_w(P_{v_i}) = P_{v_i}$; similarly, for $i < n$ we have

$$\begin{aligned} \beta_w(S_{e_i})(z) = U_w e_{(i+1)i} U_w^* &= \Big(\sum_{j=1}^n w^j e_{jj} e_{(i+1)i}\Big) U_w^* \\ &= w^{i+1} e_{(i+1)i} \Big(\sum_{k=1}^n w^{-k} e_{kk}\Big) \\ &= w^{i+1} w^{-i} e_{(i+1)i} = w S_{e_i}(z). \end{aligned}$$

When we compute $\beta_w(S_{e_n})$, however, the w^n in the variable comes into play:

$$\begin{aligned} \beta_w(S_{e_i})(z) = U_w e_{1n}(w^n z) U_w^* &= U_w (w^n z) e_{1n} U_w^* \\ &= (w^n z)(w^1 w^{-n} e_{1n}) = wz e_{1n} = w S_{e_n}(z). \end{aligned}$$

Thus β is an action of $\mathbb{T}$ on $C(\mathbb{T}, M_n(\mathbb{C}))$ such that $\pi_{S,P} \circ \gamma_w = \beta_w \circ \pi_{S,P}$ for every $w \in \mathbb{T}$, and the gauge-invariant uniqueness theorem implies that $\pi_{S,P}$ is an isomorphism.

The vertex matrix of the graph C_n is a permutation matrix E_n. Though Cuntz and Krieger did not explicitly define $\mathcal{O}_{E_n}$, they comment in [**16**] that every quotient of $C(\mathbb{T}, M_2(\mathbb{C}))$ is generated by a Cuntz-Krieger family, and Evans confirmed that a spatial construction of $\mathcal{O}_A$ yields $\mathcal{O}_{E_n} = C(\mathbb{T}, M_n(\mathbb{C}))$ [**38**, Theorem 2.2]. The above argument is from [**60**].

The Cuntz-Krieger uniqueness theorem does not apply to every graph, but when it does apply, it gives sharper results. Here we discuss a couple of examples, and we will see further applications in Theorem 4.9 and Lemma 7.10, for example.

EXAMPLE 2.15. In Example 1.24, we saw that when E is the graph

$$e \circlearrowleft v \xleftarrow{\ f\ } w,$$

$(C^*(E), s_e + s_f)$ is universal for C^*-algebras generated by an isometry. The only cycle e in E has f as an entry, so the Cuntz-Krieger uniqueness theorem applies. If $\{S, P\}$ is a Cuntz-Krieger E-family such that P_w is non-zero, then S_f is non-zero, and so is $P_v \geq S_f S_f^*$. Since P_w is the projection $1-(S_e+S_f)(S_e+S_f)^*$ onto the kernel of the adjoint $(S_e + S_f)^*$, we deduce from Corollary 2.5 that any two isometries whose adjoints have non-zero kernels (in other words, any two non-unitary isometries) generate isomorphic C^*-algebras. Thus for this graph the Cuntz-Krieger uniqueness theorem reduces to Coburn's theorem.

EXAMPLE 2.16. Fix $n > 1$, and consider the graph E consisting of a single vertex v and n loops. The cycles in E are the loops, and each loop is an entry for the others, so the Cuntz-Krieger uniqueness theorem applies. Since p_v is an identity for $C^*(E)$, the uniqueness theorem says that any two families $\{S_i\}$ of isometries such that $\sum_{i=1}^n S_i S_i^* = 1$ generate isomorphic C^*-algebras. In particular, it follows that the C^*-algebra $C^*(S_i)$ generated by such a family is *simple* in the sense that it has no non-zero ideals. This essentially unique C^*-algebra is called the *Cuntz algebra*, and is denoted by $\mathcal{O}_n$; this uniqueness theorem was first proved in [**14**].

REMARK 2.17. The first uniqueness theorem for Cuntz-Krieger algebras was proved by Cuntz and Krieger in [**16**]. Their theorem said that if A is a $\{0, 1\}$-matrix satisfying a certain condition (I), then any two Cuntz-Krieger A-families generate isomorphic C^*-algebras; they then called the essentially unique C^*-algebra generated by such a family $\mathcal{O}_A$. When we translate their result into a theorem about finite graphs, as in Remark 2.8, we recover Corollary 2.5 for finite E. This was generalised to row-finite graphs in [**96**] and [**82**], and the condition "every cycle has an entry" was introduced in [**82**], where it was called Condition (L). Theorem 2.4 is slightly more general than [**82**, Theorem 3.7], and Corollary 2.5 is [**9**, Theorem 3.1]. The universal Cuntz-Krieger algebras of finite $\{0, 1\}$-matrices were introduced in [**60**], and a gauge-invariant uniqueness theorem for these algebras was proved in [**60**, Theorem 2.3]; the version we have stated here is [**9**, Theorem 2.1].

CHAPTER 3

Proofs of the uniqueness theorems

In this chapter we shall prove the gauge-invariant and Cuntz-Krieger uniqueness theorems for a row-finite graph E. The first step in both proofs is to analyse the *fixed-point algebra*

$$C^*(E)^\gamma := \{a \in C^*(E) : \gamma_z(a) = a \text{ for all } z \in \mathbb{T}\}$$

for the gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(E)$, which is usually called the *core* of $C^*(E)$. When we understand the structure of the core, we can prove quite easily that if $\{T, Q\}$ is a Cuntz-Krieger E-family with each Q_v non-zero, then $\pi_{T,Q}$ is faithful on $C^*(E)^\gamma$. From here, gauge-invariant uniqueness follows easily, but there is still some work to do to get Cuntz-Krieger uniqueness.

The core is a $*$-subalgebra of $C^*(E)$, and the continuity of γ implies that it is a C^*-subalgebra. Since $\gamma_z(s_\mu s_\nu^*) = z^{|\mu|-|\nu|} s_\mu s_\nu^*$, a word $s_\mu s_\nu^*$ is fixed by the gauge action if and only if $|\mu| = |\nu|$; thus

$$\overline{\operatorname{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu) \text{ and } |\mu| = |\nu|\} \tag{3.1}$$

is contained in $C^*(E)^\gamma$. We want to prove that $C^*(E)^\gamma$ is precisely (3.1). To do this, we need a way of projecting elements of $C^*(E)$ into $C^*(E)^\gamma$, and this is done by averaging over γ. To make this precise, we need to be able to integrate functions with values in a C^*-algebra.

For a continuous function $f : \mathbb{T} \to \mathbb{C}$, we write

$$\int_{\mathbb{T}} f(z)\, dz := \int_0^1 f(e^{2\pi i t})\, dt.$$

LEMMA 3.1. *Suppose that A is a C^*-algebra and $f : \mathbb{T} \to A$ is a continuous function. Then there is a unique element $\int_{\mathbb{T}} f(z)\, dz$ of A such that, for every representation π of A on $\mathcal{H}$ and $h, k \in \mathcal{H}$, we have*

$$\Big(\pi\Big(\int_{\mathbb{T}} f(z)\, dz\Big)h \;\Big|\; k\Big) = \int_{\mathbb{T}} \big(\pi(f(z)h \,|\, k\big)\, dz. \tag{3.2}$$

We then have

(a) $b\big(\int_{\mathbb{T}} f(z)\, dz\big) = \int_{\mathbb{T}} bf(z)\, dz$ *for* $b \in A$;
(b) $\big\| \int_{\mathbb{T}} f(z)\, dz\big\| \leq \int_{\mathbb{T}} \|f(z)\|\, dz$;
(c) $\phi\big(\int_{\mathbb{T}} f(z)\, dz\big) = \int_{\mathbb{T}} \phi(f(z))\, dz$ *for every homomorphism* $\phi : A \to B$;
(d) *for* $w \in \mathbb{T}$, $\int_{\mathbb{T}} f(wz)\, dz = \int_{\mathbb{T}} f(z)\, dz$.

PROOF. We begin by choosing a faithful representation $\rho : A \to B(\mathcal{H})$. Then $(h, k) \mapsto \int_{\mathbb{T}} (\rho(f(z))h \,|\, k)\, dz$ is a bounded sesquilinear form on $\mathcal{H}$, and hence there is a bounded operator T on $\mathcal{H}$ such that

$$(Th \,|\, k) = \int_{\mathbb{T}} \big(\rho(f(z))h \,|\, k\big)\, dz \quad \text{for all } h, k \in \mathcal{H}.$$

For $\epsilon > 0$, we can use a partition of unity to see that there exist finitely many $f_i \in C(\mathbb{T})$ and $a_i \in A$ such that $\| \sum_i f_i(z)a_i - f(z)\| < \epsilon$ for all $z \in \mathbb{T}$, and then

$$\Big\| T - \sum_i \Big(\int_{\mathbb{T}} f_i(z)\, dz \Big) \rho(a_i) \Big\| \leq \epsilon$$

(the details are in [**114**, page 275]). This proves, first, that T belongs to the C^*-subalgebra $\rho(A)$ of $B(\mathcal{H})$, so that we can define $\int_{\mathbb{T}} f(z)\, dz$ to be $\rho^{-1}(T)$, and, second, that we then have

$$\Big\| \int_{\mathbb{T}} f(z)\, dz - \sum_i \Big(\int_{\mathbb{T}} f_i(z)\, dz \Big) a_i \Big\| \leq \epsilon,$$

which implies that if π is any other representation of A, then

$$\Big| \Big(\pi\Big(\int_{\mathbb{T}} f(z)\, dz \Big) h \;\Big|\; k\Big) - \int_{\mathbb{T}} \big(\pi(f(z)h \,|\, k\big)\, dz \Big| \leq 2\epsilon \|h\|\, \|k\| \quad \text{for all } h, k \in \mathcal{H}. \tag{3.3}$$

Since the ϵ in (3.3) is arbitrary, this implies (3.2).

Properties (a) and (b) can be verified by applying a faithful representation π to each side and using (3.2). For (c), take a faithful representation ρ of B and apply (3.2) with $\pi = \rho \circ \phi$. To verify (d), use (3.2) to reduce to the integrals of functions $g : \mathbb{T} \to \mathbb{C}$, and write $w = e^{2\pi i\theta}$. Then

$$\int_{\mathbb{T}} g(wz)\, dz = \int_0^1 g(e^{2\pi i(\theta+t)})\, dt = \int_{\theta}^{1+\theta} g(e^{2\pi i t})\, dt,$$

which equals $\int_0^1$ by periodicity. □

Proposition 3.2. *Let α be an action of $\mathbb{T}$ on a C^*-algebra A, and define $\Phi : A \to A$ by*

$$\Phi(a) = \int_{\mathbb{T}} \alpha_z(a)\, dz.$$

Then $\Phi(a) \in A^\alpha$ for every $a \in A$, and $\Phi(a) = a$ for every $a \in A^\alpha$. The map $\Phi : A \to A^\alpha$ is linear and norm-decreasing, and is faithful in the sense that $\Phi(a^*a) = 0$ implies $a = 0$.

Proof. For $a \in A$ and $w \in \mathbb{T}$, parts (c) and (d) of Lemma 3.1 imply that

$$\alpha_w(\Phi(a)) = \int_{\mathbb{T}} \alpha_w(\alpha_z(a))\, dz = \int_{\mathbb{T}} \alpha_{wz}(a)\, dz = \int_{\mathbb{T}} \alpha_z(a)\, dz = \Phi(a),$$

so $\phi(a) \in A^\alpha$. If $a \in A^\alpha$, then part (a) of Lemma 3.1 implies that $\Phi(a) = \big(\int_{\mathbb{T}} 1 \;dz\big)a = a$. The linearity of Φ follows from an application of (3.2). Since automorphisms of C^*-algebras are norm-preserving, part (b) of Lemma 3.1 gives

$$\|\Phi(a)\| = \Big\| \int_{\mathbb{T}} \alpha_z(a)\, dz \Big\| \leq \int_{\mathbb{T}} \|\alpha_z(a)\|\, dz = \int_{\mathbb{T}} \|a\|\, dz = \|a\|.$$

To prove the last assertion, suppose $\Phi(a^*a) = 0$, and choose a faithful representation π of A on $\mathcal{H}$. Then for $h \in \mathcal{H}$, we have

$$0 = \big(\pi(\Phi(a^*a))h \,|\, h\big) = \int_{\mathbb{T}} \big(\pi(\alpha_z(a^*a))h \,|\, h\big)\, dz = \int_{\mathbb{T}} \|\pi(\alpha_z(a))h\|^2\, dz. \tag{3.4}$$

Since $z \mapsto \|\pi(\alpha_z(a))h\|^2$ is a non-negative continuous function, (3.4) implies that the function is identically zero. In particular, it is zero when $z = 1$, and we deduce that $\pi(a)h = 0$ for all h. Since π is faithful, this implies that $a = 0$. □

For the rest of this chapter, Φ will denote the linear map of $C^*(E)$ onto $C^*(E)^\gamma$ obtained by applying Proposition 3.2 to the gauge action γ.

COROLLARY 3.3. *For every finite subset F of E^* and every choice of scalars $c_{\mu,\nu}$, we have*

$$\Phi\Big(\sum_{\mu,\nu\in F} c_{\mu,\nu}s_\mu s_\nu^*\Big) = \sum_{\{\mu,\nu\in F:\,|\mu|=|\nu|\}} c_{\mu,\nu}s_\mu s_\nu^*, \tag{3.5}$$

and

$$C^*(E)^\gamma = \overline{\operatorname{span}}\{s_\mu s_\nu^* : s(\mu)=s(\nu) \text{ and } |\mu|=|\nu|\}. \tag{3.6}$$

PROOF. We have already observed that the right-hand side of (3.6) is contained in $C^*(E)^\gamma$. For $\mu,\nu\in E^*$ with $s(\mu)=s(\nu)$, part (a) of Lemma 3.1 implies that

$$\begin{aligned}\Phi(s_\mu s_\nu^*) &= \int_{\mathbb{T}} z^{|\mu|-|\nu|}s_\mu s_\nu^*\,dz = \Big(\int_T z^{|\mu|-|\nu|}\,dz\Big)s_\mu s_\nu^* \\ &= \begin{cases} s_\mu s_\nu^* & \text{if } |\mu|=|\nu| \\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

which because Φ is linear gives (3.5). Since Φ is continuous, (3.5) implies that

$$\Phi(C^*(E)) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : s(\mu)=s(\nu) \text{ and } |\mu|=|\nu|\}.$$

Since we know from Proposition 3.2 that $\Phi(a)=a$ for $a\in C^*(E)^\gamma$, this implies that the right-hand side of (3.6) contains $C^*(E)^\gamma$. □

We now use Corollary 3.3 to analyse the structure of the core $C^*(E)^\gamma$. For $k\in\mathbb{N}$, let

$$\mathcal{F}_k := \overline{\operatorname{span}}\{s_\mu s_\nu^* : |\mu|=|\nu|=k, s(\mu)=s(\nu)\}.$$

If μ, ν, α and β are all paths of length k, then μ and α cannot extend each other without being equal, and Corollary 1.15 gives

$$(s_\mu s_\nu^*)(s_\alpha s_\beta^*) = \begin{cases} s_\mu s_\beta^* & \text{if } \nu=\alpha \\ 0 & \text{otherwise.}\end{cases} \tag{3.7}$$

This implies, first, that for each $v\in E^0$, $\{s_\mu s_\nu^* : |\mu|=|\nu|=k, s(\mu)=s(\nu)=v\}$ is a family of matrix units, and hence

$$\mathcal{F}_k(v) := \overline{\operatorname{span}}\{s_\mu s_\nu^* : |\mu|=|\nu|=k, s(\mu)=s(\nu)=v\}$$

is isomorphic to the C^*-algebra $\mathcal{K}(\ell^2(E^k\cap s^{-1}(v)))$ of compact operators on the ℓ^2-space of the countable set

$$E^k\cap s^{-1}(v) = \{\mu\in E^k : s(\mu)=v\}$$

(see Corollary A.9 and the following Remark). The formula (3.7) also implies that $\mathcal{F}_k(v)\mathcal{F}_k(w)=0$ when $v\neq w$, and Corollary A.11 implies that

$$\mathcal{F}_k \cong \textstyle\bigoplus_{v\in E^0}\mathcal{F}_k(v) \cong \bigoplus_{v\in E^0}\mathcal{K}(\ell^2(E^k\cap s^{-1}(v))), \tag{3.8}$$

where the direct sum is the C^*-algebraic direct sum consisting of elements $\{a_v : v\in E^0\}$ such that $v\mapsto \|a_v\|$ vanishes at infinity.

When the graph E does not contain sources, and $\mu,\nu\in E^k\cap s^{-1}(v)$, then the Cuntz-Krieger relation at v implies that

$$s_\mu s_\nu^* = s_\mu p_v s_\nu^* = \sum_{r(e)=v} s_\mu s_e s_e^* s_\nu^* = \sum_{r(e)=v} s_{\mu e}s_{\nu e}^*,$$

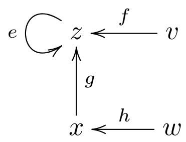
so $\mathcal{F}_k \subset \mathcal{F}_{k+1}$, and we can deduce from Corollary 3.3 that

$$C^*(E)^\gamma = \overline{\bigcup_k \mathcal{F}_k} = \overline{\bigcup_k \big(\bigoplus_{v\in E^0} \mathcal{F}_k(v)\big)}.$$

For general row-finite E, we use a device introduced by Yeend in the context of higher-rank graphs (see [**111**]). For $k \in \mathbb{N}$, let

$$E^{\leq k} := \{\mu \in E^* : |\mu| = k, \text{ or } |\mu| < k \text{ and } s(\mu) \text{ is a source}\}.$$

EXAMPLE 3.4. For the following directed graph E



we have $E^2 = \{ef, ee, eg, gh\}$ and $E^{\leq 2} = \{ef, ee, eg, gh, f, h, v, w\}$.

If ν and α are paths in $E^{\leq k}$ and ν is shorter than α, then $s(\nu)$ is a source and α cannot extend ν. Thus the formula (3.7) still holds for $\mu, \nu, \alpha, \beta \in E^{\leq k}$. This implies, first, that

$$\mathcal{F}_{\leq k}(v) := \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^{\leq k},\ s(\mu) = s(\nu) = v\}$$

is isomorphic to $\mathcal{K}(\ell^2(E^{\leq k} \cap s^{-1}(v)))$, and, second, that

$$\mathcal{F}_{\leq k} := \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^{\leq k}\}$$

is the direct sum of the $\mathcal{F}_{\leq k}(v)$. If v is not a source, then the Cuntz-Krieger relation at v shows that $\mathcal{F}_{\leq k}(v) = \mathcal{F}_k(v) \subset \mathcal{F}_{k+1} \subset \mathcal{F}_{\leq k+1}$; if v is a source, then $E^{\leq k} \cap s^{-1}(v) \subset E^{\leq k+1} \cap s^{-1}(v)$, and $\mathcal{F}_{\leq k}(v) \subset \mathcal{F}_{\leq k+1}(v)$. So $\mathcal{F}_{\leq k} \subset \mathcal{F}_{\leq k+1}$, and

$$C^*(E)^\gamma \subset \overline{\bigcup_k \mathcal{F}_{\leq k}} = \overline{\bigcup_k \big(\bigoplus_{v\in E^0} \mathcal{F}_{\leq k}(v)\big)}. \tag{3.9}$$

LEMMA 3.5. *Suppose $\{T, Q\}$ is a Cuntz-Krieger E-family in a C^*-algebra B such that $Q_v \neq 0$ for all $v \in E^0$. Then $\pi_{T,Q}$ is isometric on $C^*(E)^\gamma$.*

PROOF. For every $\mu \in E^{\leq k}$, we have $T_\mu^* T_\mu = Q_{s(\mu)}$, so every matrix unit $s_\mu s_\nu^*$ in every $\mathcal{F}_{\leq k}(v)$ has non-zero image $T_\mu T_\nu^*$ under $\pi_{T,Q}$. Thus $\pi_{T,Q}$ is injective on each $\mathcal{F}_{\leq k}(v)$, and hence also on the direct sum $\mathcal{F}_{\leq k} = \bigoplus_v \mathcal{F}_{\leq k}$. Since $\mathcal{F}_{\leq k}$ is a C^*-algebra, every injective homomorphism on $\mathcal{F}_{\leq k}$ is isometric. Thus $\pi_{T,Q}$ is isometric on $\bigcup_k \big(\bigoplus_{v\in E^0} \mathcal{F}_{\leq k}(v)\big)$, and hence by (3.9) on $C^*(E)^\gamma$. □

REMARK 3.6. If E has sources, the inclusion in (3.9) will be proper, because $\mathcal{F}_{\leq k}(v)$ will include elements $s_\mu s_\nu^*$ where $|\mu| \neq |\nu|$. If we want to describe the structure of the core, we need to introduce the subalgebras

$$\mathcal{F}_{k,l}(v) := \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^{\leq k},\ |\mu| = |\nu| = l,\ s(\mu) = s(\nu) = v\}.$$

Then $G_k(v) := \mathcal{F}_{\leq k}(v) \cap C^*(E)^\gamma$ is the direct sum of the $\mathcal{F}_{k,l}(v)$ for $l \leq k$, and $C^*(E)^\gamma$ is the closure of $\bigcup_{k=1}^\infty (\bigoplus_v G_k(v))$.

At this stage we have the tools to prove the gauge-invariant uniqueness theorem.

PROOF OF THEOREM 2.2. From parts (c) and (b) of Lemma 3.1 we have

$$\|\pi_{T,Q}(\Phi(a))\| \leq \int_{\mathbb{T}} \|\pi_{T,Q}(\gamma_z(a))\|\, dz = \int_{\mathbb{T}} \|\beta_z(\pi_{T,Q}(a))\|\, dz,$$

which, since automorphisms of C^*-algebras are norm-preserving, implies that

$$\|\pi_{T,Q}(\Phi(a))\| \leq \int_{\mathbb{T}} \|\pi_{T,Q}(a)\|\,dz = \|\pi_{T,Q}(a)\|. \tag{3.10}$$

Now putting the bits together gives

$$\begin{aligned}\pi_{T,Q}(a) = 0 &\Longleftrightarrow \pi_{T,Q}(a^*a) = 0 \\ &\Longrightarrow \pi_{T,Q}(\Phi(a^*a)) = 0 \qquad \text{(by (3.10))} \\ &\Longrightarrow \Phi(a^*a) = 0 \qquad \text{(because } \pi_{T,Q} \text{ is faithful on } C^*(E)^\gamma) \\ &\Longrightarrow a^*a = 0 \qquad \text{(by Lemma 3.2)} \\ &\Longrightarrow a = 0,\end{aligned} \tag{3.11}$$

so that $\pi_{T,Q}$ is injective. Since the range of any homomorphism between C^*-algebras is a C^*-algebra, and $\pi_{T,Q}(C^*(E))$ is generated by $\{\pi_{T,Q}(s), \pi_{T,Q}(p)\} = \{T, Q\}$, the range of $\pi_{T,Q}$ is $C^*(T, Q)$. □

The action β was only used to get the estimate (3.10). Thus to prove the Cuntz-Krieger uniqueness theorem, it suffices to prove a similar estimate. As a first step towards this, we look at a consequence of the hypothesis that every cycle has an entry.

LEMMA 3.7. *Suppose that E has no sources, and that every cycle in E has an entry. Then for each $v \in E^0$ and each $n \in \mathbb{N}$, there is a path $\lambda \in E^*$ such that $r(\lambda) = v$, $|\lambda| \geq n$ and $\lambda_k \neq \lambda_{|\lambda|}$ for $k < |\lambda|$ (we say that the path λ is* non-returning*).*

PROOF. If there is a path $\lambda \in E^n$ with $r(\lambda) = v$ and no repeated vertices, this will suffice. Otherwise, every path of length n which ends at v contains a return path. Choose a shortest path α such that $r(\alpha) = v$ and there is a cycle β based at $s(\alpha)$. Then β has an entry e, and for sufficently many repetitions of β, $\lambda = \alpha\beta\beta\cdots\beta\beta' e$ has the required properties, where β' is the segment of β from $r(e)$ to $s(\beta)$. □

PROOF OF THEOREM 2.4. By representing B on Hilbert space, we may assume that $\{T, Q\}$ is a Cuntz-Krieger E-family of operators on a Hilbert space $\mathcal{H}$. To avoid complicating the notation, it is convenient to first reduce to the case where E has no sources. Let E_+ be the graph obtained by adding a head to every source of E, as in Corollary 2.11. Then by enlarging the Hilbert space, we can find a Cuntz-Krieger E_+-family $\{S, P\}$ such that $S_e = T_e$ for $e \in E^1$ and $P_v = Q_v$ for $v \in E^0$. (For each source w, add the direct sum $\bigoplus_{i=1}^{\infty} \mathcal{H}_{i,w}$ of copies $\mathcal{H}_{i,w}$ of $P_w\mathcal{H}$, and take the partial isometry S_{e_i} associated to the edge e_i on the head to be the identity map of $\mathcal{H}_{i,w}$ onto $\mathcal{H}_{i-1,w}$, where by $\mathcal{H}_{0,w}$ we mean the original space $P_w\mathcal{H}$.) Then each P_v is non-zero. So if we know the theorem holds for E_+, then $\pi_{S,P}$ is faithful on $C^*(E_+)$, and so is $\pi_{T,Q}$, which is composition of $\pi_{S,P}$ with the injection ϕ of Corollary 2.11.

So we may suppose that E has no sources. As we observed after the proof of the gauge-invariant uniqueness theorem, it suffices to prove that $\|\pi_{T,Q}(\Phi(a))\| \leq \|\pi_{T,Q}(a)\|$ for all $a \in C^*(E)$; by continuity it suffices to do this for a of the form $a = \sum_{(\mu,\nu)\in F} c_{\mu,\nu} s_\mu s_\nu^*$, where F is a finite set of pairs (μ, ν) with $s(\mu) = s(\nu)$. The strategy is to find a projection Q which satisfies

$$\|Q\pi_{T,Q}(\Phi(a))Q\| = \|\pi_{T,Q}(\Phi(a))\|, \quad \text{and} \tag{3.12}$$

$$QT_\mu T_\nu^* Q = 0 \quad \text{when } (\mu,\nu) \in F \text{ and } |\mu| \neq |\nu|. \tag{3.13}$$

(Skip ahead to (3.15) if you want to see how this will help.)

Let $k = \max\{|\mu|, |\nu| : (\mu,\nu) \in F\}$. Because the graph E has no sources, we may suppose by applying the Cuntz-Krieger relations and changing F that $k = \min\{|\mu|, |\nu|\}$ for every pair (μ,ν) with $c_{\mu,\nu} \neq 0$. In particular, if $c_{\mu,\nu} \neq 0$ and $|\mu| = |\nu|$, then $|\mu| = |\nu| = k$. Since we know from (3.5) that

$$\Phi(a) = \sum_{\{(\mu,\nu)\in F \colon |\mu|=|\nu|\}} c_{\mu,\nu} s_\mu s_\nu^*,$$

we deduce that $\Phi(a)$ belongs to $\mathcal{F}_k = \overline{\operatorname{span}}\{s_\mu s_\nu^* : |\mu| = |\nu| = k\}$. Since $\mathcal{F}_k$ is the C^*-algebraic direct sum $\bigoplus_{v\in E^0} \mathcal{F}_k(v)$, there exists $v \in E^0$ such that

$$\|\Phi(a)\| = \Bigg\| \sum_{\{(\mu,\nu)\in F \colon |\mu|=|\nu|, s(\mu)=s(\nu)=v\}} c_{\mu,\nu} s_\mu s_\nu^* \Bigg\|. \tag{3.14}$$

We write

$$b_v := \sum_{\{(\mu,\nu)\in F \colon |\mu|=|\nu|, s(\mu)=s(\nu)=v\}} c_{\mu,\nu} s_\mu s_\nu^*.$$

Let G be the set of paths which arise either as μ or ν for some $(\mu,\nu) \in F$ satisfying $|\mu| = |\nu|$ and $s(\mu) = s(\nu) = v$. Notice that $\operatorname{span}\{s_\mu s_\nu^* : \mu,\nu \in G\}$ is a finite-dimensional matrix algebra containing b_v.

For the vertex v satisfying (3.14), and $n > \max\{|\mu|, |\nu| : (\mu,\nu) \in F\}$, we choose $\lambda \in E^*$ as in Lemma 3.7. We claim that

$$Q := \sum_{\tau\in G} T_{\tau\lambda} T_{\tau\lambda}^*$$

satisfies (3.12) and (3.13)

Suppose $(\mu,\nu) \in F$ satisfies $|\mu| = |\nu|$ and $\tau \in G$. Then $T_{\tau\lambda}^* T_\mu$ is non-zero if and only if $\tau = \mu$, and hence

$$\begin{aligned}
QT_\mu T_\nu^* Q &= \begin{cases} (T_{\mu\lambda} T_{\mu\lambda}^* T_\mu)(T_\nu^* T_{\nu\lambda} T_{\nu\lambda}^*) & \text{if } \mu,\nu \in G \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} T_{\mu\lambda}(T_\lambda^* T_\lambda) T_{\nu\lambda}^* & \text{if } \mu,\nu \in G \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} T_{\mu\lambda} T_{\nu\lambda}^* & \text{if } \mu,\nu \in G \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

For $\tau \in G$, the initial projection $Q_{s(\tau\lambda)}$ of $T_{\tau\lambda}$ is non-zero, and hence the final projection $T_{\tau\lambda} T_{\tau\lambda}^*$ is also non-zero. Thus $\{QT_\mu T_\nu^* Q : \mu,\nu \in G\}$ is a collection of non-zero matrix units, and the map $b \mapsto Q\pi_{T,Q}(b)Q$ is a faithful representation of the matrix algebra $\operatorname{span}\{s_\mu s_\nu^* : \mu,\nu \in G\}$. Since faithful representations of C^*-algebras are isometric, this implies in particular that

$$\|\pi_{T,Q}(\Phi(a))\| = \|\Phi(a)\| = \|b_v\| = \|Q\pi_{T,Q}(b_v)Q\| = \|Q\pi_{T,Q}(\Phi(a))Q\|,$$

and we have verified (3.12).

Now suppose that $(\mu,\nu)\in F$ satisfies $|\mu|\neq|\nu|$. Either μ or ν has length k, say $|\mu|=k$. As before, $T_{\tau\lambda}^*T_\mu$ is non-zero if and only if $\tau=\mu$. Thus

$$QT_\mu T_\nu^*Q=\sum_{\tau\in G}T_{\mu\lambda}T_{\mu\lambda}^*T_\mu T_\nu^*T_{\tau\lambda}T_{\tau\lambda}^*=\sum_{\tau\in G}T_{\mu\lambda}(T_{\nu\lambda}^*T_{\tau\lambda})T_{\tau\lambda}^*.$$

For $T_{\nu\lambda}^*T_{\tau\lambda}$ to be non-zero, $\nu\lambda$ must extend $\tau\lambda$, which is impossible because $0<|\nu|-|\tau|<|\lambda|$ and λ is non-returning. Thus $QT_\mu T_\nu^*Q=0$, and we have verified (3.13).

We now estimate, using (3.12) and (3.13):

$$\begin{aligned}\|\pi_{T,Q}(\Phi(a))\|&=\|Q\pi_{T,Q}(\Phi(a))Q\|\\&=\Big\|Q\Big(\sum_{\{(\mu,\nu)\in F:\,|\mu|=|\nu|\}}c_{\mu,\nu}T_\mu T_\nu^*\Big)Q\Big\|\\&=\Big\|Q\Big(\sum_{(\mu,\nu)\in F}c_{\mu,\nu}T_\mu T_\nu^*\Big)Q\Big\|\\&\leq\Big\|\sum_{(\mu,\nu)\in F}c_{\mu,\nu}T_\mu T_\nu^*\Big\|\\&=\|\pi_{T,Q}(a)\|.\end{aligned}\tag{3.15}$$

This estimate allows us to run the argument of (3.11), and this completes the proof. □

Remark 3.8. This proof of Theorem 2.4 is based on the arguments of Cuntz and Krieger [**16**], as adapted to the C^*-algebras of row-finite graphs in [**9**], with Lemma 3.7 borrowed from [**73**]. The original proof in [**82**] used the groupoid model for $C^*(E)$ constructed in [**83**].

CHAPTER 4

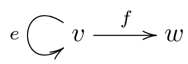
Simplicity and ideal structure

In this chapter we investigate the ideal structure of the C^*-algebra of a row-finite graph E, using the uniqueness theorems which we proved in the previous chapter. Our first result gives a condition on E which ensures that $C^*(E)$ is simple. We will later prove that this condition is also necessary (see Theorem 4.14).

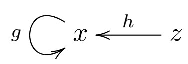
For vertices v, w in a directed graph E, we write $w \le v$ to mean that there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. This relation $\le$ is transitive in the sense that $w \le v$ and $v \le x$ imply $w \le x$, and is reflexive because we can take μ to be the path v of length 0. The relation $\le$ is not a partial order, because distinct vertices v, w on a cycle satisfy $v \le w \le v$; it is what is known as a *preorder*.

We denote by E^∞ the set of infinite paths $\lambda = \lambda_1\lambda_2\cdots$, and by $E^{\le\infty}$ the set obtained by adding to E^∞ the finite paths which begin at sources. We say that the graph E is *cofinal* if for every $\mu \in E^{\le\infty}$ and $v \in E^0$ there exists a vertex w on μ such that $v \le w$.

Example 4.1. The graph



is cofinal, whereas



is not: one cannot reach z from any point on the infinite path $ggg\cdots$.

Proposition 4.2. *Suppose E is a row-finite graph in which every cycle has an entry. If E is cofinal, then $C^*(E)$ is simple.*

A directed graph E is called *transitive* if for every pair v, w of vertices, we have $v \le w$ and $w \le v$. If E is transitive and E is not one of the graphs C_n consisting of a single cycle, then E satisfies the hypotheses of Proposition 4.2. So there are lots of graphs E for which $C^*(E)$ is simple.

Proof of Proposition 4.2. Since every ideal in a C^*-algebra is the kernel of a representation, it suffices to prove that every non-zero representation $\pi_{S,P}$ of $C^*(E)$ is faithful. Suppose $\{S, P\}$ is a Cuntz-Krieger E-family such that $\pi_{S,P}$ is non-zero. If all the vertex projections P_v were 0, then the relation $S_e^*S_e = P_{s(e)}$ would force $S_e = 0$ for all e, and $\pi_{S,P}$ would be identically 0. So at least one P_v is non-zero. We want to prove that all the vertex projections are non-zero.

Let $w \in E^0$. If the vertex v for which $P_v \ne 0$ is not a source, the Cuntz-Krieger relation at v implies that there exists $e \in E^1$ such that $r(e) = v$ and $S_eS_e^* \ne 0$. Then $P_{s(e)} = S_e^*S_e \ne 0$, and if $s(e)$ is not a source, we can repeat this argument at $s(e)$. This process either ends at a source or continues to yield an infinite path;

either way we obtain a path $\mu \in E^{\leq\infty}$ such that $r(\mu) = v$ and $P_x \neq 0$ for every vertex x on μ. By cofinality, there exists $\alpha \in E^*$ with $r(\alpha) = w$ and $s(\alpha)$ a vertex on μ. But now $S_\alpha^* S_\alpha = P_{s(\alpha)} \neq 0$, so $S_\alpha S_\alpha^* \neq 0$. Since $P_w S_\alpha S_\alpha^* = S_\alpha S_\alpha^*$, it follows that P_w is non-zero.

Thus all the vertex projections P_w are non-zero, and the Cuntz-Krieger uniqueness theorem implies that $\pi_{S,P}$ is faithful, as required. □

Remark 4.3. There has recently been a great deal of interest in the classification of simple C^*-algebras, and it is natural to ask when these simple graph algebras satisfy the other hypotheses required in the classification program. All graph algebras are nuclear: one way to see this is to prove that $C^*(E) \rtimes_\gamma \mathbb{T}$ is AF, as we will do in Chapter 7, and then the Takesaki-Takai duality theorem implies that $C^*(E)$ is stably isomorphic to $(C^*(E) \rtimes_\gamma \mathbb{T}) \rtimes_{\widehat{\gamma}} \mathbb{Z}$. Indeed, this argument shows that $C^*(E)$ belongs to the *bootstrap class* $\mathcal{N}$ of Rosenberg and Schochet (see [**125**] or [**123**, §2.4]).

A simple graph algebra $C^*(E)$ is *purely infinite* in the sense of [**123**, §4.1] if and only if for every $v \in E^0$ there is a cycle μ with $r(\mu) \geq v$ (see [**82**, Theorem 3.9] and [**9**, Proposition 5.3]). It follows from this and Remark 2.13 that there is a dichotomy for simple graph algebras: they are either AF or purely infinite [**82**, Corollary 3.11].

A graph algebra $C^*(E)$ has an identity if and only if E^0 is finite: if E^0 is finite, $\sum_{v\in E^0} p_v$ is an identity, and if $C^*(E)$ has an identity, the sums discussed in Lemma 2.10 must converge in norm, which is only possible if the sums are finite, or in other words if E^0 is finite. Criteria for the stability of $C^*(E)$ are given in [**143**], which improves earlier results in [**54**]. In particular, the C^*-algebras of infinite transitive graphs are always stable.

The simple C^*-algebras are the ones with trivial ideal structure. Our next goal is to describe the ideals of $C^*(E)$ when $C^*(E)$ is not simple. By convention, when we talk about ideals in C^*-algebras, we mean closed two-sided ideals unless otherwise stated.

Suppose I is an ideal in $C^*(E)$, and consider

$$H_I := \{v \in E^0 : p_v \in I\}.$$

The idea is that the set H_I determines the ideal I. To see why, consider the quotient map $q : C^*(E) \to C^*(E)/I$. The projections $\{q(p_v) : v \notin H_I\}$ are all non-zero. If $s(e) \notin H_I$, then $q(s_e)$ has non-zero initial projection, and $q(p_{r(e)}) \geq q(s_e)q(s_e)^*$ is also non-zero, so that $r(e)$ does not belong to H_I either. Thus

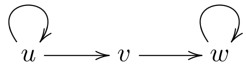
$$E\backslash H_I := \{E^0\backslash H_I, s^{-1}(E^0\backslash H_I), r, s\}$$

is a graph, and $\{q(s_e), q(p_v) : s(e) \notin H_I, v \notin H_I\}$ is a Cuntz-Krieger family for $E\backslash H_I$ with every vertex projection non-zero. If every cycle in this graph has an entry, then the Cuntz-Krieger uniqueness theorem implies that $C^*(E)/I$ is isomorphic to $C^*(E\backslash H_I)$. So we need to identify the subsets H of E^0 which arise as H_I, and find a verifiable condition on E which ensures that every cycle in every $E\backslash H$ has an entry.

The characterisation of the sets H_I uses the preorder $\leq$ on E^0. We say that a subset H of E^0 is *hereditary* if $w \in H$ and $w \leq v$ imply $v \in H$; we say that H is *saturated* if $r^{-1}(v) \neq \emptyset$ and $\{s(e) : r(e) = v\} \subset H$ imply $v \in H$. (We warn that in Chapter 5 we will see that a different definition of saturated is needed when E

is not row-finite.) In every row-finite graph E, E^0 and $\emptyset$ are saturated hereditary sets; we shall refer to the others as the *non-trivial* saturated hereditary sets.

EXAMPLE 4.4. In the following graph E



the sets $\{u\}$ and $\{u, v\}$ are non-trivial hereditary subsets of E^0, but $\{v\}$ is not. The set $\{u, v\}$ is saturated, but $\{u\}$ is not because v satisfies $\{s(e) : r(e) = v\} \subset \{u\}$. If $H = \{u, v\}$, then $E\backslash H$ consists of a single loop at w.

LEMMA 4.5. *Suppose I is a non-zero ideal in the C^*-algebra of the row-finite graph E. Then H_I is saturated and hereditary.*

PROOF. To see that H_I is hereditary, suppose $w \in H_I$ and $w \le v$. Then there exists $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$, and

$$p_w \in I \Longrightarrow s_\mu = p_{r(\mu)} s_\mu = p_w s_\mu \in I \Longrightarrow p_v = s_\mu^* s_\mu \in I \Longrightarrow v \in H_I.$$

To see that H_I is saturated, suppose that $r^{-1}(v) \neq \emptyset$ and $\{s(e) : r(e) = v\} \subset H_I$. Then for every e with $r(e) = v$, $s_e = s_e p_{s(e)}$ belongs to I, and v is not a source, so

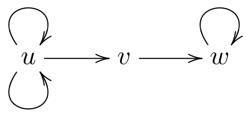
$$p_v = \sum_{r(e)=v} s_e s_e^* \in I,$$

which implies that v belongs to H_I. □

We now introduce our hypothesis on E. We say that E *satisfies Condition* (K) if for every vertex v, either there is no cycle based at v, or there are two distinct paths μ, ν such that $s(\mu) = v = r(\mu)$, $s(\nu) = v = r(\nu)$, $r(\mu_i) \neq v$ for $i < |\mu|$, and $r(\nu_i) \neq v$ for $i < |\nu|$ (we call these distinct *return paths*). If E is a graph which satisfies Condition (K) then every cycle in E has an entry, but the converse is false, as the next examples show.

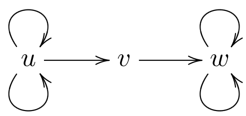
EXAMPLES 4.6. In the graph

(4.1)

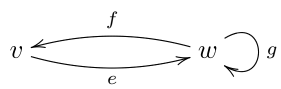


every cycle has an entry, but Condition (K) is not satisfied at the vertex w. When we add another loop at w,

(4.2)



Condition (K) is satisfied. Notice that there is no problem at v because it does not lie on any cycles. It is important that we are not insisting that we can choose the return paths to be cycles: for example, in the graph



$\mu = fe$ and $\nu = fge$ are return paths at v, but there is only one cycle based at v.

If H is a hereditary set, then $E\backslash H := \{E^0\backslash H, s^{-1}(E^0\backslash H), r, s\}$ is a graph. We now check that Condition (K) does what we want:

LEMMA 4.7. *The graph E satisfies Condition* (K) *if and only if for every saturated hereditary subset H of E^0, every cycle in $E\backslash H := \{E^0\backslash H, s^{-1}(E^0\backslash H), r, s\}$ has an entry.*

EXAMPLE 4.8. In the graph (4.1), the set $H = \{u, v\}$ is saturated and hereditary, but $E\backslash H$ consists of a single loop based at w. Thus the single cycle in $E\backslash H$ does not have an entry in $E\backslash H$.

PROOF OF LEMMA 4.7. Suppose E satisfies Condition (K), H is saturated and hereditary, and μ is a cycle in $E\backslash H$ with $v = s(\mu)$. By Condition (K), there is a distinct return path ν in E with $s(\nu) = v$. Choose i such that $\mu_j = \nu_j$ for $j < i$ and $\mu_i \neq \nu_i$. Then ν_i is certainly an entry to μ in E. However, since $v \geq s(\nu_i)$ and H is hereditary, $s(\nu_i)$ cannot be in H. So $\nu_i \in (E\backslash H)^1 = s^{-1}(E^0\backslash H)$, and is an entry to μ in $E\backslash H$.

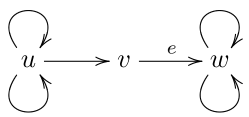
Conversely, suppose that for every saturated hereditary subset H of E^0, every cycle in $E\backslash H$ has an entry in $E\backslash H$. Suppose $v \in E^0$ and there is a cycle μ in E with $s(\mu) = v$. We have to show that v lies on another return path ν. We claim that $H := \{w : v \not\geq w\}$ is hereditary and saturated. If $w \in H$ and $z \geq w$, then $v \geq z$ implies $v \geq w$, so we must have $v \not\geq z$ and $z \in H$. Thus H is hereditary. Now suppose that $z \in E^0$ is not a source and satisfies $\{s(e) : r(e) = z\} \subset H$. If $z \notin H$, then there is a path α such that $s(\alpha) = v$ and $r(\alpha) = z$; but then $s(\alpha_1) \in H$, and $v \geq s(\alpha_1)$ implies $v \in H$, which is a contradiction. So $z \in H$, and H is saturated, as claimed.

The cycle μ lies in $E\backslash H$. So μ has an entry e in $E\backslash H$, say with $r(e) = r(\mu_i)$. Then $s(e)$ is not in H, and hence $v \geq s(e)$; choose a shortest path β with $s(\beta) = v$ and $r(\beta) = s(e)$. Now $\nu = \mu_1 \cdots \mu_{i-1} e\beta$ is the required return path. □

We can now state our main classification theorem. If H is a saturated and hereditary subset of E^0, then $E\backslash H$ is the graph in Lemma 4.7, and E_H denotes the graph $(H, r^{-1}(H), r_E, s_E)$.

THEOREM 4.9. *Suppose E is a row-finite graph which satisfies Condition* (K). *For $H \subset E^0$, let I_H be the ideal generated by $\{p_v : v \in H\}$. Then $H \mapsto I_H$ is a bijection between the saturated hereditary subsets of E^0 and the closed ideals in $C^*(E)$, with inverse given by $I \mapsto H_I = \{v : p_v \in I\}$. The quotient $C^*(E)/I_H$ is isomorphic to $C^*(E\backslash H)$, and $C^*(E_H)$ is isomorphic to the full corner $p_H I_H p_H$ associated to the projection p_H defined in Lemma 2.10.*

EXAMPLE 4.10. Consider the saturated and hereditary set $H = \{u, v\}$ in the graph



of (4.2). When we split E into two graphs E_H and $E\backslash H$, as in Theorem 4.9, the edge e does not belong to either graph: it is not in E_H because $r(e) = w$ is not in H, and it is not in $E\backslash H$ because $s(e) = v$ does not belong to $E^0\backslash H$.

Proof of Theorem 4.9. Let I be an ideal in $C^*(E)$. Then we know from Lemma 4.5 that $H = H_I$ is saturated and hereditary. We claim that $I = I_H$. Since all the generators of I_H are by definition in I, we trivially have $I_H \subset I$. Consider the quotient maps

$$q^I : C^*(E) \to C^*(E)/I, \quad q^{I_H} : C^*(E) \to C^*(E)/I_H, \quad \text{and}$$
$$q^{I/I_H} : C^*(E)/I_H \to C^*(E)/I = (C^*(E)/I_H)/(I/I_H);$$

note that $q^I = q^{I/I_H} \circ q^{I_H}$. Since q^I and q^{I_H} kill exactly the same vertex projections, and hence exactly the same partial isometries s_e, both $\{q^I(s_e), q^I(p_v)\}$ and $\{q^{I_H}(s_e), q^{I_H}(p_v)\}$ are Cuntz-Krieger $(E\backslash H)$-families which generate the respective quotients. Let $\pi : C^*(E\backslash H) \to C^*(E)/I_H$ and $\rho : C^*(E\backslash H) \to C^*(E)/I$ be the corresponding homomorphisms. Then ρ and $q^{I/I_H} \circ \pi$ are homomorphisms which agree on the generators of $C^*(E\backslash H)$, and hence are equal. By Lemma 4.7, we can apply the Cuntz-Krieger uniqueness theorem to $E\backslash H$, and deduce that ρ is injective. Since π is surjective and $\rho = q^{I/I_H} \circ \pi$, this implies that q^{I/I_H} is injective. We deduce that $I = I_H$, as claimed, and that $C^*(E)/I_H$ is isomorphic to $C^*(E\backslash H)$.

Since $I = I_{H_I}$, $H \mapsto I_H$ is surjective. To see that it is injective, we need to show that if H is saturated and hereditary, then $H = \{v : p_v \in I_H\}$. We trivially have $H \subset \{v : p_v \in I_H\}$. To prove the reverse inclusion, consider the canonical $(E\backslash H)$-family $\{t, q\}$ which generates $C^*(E\backslash H)$. We claim that when we define $t_e = 0$ for $s(e) \in H$ and $q_v = 0$ for $v \in H$, $\{t, q\}$ becomes a Cuntz-Krieger E-family. When $s(e) \in H$, we have $t_e^* t_e = 0 = q_{s(e)}$, and the relations at $v \in H$ are also trivially satisfied because hereditarity of H implies that every edge e with $r(e) = v$ has $s(e) \in H$, and hence $t_e = 0$. If v is not a source in $E\backslash H$, the new edges with $r(e) = v$ all have $t_e t_e^* = 0$, so the relation at v is unchanged. If v is a source in $E\backslash H$, all edges e in E with $r(e) = v$ have $s(e) \in H$, which unless v is a source in E implies $v \in H$ by saturation; thus v is a source in E, and there is still no Cuntz-Krieger relation at v. So $\{t, q\}$ is a Cuntz-Krieger E-family, as claimed. Now the universal property of $C^*(E)$ gives a homomorphism $\pi_{t,q}$ of $C^*(E)$ into $C^*(E\backslash H)$. Since $\pi_{t,q}(p_v) = 0$ for $v \in H$, $\ker \pi_{t,q} \supset I_H$, and

$$v \notin H \Longrightarrow q_v \neq 0 \Longrightarrow \pi_{t,q}(p_v) \neq 0 \Longrightarrow p_v \notin \ker \pi_{t,q} \Longrightarrow p_v \notin I_H,$$

so that $H \supset \{v : p_v \in I_H\}$. Thus $H = \{v : p_v \in I_H\}$, and $H \mapsto I_H$ is injective.

For the last statement, suppose H is saturated and hereditary. We claim that

$$I_H = \overline{\operatorname{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu) \in H\}. \tag{4.3}$$

To see this, notice that if $s(\mu) = s(\nu) \in H$ and α, β are any paths in E, then the formula (1.4) shows that the product $(s_\mu s_\nu^*)(s_\alpha s_\beta^*)$ has the form $s_\sigma s_\tau^*$ for some $\sigma, \tau \in E^*$ with $s(\mu) \leq s(\sigma) = s(\tau)$; since H is hereditary, $s(\sigma)$ then belongs to H, and the product belongs to the right-hand side of (4.3). Thus the right-hand side of (4.3) is an ideal. Since this ideal contains the generators of I_H, and every spanning element $s_\mu s_\nu^* = s_\mu p_{s(\mu)} s_\nu^*$ belongs to I_H, it must be I_H, as claimed.

Since compression by the projection p_H is linear and continuous, it follows from (4.3) that

$$p_H I_H p_H = \overline{\operatorname{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu) \in H, r(\mu) \in H, r(\nu) \in H\}. \tag{4.4}$$

Since H is hereditary, $\{s_e : r(e) \in H\} \cup \{p_v : v \in H\}$ is a Cuntz-Krieger E_H-family in $p_H I_H p_H$, and (4.4) implies that this family generates $p_H I_H p_H$. Every cycle in E_H has an entry in E, and by hereditarity this is also an entry in E_H. Thus the

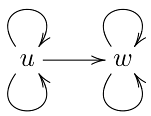
Cuntz-Krieger uniqueness theorem implies that $p_H I_H p_H$ is isomorphic to $C^*(E_H)$. The corner is full because $\{p_v : v \in H\}$ generates I_H. □

REMARK 4.11. In the last two paragraphs, we did not use that H was saturated. In fact, if X is any hereditary subset of E^0, then the smallest saturated set ΣX containing X is also hereditary, and there is an isomorphism of $C^*(E_X)$ onto the corner $p_X I_{\Sigma X} p_X$, which is again full.

REMARK 4.12. When E does not satisfy Condition (K), the saturated hereditary subsets of E^0 parametrise the *gauge-invariant ideals* I for which $\gamma_z(I) \subset I$ for every $z \in \mathbb{T}$. To prove this, follow the argument of Theorem 4.9, but use gauge-invariant ideals everywhere, and use the gauge-invariant uniqueness theorem everywhere we used the Cuntz-Krieger uniqueness theorem. See [**9**, Theorem 4.1] for details. Of importance for us will be the observation that if H is a non-trivial saturated hereditary subset of E^0, then I_H is a non-zero gauge-invariant ideal in $C^*(E)$ with non-zero quotient $C^*(E \backslash H)$.

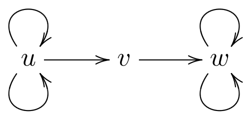
Many important directed graphs do not satisfy Condition (K), such as the graphs used to model non-commutative spheres in [**56**] (see Example 1.25), and it is of considerable interest to learn more about the ideal structure of their C^*-algebras. The ideal structure of graph algebras has now been completely determined by Hong and Szymański [**59**]; since they deal with graphs which are not row-finite, we postpone a discussion of their work to Chapter 5. Their results, however, contain important new information even for row-finite graphs (see [**59**, Corollary 3.5]).

EXAMPLE 4.13. In the following graph E



the only non-trivial hereditary set is $\{u\}$ and it is saturated. So the ideals in $C^*(E)$ are $\{0\} = I_\emptyset$, $I_{\{u\}}$ and $C^*(E) = I_{E^0}$. The quotient $C^*(E)/I_{\{u\}}$ is the C^*-algebra of the graph consisting of two loops based at w, and hence is the Cuntz algebra $\mathcal{O}_2$. The corner $p_u I_{\{u\}} p_u$ is also $\mathcal{O}_2$. The ideal $I_{\{u\}}$ itself is much larger.

If we add an extra vertex to E to obtain the following graph F



then the non-trivial hereditary sets are $\{u\}$ and $\{u, v\}$, and only $\{u, v\}$ is saturated. So the ideals in $C^*(E)$ are $\{0\} = I_\emptyset$, $I_{\{u,v\}}$ and $C^*(E) = I_{E^0}$. The quotient $C^*(E)/I_{\{u,v\}}$ is still $\mathcal{O}_2$, and $p_u I_{\{u,v\}} p_u$ is still $\mathcal{O}_2$, but the ideal $p_{\{u,v\}} I_{\{u,v\}} p_{\{u,v\}}$ discussed in Theorem 4.9 is larger. (Indeed, with a bit of work one can see that it is isomorphic to the C^*-algebra $M_2(\mathcal{O}_2)$ of 2×2 matrices over $\mathcal{O}_2$.) This illustrates why one might prefer to work with ideals generated by more general hereditary sets, as in Remark 4.11.

Theorem 4.9 gives us another criterion for simplicity: $C^*(E)$ is simple if and only if every cycle has an entry and E^0 has no non-trivial saturated hereditary sets.

We can also use the ideals I_H to see that the cofinality criterion of Proposition 4.2 is necessary and sufficient.

THEOREM 4.14. *Suppose E is a row-finite graph. Then $C^*(E)$ is simple if and only if every cycle in E has an entry and E is cofinal.*

PROOF. We proved the "if" direction in Proposition 4.2. So suppose that $C^*(E)$ is simple.

To see cofinality, suppose $\mu \in E^{\leq\infty}$, and consider

$$H_\mu := \{w \in E^0 : w \not\leq v \text{ for every } v \text{ on } \mu\}.$$

Then H_μ is saturated and hereditary. If H_μ were non-trivial, then I_{H_μ} would be a proper ideal in $C^*(E)$ (see Remark 4.12); thus H_μ is either empty or all of E^0. Since $r(\mu)$ does not belong to H_μ, we must have H_μ empty. Thus we can reach every vertex in E from some vertex on μ, and E is cofinal.

Now suppose that μ is a cycle in E which does not have an entry. Then $X := \{s(\mu_i) : 1 \leq i \leq |\mu|\}$ is a nonempty hereditary set, so its saturation ΣX must be all of E^0, and $I_{\Sigma X} = C^*(E)$ by simplicity. Thus by Remark 4.11, $C^*(E_X)$ is isomorphic to the corner $p_X C^*(E) p_X$. However, the graph E_X is a cycle, so we know from Example 2.14 that $C^*(E_X)$ is isomorphic to $C(\mathbb{T}, M_{|\mu|}(\mathbb{C}))$. Let J be a proper ideal in $p_X C^*(E) p_X$, such as the set of functions which vanish at 1. Then

$$C^*(E)JC^*(E) := \overline{\operatorname{span}}\{ajb : a, b \in C^*(E), j \in J\}$$

is a non-zero ideal in $C^*(E)$, and hence is all of $C^*(E)$. Thus

$$p_X C^*(E)JC^*(E)p_X = p_X C^*(E)p_X J p_X C^*(E) p_X = J$$

is all of $p_X C^*(E) p_X$, and we have a contradiction. So every cycle in E has an entry. □

REMARK 4.15. The ideal structure of the Cuntz-Krieger algebras $\mathcal{O}_A$ was analysed by Cuntz in [**15**] under a hypothesis (II) on A, which is equivalent to asking that the associated graph E_A satisfies Condition (K). The ideal theory of $\mathcal{O}_A$ for arbitrary finite A was studied in [**60**]. Theorem 4.9 is [**9**, Theorem 4.4], which slightly generalises previous results in [**83**] and [**67**]. The proof given here is based on that of [**9**], and is substantially different from those of [**15**] and [**60**], which used a complicated approximate identity argument, from that of [**83**], which used the groupoid model for $C^*(E)$ and results of Renault [**115**], and from that of [**67**], which viewed $C^*(E)$ as a Cuntz-Pimsner algebra.

REMARK 4.16. When E is a finite graph with no sinks or sources, every vertex v connects to a cycle μ. If E is cofinal, then the infinite path $\mu\mu\cdots$ connects to every vertex w, and hence so does v. So Theorem 4.14 says that $C^*(E)$ is simple if and only if E is transitive and not a single cycle C_n. This simplicity criterion for Cuntz-Krieger algebras was obtained in [**16**]. Simplicity criteria for various classes of row-finite graphs were obtained in [**96, 83, 67**], and cofinality was introduced in [**83**]. Theorem 4.14 is a mild improvement on [**9**, Proposition 5.1], in that we have amended the definition of cofinality to accommodate graphs with sources.

CHAPTER 5

Arbitrary graphs

In this chapter we consider infinite graphs which are not row-finite, so that vertices can receive infinitely many edges. We analyse the C^*-algebras of such graphs using a method of Drinen and Tomforde which reduces to the row-finite case.

The obvious problem with an arbitrary directed graph E is how to make sense of the Cuntz-Krieger relation

$$P_v = \sum_{\{e \in E^1 : r(e) = v\}} S_e S_e^* \tag{5.1}$$

when $r^{-1}(v)$ is infinite, in which case we call v an *infinite receiver*. The first adjustment we make is to assume that the range projections $S_e S_e^*$ are mutually orthogonal and satisfy $P_{r(e)} S_e S_e^* = S_e S_e^*$; when $r^{-1}(v)$ is finite and (5.1) holds, this is automatic because the sum of projections is a projection if and only if the projections are mutually orthogonal (Corollary A.3). When the $S_e S_e^*$ are mutually orthogonal, each finite partial sum $\sum_{e \in F} S_e S_e^*$ is a projection, and each difference

$$\Big(\sum_{e \in F} S_e S_e^*\Big) - \Big(\sum_{e \in G} S_e S_e^*\Big) = \sum_{e \in F \setminus G} S_e S_e^*$$

between finite partial sums with $G \subset F \subset r^{-1}(v)$ is a projection. Thus, since all non-zero projections have norm 1, these partial sums cannot converge in $C^*(E)$ when v is an infinite receiver. They can't converge strictly in the multiplier algebra $M(C^*(E))$ either, because then the partial sums would have to converge in norm when we multiply by $P_v \in C^*(E)$, and this doesn't change anything because $P_v S_e S_e^* = S_e S_e^*$ for every $e \in r^{-1}(v)$.

It turns out that the right thing to do is to abandon (5.1) altogether when $r^{-1}(v)$ is infinite. That this is right was discovered independently by several different authors who approached graph algebras from different points of view (see, for example, [**46, 101, 116, 132**]). For motivation, we recall a theorem of Cuntz [**14**], which says that all countably infinite families of isometries $\{S_i : i \in \mathbb{N}\}$ on Hilbert space with mutually orthogonal ranges generate isomorphic C^*-algebras. This can be viewed as a uniqueness theorem for the graph E which consists of one vertex and infinitely many loops; the graph algebra $C^*(E)$ is usually denoted by $\mathcal{O}_\infty$, and is called the *infinite Cuntz algebra*. For operators on Hilbert space, the sum $\sum_{i=1}^\infty S_i S_i^*$ converges in the strong operator topology to a projection P_∞, but Cuntz's theorem says that the structure of the C^*-algebra $C^*(S_i)$ does not depend on whether $1 = P_\infty$ or not. This suggests that when $|r^{-1}(v)| = \infty$, whether or not (5.1) holds may not affect $C^*(S, P)$.

Let E be a directed graph. A *Cuntz-Krieger E-family* consists of mutually orthogonal projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ with mutually orthogonal ranges such that

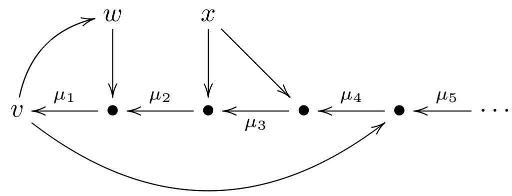
- $S_e^* S_e = P_{s(e)}$;
- $P_{r(e)} S_e S_e^* = S_e S_e^*$ for every $e \in E^1$;
- $P_v = \sum_{\{e \in E^1 : r(e) = v\}} S_e S_e^*$ for every $v \in E^0$ such that $0 < |r^{-1}(v)| < \infty$.

The C^*-algebra of E is the universal C^*-algebra $C^*(E)$ generated by a Cuntz-Krieger E-family $\{s, p\}$. One can construct such an algebra by generalising the construction of Proposition 1.21, and it is unique up to isomorphism.

We will study $C^*(E)$ by constructing a row-finite graph F whose C^*-algebra contains $C^*(E)$ as a full corner. We will describe the properties we want F to have, and then it will be easy to construct F.

Let F be a directed graph. We say that an infinite path $\mu \in F^\infty$ is *collapsible* if μ has no exits except at $r(\mu)$, $r^{-1}(r(\mu_i))$ is finite for every i, and $r^{-1}(r(\mu)) = \{\mu_1\}$[1]. As the name suggests, we are going to form a new graph by collapsing the path μ to a single vertex (see Proposition 5.2).

EXAMPLE 5.1. The path μ in the following graph F is collapsible:



Let μ be a collapsible path in a graph F. We set $s_\infty(\mu) = \{s(\mu_i) : i \geq 1\}$ and

$$F^*(\mu) := \{\nu \in F^* : |\nu| > 1 \text{ and } \nu = \mu_1\mu_2 \cdots \mu_{|\nu|-1}e \text{ for some } e \neq \mu_{|\nu|}\}.$$

Notice that since μ has no exits except at $r(\mu) = r(\mu_1)$, $F^*(\mu)$ consists of all the paths which start outside $s_\infty(\mu)$, immediately enter $s_\infty(\mu)$, and end at $r(\mu)$. The graph F_μ in the proposition is obtained from F by removing everything on the path μ except its range, and replacing every path ν which immediately enters μ and ends at $r(\mu)$ with a single edge e_ν.

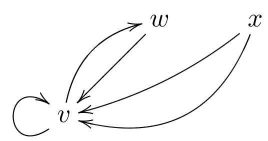
PROPOSITION 5.2. *Suppose that μ is a collapsible path in a row-finite graph F, and define $s_\infty(\mu)$, $F^*(\mu)$ as above. Define*

$$F_\mu^0 := F^0 \setminus s_\infty(\mu) \quad \text{and} \quad F_\mu^1 := \big(F^1 \setminus (r^{-1}(s_\infty(\mu)) \cup \{\mu_1\})\big) \cup \{e_\nu : \nu \in F^*(\mu)\},$$

and extend the range and source maps to F_μ^1 by setting $r(e_\nu) := r(\nu) = r(\mu)$ and $s(e_\nu) := s(\nu)$. Let $p := \sum_{v \in F_\mu^0} p_v$, as in Lemma 2.10. Then $pC^*(F)p$ is a full corner in $C^*(F) = C^*(s, p)$, and there is an isomorphism ϕ of $C^*(F_\mu) = C^*(t, q)$ onto $pC^*(F)p$ such that $\phi(q_w) = p_w$ for $w \in F_\mu^0$, and

$$\phi(t_f) = s_f \text{ for } f \in F^1 \setminus (r^{-1}(s_\infty(\mu)) \cup \{\mu_1\}), \quad \phi(t_{e_\nu}) = s_\nu \text{ for } \nu \in F^*(\mu).$$

EXAMPLE 5.3. When μ is the path in Example 5.1, F_μ looks like



[1]This last point shouldn't be necessary but it does seem to simplify the formulas.

In the proof of Proposition 5.2, we need to extend a Cuntz-Krieger F_μ-family to a Cuntz-Krieger F-family. This lemma is also important in applications of the proposition, so we state it separately; it is essentially Lemma 2.10 of [**30**].

LEMMA 5.4. *Suppose that μ is a collapsible path in F and $\{T, Q\}$ is a Cuntz-Krieger F_μ-family on a Hilbert space $\mathcal{H}$. Then there is a Hilbert space $\mathcal{H}_\mu$ and a Cuntz-Krieger F-family $\{S, P\}$ on $\mathcal{H}\oplus\mathcal{H}_\mu$ such that $Q_w = P_w$ for $w \in F_\mu^0$, $T_f = S_f$ for $f \in F^1 \setminus (r^{-1}(s_\infty(\mu)) \cup \{\mu_1\})$, and $T_{e_\nu} = S_\nu$ for $\nu \in F^*(\mu)$.*

PROOF. Let $\mathcal{H}_1$ be a copy of $Q_{r(\mu)}(\mathcal{H})$, and for $i > 1$, let $\mathcal{H}_i$ be a copy of the range of

$$Q_{r(\mu)} - \sum_{\{\nu\in F^*(\mu):\,|\nu|\leq i\}} T_{e_\nu} T_{e_\nu}^*.$$

Then we take $\mathcal{H}_\mu := \bigoplus_{i=1}^\infty \mathcal{H}_i$. For $e \in F_\mu^1 \cap F^1$ and $v \in F_\mu^0 \subset F^0$, we take $S_e = T_e$ and $P_v = Q_v$. For $i \geq 1$, we take $P_{s(\mu_i)}$ to be the projection on the new summand $\mathcal{H}_i$. We define S_{μ_1} to be the identity from $\mathcal{H}_1 = P_{r(\mu)}\mathcal{H}$ to $P_{r(\mu)}\mathcal{H} \subset \mathcal{H}$, and for $i > 1$, we define S_{μ_i} to be the partial isometry with initial space $\mathcal{H}_i$ which takes an element to the corresponding element of the summand $\mathcal{H}_{i-1}$. For $e \in r^{-1}(s_\infty(\mu))\setminus\{\mu_i\}$ with $r(e) = s(\mu_i)$, say, we write $\nu = \mu_1 \cdots \mu_i e$, and take S_e to be the partial isometry T_{e_ν}, viewed as a map of $P_{s(e)}\mathcal{H}$ into $\mathcal{H}_i$. It is straightforward to check that $\{S, P\}$ is a Cuntz-Krieger F-family, and that $\{S, P\}$ has the required relationship to $\{T, Q\}$. □

PROOF OF PROPOSITION 5.2. We need to check that the s_f, s_ν and p_w form a Cuntz-Krieger F_μ-family. There is no problem with the initial projections. Since the only vertex in F_μ^0 where we have changed $r^{-1}(v)$ is $r(\mu)$, we just need to verify the relations at $r(\mu)$. The relations in F at the vertices $r(\mu_i)$ for $i < n$ give

$$p_{r(\mu)} = s_{\mu_1\cdots\mu_n} s_{\mu_1\cdots\mu_n}^* + \sum_{\{\nu\in F^*(\mu):|\nu|\leq n\}} s_\nu s_\nu^*. \tag{5.2}$$

If $r(\mu)$ receives finitely many edges in F_μ, then for large enough n, (5.2) is the Cuntz-Krieger relation for $p_{r(\mu)}$ in F_μ. If it receives infinitely many edges in F_μ, then we can still deduce from (5.2) that the range projections of the partial isometries corresponding to edges in $r^{-1}(r(\mu))$ are mutually orthogonal and dominated by $p_{r(\mu)}$, which is all the Cuntz-Krieger relations demand. Now the universal property of $C^*(F_\mu)$ gives the existence of $\phi : C^*(F_\mu) \to C^*(F)$.

To see that ϕ is injective, it suffices to prove that every representation of $C^*(F_\mu)$ factors through ϕ. So suppose $\pi = \pi_{T,Q}$ is a representation of $C^*(F_\mu)$ on a Hilbert space $\mathcal{H}$. Let $\{S_e, P_v\}$ be a Cuntz-Krieger F-family on $\mathcal{H}\oplus\mathcal{H}_\mu$ with the properties described in Lemma 5.4. These properties say precisely that the representation $\pi_{S,P}$ of $C^*(F)$ satisfies $\pi_{S,P} \circ \phi = \pi_{T,Q}$. So ϕ is injective.

Because $p\phi(t_f) = \phi(t_f) = \phi(t_f)p$ for all $f \in F_\mu^1$, the range of ϕ is contained in the corner. Since $a \mapsto pap$ is continuous and linear,

$$\begin{aligned} pC^*(F)p &= \overline{\operatorname{span}}\{ps_\alpha s_\beta^* p : s(\alpha) = s(\beta)\} \\ &= \overline{\operatorname{span}}\{s_\alpha s_\beta^* : r(\alpha) \in F_\mu^0, r(\beta) \in F_\mu^0 \text{ and } s(\alpha) = s(\beta)\}. \end{aligned} \tag{5.3}$$

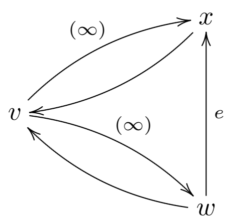
Every path α in F which begins and ends in F_μ^0 factors as a product of edges in F_μ^1 and paths in $F^*(\mu)$, and then s_α belongs to the range of ϕ. If $s(\alpha) = s(\beta) = s(\mu_n)$, then it follows from (5.2) that $s_\alpha s_\beta^* = s_{\alpha'} s_{\mu_1\cdots\mu_n} s_{\mu_1\cdots\mu_n}^* s_{\beta'}^*$ also belongs to the range of ϕ. So it follows from (5.3) that the range of ϕ is the corner, as claimed.

To see that $pC^*(F)p$ is not contained in a proper ideal, so that it is a full corner, note that $p_{r(\mu)}$ and $s_{\mu'} = p_{r(\mu)}s_{\mu'}$ belong to the ideal I generated by $pC^*(F)p$ for every finite final subpath $\mu' = \mu_1 \cdots \mu_n$ of μ. Thus $p_{s(\mu_n)} = s^*_{\mu'}s_{\mu'}$ belongs to I, and I contains all the generators of $C^*(F)$. □

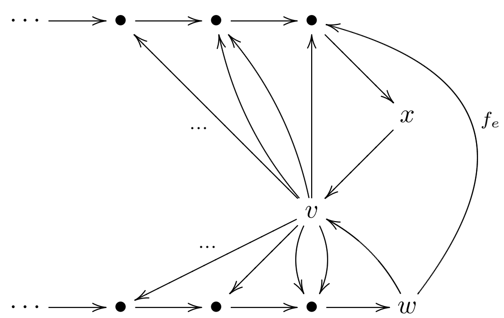
Remark 5.5. Collapsing a path, or the reverse process of inserting a collapsible path, is one of several constructions which one can perform on a graph without changing the graph algebra in an essential way. The process of inserting a collapsible path at a vertex v of E (that is, passing from E to a graph F with a collapsible path μ such that $E = F_\mu$) is an example of an *in-delay* at the vertex v. This and other constructions, some of which embed the graph algebra as a corner in another graph algebra and some of which don't change it at all, are discussed in [**27**] and [**8**]. Of course, one can then combine and iterate these constructions. Conversely, one can also ask what other kinds of subgraphs can be collapsed, and a general theorem along these lines is proved in [**13**].

When we have a collection M of disjoint collapsible paths, we can carry out the construction of Proposition 5.2 on all the paths in M simultaneously, yielding a graph F_{M} which will no longer be row-finite. Conversely, a *Drinen-Tomforde desingularisation* of a directed graph E is a row-finite graph F with a collection M of disjoint collapsible paths such that $F_{\mathrm{M}} = E$. It is easy to see that every graph has a Drinen-Tomforde desingularisation: just add a head μ at each infinite receiver v, and replace each edge $e \in E^1$ with $r(e) = v$ by an edge $f_e \in F$ with $s_F(f_e) = s(e)$ and range some $s(\mu_i)$, in such a way that each $s(\mu_i)$ receives only finitely many of these new edges. Then the new graph F will be row-finite, each new path will be collapsible, and on collapsing them we recover the original graph.

Example 5.6. Consider the following graph E:



where the symbol (∞) indicates that there are infinitely many edges going in the prescribed direction. Then a desingularisation of E might look like



Here M would consist of the two new infinite paths ending at x and w, and the only constraints on where we relocate the infinitely many edges from v to x, for example, are that their sources must be v, their range must lie on the new infinite path ending at x, and at most finitely many can point at any one vertex.

In view of the discussion preceding Example 5.6, applying the construction of Proposition 5.2 to the paths in M gives the following theorem of Drinen and Tomforde [**30**, Theorem 2.11].

THEOREM 5.7 (Drinen and Tomforde). *Let E be a directed graph, and let (F, M) be a Drinen-Tomforde desingularisation of E. Then there is an isomorphism of $C^*(E)$ onto the corner $pC^*(F)p$ associated to $p := \sum_{v\in E^0} p_v$, and this corner is full.*

The charm of Drinen and Tomforde's construction is that it is very concrete, and it is easy to see how properties of E are reflected in a desingularisation. Thus Theorem 5.7 allows us to extend many results about row-finite graphs to arbitrary graphs. The following version of the Cuntz-Krieger uniqueness theorem was first proved in [**46**, Theorem 2] using results from [**40**].

COROLLARY 5.8. *Suppose that E is a directed graph in which every cycle has an entry, and that $\{S, P\}$ and $\{T, Q\}$ are two Cuntz-Krieger E-families in which all the projections P_v and Q_v are non-zero. Then there is an isomorphism ψ of $C^*(S, P)$ onto $C^*(T, Q)$ such that $\psi(S_e) = T_e$ for evry $e \in E^1$ and $\psi(P_v) = Q_v$ for every $v \in E^0$.*

PROOF. (See [**30**, Corollary 2.12].) Let (F, M) be a Drinen-Tomforde desingularisation of E. Because the paths $\mu \in \mathrm{M}$ have no exits except at $r(\mu)$, the cycles in F are in one-to-one correspondence with the cycles in E. The cycles in F which pass through some $r(\mu)$ certainly have entries. The cycles in F which avoid all the $r(\mu)$ are cycles in E, and therefore have entries. So every cycle in F has a an entry.

We now use Lemma 5.4 to extend $\{S, P\}$ and $\{T, Q\}$ to Cuntz-Krieger F-families $\{S', P'\}$ and $\{T', Q'\}$. Since $P_w \neq 0$ for all $w \in E^0$, we have $S_e \neq 0$ for all $e \in E^1$. Every new edge in F occurs in at least one $\nu \in F^*(\mu)$, and $S'_\nu = S_{e_\nu} \neq 0$, so every new S'_e and every new P'_v are non-zero; similarly, every new Q'_v is non-zero. Thus the Cuntz-Krieger uniqueness theorem (Corollary 2.5) implies that there is an isomorphism θ of $C^*(S', P')$ onto $C^*(T', Q')$ which maps generators to generators. Since Lemma 5.4 also tells us how to recover the original Cuntz-Krieger families, θ maps $C^*(S, P)$ onto $C^*(T, Q)$, and $\psi := \theta|_{C^*(S,P)}$ has the required properties. □

This is a fairly typical application of desingularisations, and Drinen and Tomforde used similar arguments to extend many results about row-finite graphs. In particular, they proved that $C^*(E)$ is simple if and only if every cycle in E has an entry, E is cofinal, and we can reach every vertex in E from every infinite receiver [**30**, Corollary 2.15]. They also recovered the characterisation of purely infinite graph algebras of [**46**, Theorem 4], extended the characterisation of AF graph algebras [**30**, Corollary 2.13], and showed that a simple graph algebra $C^*(E)$ is always either purely infinite or AF [**30**, Remark 2.16], extending the dichotomy of [**82**].

REMARK 5.9. It is not immediately obvious how one might extend the gauge invariant uniqueness theorem to arbitrary graphs using a desingularisation, because the isomorphism of Theorem 5.7 is not equivariant for the respective gauge actions. One can prove such a gauge-invariant uniqueness theorem by writing $C^*(E)$ as an inductive limit $\overline{\bigcup_n C^*(F_n)}$ in which each F_n is a finite graph, as in [**113**, §1] (see [**7**, Theorem 2.1]). However, it might be interesting to have a proof of [**7**, Theorem 2.1] which uses a desingularisation. (The proof of [**140**, Theorem 6.8] works for adding a head, but does not seem to work when there are edges entering the head, as there are in a desingularisation.)

Theorem 5.7 is powerful because full corners pAp in a C^*-algebra inherit many properties of the ambient algebra A. In particular, as the next lemma shows, they have the same ideal theory. Recall that, by convention, all our ideals are closed and two-sided.

LEMMA 5.10. *Suppose that pAp is a full corner in a C^*-algebra A. Then the map $I \mapsto pIp$ is a bijection between the sets of ideals in A and in pAp; its inverse takes an ideal J in pAp to*

$$\overline{AJA} := \overline{\operatorname{span}}\{abc : b \in J \text{ and } a, c \in A\}. \tag{5.4}$$

PROOF. Suppose I is an ideal in A. Then the continuity of $a \mapsto pap$ implies that pIp is closed in pAp, and $(pAp)(pIp)(pAp) = p(Ap)I(pA)p \subset pIp$, so pIp is an ideal in pAp. Since $I = \overline{AIA}$, we have

$$\overline{A(pIp)A} = \overline{Ap(AIA)pA} = \overline{\overline{ApA}I\overline{ApA}} = \overline{AIA} = I. \tag{5.5}$$

On the other hand, if J is an ideal in pAp, then

$$p\overline{AJA}p = \overline{pAJAp} = \overline{pA(pAp)J(pAp)Ap} = \overline{(pAp)J(pAp)} = J. \tag{5.6}$$

Together, (5.5) and (5.6) give the result. □

Lemma 5.10 suggests that we should be able to identify the ideals in a graph algebra by applying Theorem 4.9 to a desingularisation of the graph. Before we start, we consider an instructive example.

EXAMPLE 5.11. We consider the following bi-infinite graph E in which there are infinitely many edges $\{e_i : i \geq 1\}$ from v_1 to w:

$$\cdots \longrightarrow v_3 \longrightarrow v_2 \longrightarrow v_1 \xrightarrow{(\infty)} w \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

The set $H := \{v_i : i \geq 1\}$ is hereditary, and each $s_{e_i} = s_{e_i}p_{s(e_i)} = s_{e_i}p_{v_1}$ belongs to the ideal I_H generated by $\{p_{v_i} : i \geq 1\}$. If there were only finitely many edges $\{e_i\}$ from v_1 to w, then $p_w = \sum_i s_{e_i}s_{e_i}^*$ would belong to I_H; as it is, there is no relation at p_w, so p_w does not belong to I_H. So if there is to be an analogue of Theorem 4.9 for non-row-finite graphs, we will have to amend the definition of saturated set to include sets like H.

Suppose that E is a graph satisfying Condition (K), and that (F, M) is a Drinen-Tomforde desingularisation of E. Since the return paths in E are in one-to-one correspondence with return paths in F, F also satisfies Condition (K). Theorem 4.9 tells us that the ideals in $C^*(F)$ are parametrised by the saturated herditary subsets of F^0. Thus we need to describe the saturated hereditary subsets of F^0 in terms of the original graph E.

Two vertices $v, w \in E^0$ satisfy $v \geq w$ in F if and only if $v \geq w$ in E, so for every hereditary set K in F, $H = K \cap E^0$ is hereditary in E. Conversely, if H is a hereditary subset of E^0, then

$$K_H := H \cup \{r_F(\mu_i) : i \geq 1,\ \mu \in \mathrm{M} \text{ and } r_F(\mu) \in H\}$$

is a hereditary subset of F^0. If $v = r_F(\mu_j)$ is a vertex on one of the collapsible paths $\mu \in \mathrm{M}$, then

$$s_F(r_F^{-1}(v)) \subset K_H \Longrightarrow s_F(\mu_j) \in K_H \Longrightarrow \{r_F(\mu_i) : i \geq 1\} \subset K_H \Longrightarrow v \in K_H.$$

The ranges of collapsible paths in M are precisely the infinite receivers in E. Thus K_H will be saturated in F if and only if

$$v \in E^0,\ |r_E^{-1}(v)| < \infty \text{ and } s_E(r_E^{-1}(v)) \subset H \Longrightarrow v \in H. \tag{5.7}$$

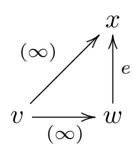
We say that a hereditary subset H of E^0 is *saturated* when (5.7) holds. For example, the set $H = \{v_i : i \geq 1\}$ in Example 5.11 is saturated.

There may be other saturated hereditary sets K in the desingularisation F with $K \cap E^0 = H$. Indeed, if $v = r_F(\mu)$ is an infinite receiver in $E^0 \setminus H$ and all but a finite positive number of edges in $r_E^{-1}(v)$ have sources in H, and

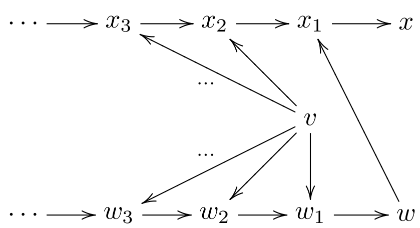
$$n_H(v) = \max\{i : s_F^{-1}(r_F(\mu_i)) \cap (E^0 \setminus H) \neq \emptyset\},$$

then $K_H \cup \{s_F(\mu_i) : i \geq n_H(v)\}$ is a saturated hereditary set in F whose intersection with E^0 is H. We call such vertices v *breaking vertices for* H, and we write B_H for the set of breaking vertices for H.

Example 5.12. Consider the following graph E



and the following desingularisation F of E:



The set $\{v\}$ is saturated and hereditary in E, and x is a breaking vertex for $\{v\}$ because all but the single edge e entering x come from $\{v\}$. The vertex w is not breaking for $\{v\}$ because all the edges into w come from $\{v\}$, and similarly x is not breaking for the saturated hereditary set $\{v, w\}$. The non-trivial saturated hereditary sets in the desingularisation F are $\{v\}$, $\{v\} \cup \{x_i : i \geq 2\}$ and $\{v, w\} \cup \{w_i : i \geq 1\}$.

If H is saturated and hereditary in E and $S \subset B_H$, the set

$$K_{H,S} := K_H \cup \{s(\mu_i) : \mu \in \mathrm{M},\ r(\mu) \in S,\ i \geq n_H(r(\mu))\}$$

is saturated and hereditary in F, and every saturated hereditary subset of F has this form. Let $J_{H,S}$ denote the ideal of $C^*(F) = C^*(t, q)$ generated by $\{q_w : w \in K_{H,S}\}$.

It now follows from Theorem 4.9 and Lemma 5.10 that the ideals in $pC^*(F)p$ are

$$\{pJ_{H,S}p : H \text{ is saturated and hereditary in } E \text{ and } S \subset B_H\}.$$

To find generators for $pJ_{H,S}p$, we note first that $pq_wp = q_w$ for $w \in E^0$, so $\{q_w : w \in H\} \subset pJ_{H,S}p$. Now suppose $v \in E^0 \setminus H$. Then we also know from Theorem 4.9

that q_v is not in $J_{H,S}$. However, if $v = r(\mu) \in S$, $J_{H,S}$ will contain a subprojection of q_v. To determine which subprojection, write $n(v)$ for $n_H(v)$ and notice that the range projection of $t_{\mu_1\cdots\mu_{n(v)}}$ is in $J_{H,S}$, as is the range projection of t_ν for any path ν of the form $\mu_1\cdots\mu_i f$ with $i < n(v)$ and $s(f) \in H$. Since

$$q_v = t_{\mu_1\cdots\mu_{n(v)}} t^*_{\mu_1\cdots\mu_{n(v)}} + \sum_{i=1}^{n(v)-1} \sum_{r(f)=s(\mu_i)} t_{\mu_1\cdots\mu_i f} t^*_{\mu_1\cdots\mu_i f}$$

(which is (5.2) in our present notation), we deduce that

$$pq_v p = q_v - \sum_{i=1}^{n(v)-1} \sum_{\{f\,:\,r(f)=s(\mu_i),\ s(f)\notin H\}} t_{\mu_1\cdots\mu_i f} t^*_{\mu_1\cdots\mu_i f}. \tag{5.8}$$

Thus $pJ_{H,S}p$ is generated by $\{q_v : v \in H\}$ and the *gap projections* (5.8). When e is an edge in E with $r_E(e) = v$ and $s_E(e) \notin H$, $e = e_\nu$ for a unique path $\nu = \mu_1 \cdots \mu_i f$, and the isomorphism ϕ of Theorem 5.7 carries s_e into $t_{\mu_1\cdots\mu_i f}$. Thus the corresponding ideal $I_{H,S} := \phi^{-1}(pJ_{H,S}p)$ of $C^*(E)$ is generated by

$$\{p_v : v \in H\} \cup \Big\{ p_v - \sum_{\{e\in E^1\,:\,r(e)=v,\ s(e)\notin H\}} s_e s_e^* : v \in S \Big\}. \tag{5.9}$$

Thus we have:

THEOREM 5.13. *Suppose that E is a directed graph which satisfies Condition* (K)*. Then the ideals in $C^*(E)$ are parametrised by the saturated hereditary subsets H of E^0 and sets S of breaking vertices for H. The ideal $I_{H,S}$ corresponding to such a pair (H,S) is generated by* (5.9).

This is Theorem 3.5 of [**30**], and the proof there is essentially the one we have just given; it was obtained independently by different methods in [**7**, Corollary 3.8].

We finish this chapter by surveying what else is known about the ideal structure of graph algebras. When E does not satisfy Condition (K), the ideals $I_{H,S}$ generated by (5.9) are the ideals in $C^*(E)$ which are invariant under the gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(E)$ [**7**, Theorem 3.6], and one can identify the quotient $C^*(E)/I_{H,S}$ as a graph algebra [**7**, Corollary 3.5]. Using these results, Hong and Szymański have obtained a complete description of the ideal structure of $C^*(E)$ [**59**]. To describe their achievement, we have to digress briefly.

The set of ideals in a C^*-algebra A is partially ordered under inclusion, and is then a *lattice*: every pair of ideals I, J has a *greatest lower bound*, namely $I \cap J$, and a *least upper bound*, namely the ideal $I \vee J$ generated by $I \cup J$. As Example 5.15 shows, even when there are very few ideals, this lattice structure can vary wildly.

The subsets of any set also form a lattice under inclusion, and when E is a row-finite directed graph, the saturated hereditary subsets of E^0 form a sublattice. The bijection $H \mapsto I_H$ of Theorem 4.9 preserves inclusion, and hence is a lattice isomorphism. Drinen and Tomforde also describe the partial order on pairs (H,S) which makes their bijection a lattice isomorphism [**30**, §3].

An efficient way to encode the ideal structure of a C^*-algebra is to describe its primitive ideal space. An ideal is *primitive* if it is the kernel of an irreducible representation of A. We denote by $\operatorname{Prim} A$ the set of primitive ideals of A. Every ideal is the intersection of the primitive ideals containing it [**114**, Proposition A.17], and the sets $h(I) := \{P \in \operatorname{Prim} A : I \subset P\}$ are the closed sets in a topology on Prim A [**114**, Proposition A.27]; the topological space Prim A is then called the *primitive ideal space of* A. The map h is inclusion-reversing, so if one can identify the set Prim A and the topology on it in familiar terms, one has a complete understanding of the lattice of ideals in A.

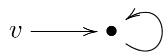
EXAMPLE 5.14. Let X be a compact Hausdorff space. Every irreducible representation of $C(X, M_n(\mathbb{C}))$ is equivalent to an evaluation map $\varepsilon_x : f \mapsto f(x)$, so the map $x \mapsto \ker \varepsilon_x$ is a bijection of X onto the set of primitive ideals. This map is in fact a homeomorphism of X onto $\operatorname{Prim} C(X, M_n(\mathbb{C}))$. This and other related examples are discussed in [**114**, Examples A.23–25].

EXAMPLE 5.15. The primitive ideals in $\mathbb{C} \oplus \mathbb{C}$ are $I = \{(w, 0)\}$, which is the kernel of $(w, z) \mapsto z$, and $J = \{(0, z)\}$. Notice that $I \cap J = \{0\}$, so $\{I\}$ and $\{J\}$ are closed and open sets in $\operatorname{Prim}(\mathbb{C} \oplus \mathbb{C})$. The algebra $B(\mathcal{H})$ also has two primitive ideals, namely $\{0\}$ and $\mathcal{K}(\mathcal{H})$ (this is proved in [**114**, Example A.32], for example). But $\{0\} \subset \mathcal{K}(\mathcal{H})$, so the closure of $\{0\}$ in $\operatorname{Prim} B(\mathcal{H})$ is the whole space. One can quickly check that the lattice operation $\vee$ is quite different too.

If E is a row-finite graph which satisfies Condition (K), then the saturated hereditary subsets of E^0 for which I_H is primitive are identified in [**9**, Proposition 6.1], and a description of the topology on the space χ_E of such sets is given in [**9**, Theorem 6.3 and Corollary 6.5]. (In fact it is more convenient to describe the topology on $\{E^0 \setminus H : I_H \text{ is hereditary}\}$.) These results were extended to arbitrary directed graphs which satisfy (K) in [**30**, §4], using a desingularisation.

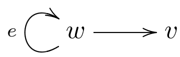
When E does not satisfy Condition (K), vertices which lie on just one return path in E give rise to a family of primitive ideals parametrised by the unit circle $\mathbb{T}$. To see why, consider some examples.

EXAMPLE 5.16. Suppose that E is a row-finite graph, and we form a new graph F by adding



to a vertex v of E. Then by taking $p_x = 0$ for all $x \in E^0$, we get a homomorphism of $C^*(F)$ onto the C^*-algebra $C^*(C_1)$ of a single loop, which is isomorphic to $C(\mathbb{T})$ (see Example 2.14). Thus we get an embedding of $\mathbb{T}$ as a closed subset of $\operatorname{Prim} C^*(F)$ (by [**114**, Proposition A.27(c)], for example).

If we form a new graph G by adding



to a vertex v of E, then the ideal $I_{\{w\}}$ in $C^*(G)$ contains $C^*(C_1) = C(\mathbb{T})$ as a full corner (see Remark 4.11). By Lemma 5.10, this implies that the primitive ideals of $I_{\{w\}}$ are parametrised by $\mathbb{T}$, and in fact we then have a homeomorphism of $\mathbb{T}$ onto an open subset of $\operatorname{Prim} C^*(G)$ (see [**114**, Proposition A.27(b)]). If we add a cycle C_n with n edges, then $C^*(C_n)$ is isomorphic to $C(\mathbb{T}, M_n(\mathbb{C}))$, and we again get a copy of $\mathbb{T}$ in the primitive ideal space. In general, vertices which lie on just one return path in E give rise to ideals in quotients of $C^*(E)$ with primitive ideal space $\mathbb{T}$.

So the problem is how to accommodate these circles in the description of $\operatorname{Prim} C^*(E)$. For the C^*-algebras of finite graphs, or more precisely for Cuntz-Krieger algebras, this problem was solved in [**60**]. The arguments used in [**60**], however, used the finiteness of the graph in crucial ways, and the corresponding problem for infinite graphs (and even for row-finite graphs) proved to be harder. It is this problem which was eventually settled by Hong and Szymański in [**59**, Theorem 3.4]. Their theorem has already had several applications [**55, 57**].

Thus we now have a complete description of the ideal structure of $C^*(E)$ for an arbitrary directed graph E. Graph algebras are one of the very few classes of non-simple C^*-algebras for which we can make such a claim.

Remark 5.17. The connection between graph algebras and the Cuntz-Krieger algebras of $\{0,1\}$-matrices described in Remark 2.8 extends without difficulty to row-finite graphs. For more general graphs, however, the connection breaks down, because it would involve infinite sums of projections and partial isometries. Exel and Laca have introduced a different notion of Cuntz-Krieger algebras for infinite $\{0,1\}$-matrices which includes most graph algebras, and extended the basic theory of Cuntz-Krieger algebras to their *Exel-Laca algebras* ([**40**], see also [**135**]). Exel and Laca used crossed products by partial actions as their main tool; since then, Exel-Laca algebras have been studied using Cuntz-Pimsner algebras [**134, 126**], using a groupoid model [**116**], and as inductive limits of the C^*-algebras of finite graphs with sources [**113**]. More recently, Tomforde has described a generalisation of directed graphs called *ultragraphs*, in which the source of an edge is a set of vertices rather than a single vertex. He has shown that ultragraphs have C^*-algebras which behave very much like graph algebras, but which include the Exel-Laca algebras. Tomforde has proved uniqueness theorems for these *ultragraph algebras*, has found criteria for simplicity and pure-infiniteness of ultragraph algebras, and has shown that simple ultragraph algebras are either AF or purely infinite [**140, 141**]. He has also shown by example that the class of ultragraph algebras is strictly larger than the class of Exel-Laca algebras [**140**, §5].

CHAPTER 6

Applications to non-abelian duality

Recall that an *action* of a locally compact group G on a C^*-algebra A is a homomorphism α of G into the group $\operatorname{Aut} A$ of automorphisms of A such that, for each fixed $a \in A$, the map $s \mapsto \alpha_s(a)$ is continuous from G to A. For example, if E is a directed graph, the gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} C^*(E)$ is an action of the compact group $\mathbb{T}$. Other examples are the actions $\mathrm{lt} : G \to \operatorname{Aut} C_0(X)$ and $\mathrm{rt} : G \to \operatorname{Aut} C_0(X)$ induced by left and right actions of G on a locally compact space X: $\mathrm{lt}_s(f)(x) := f(s^{-1} \cdot x)$ and $\mathrm{rt}_s(f)(x) := f(x \cdot s)$.

A triple (A, G, α) consisting of a C^*-algebra A, a locally compact group G and an action $\alpha : G \to \operatorname{Aut} A$ is a *dynamical system*. A *covariant representation* of (A, G, α) on a Hilbert space $\mathcal{H}$ is a pair (π, U) consisting of a non-degenerate representation $\pi : A \to B(\mathcal{H})$ and a strongly continuous unitary representation $U : G \to U(\mathcal{H})$ such that

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^* \quad \text{for } a \in A \text{ and } s \in G.$$

Non-trivial examples are the *regular representations* $(\tilde{\pi}, \lambda)$ induced by a representation $\pi : A \to B(\mathcal{H})$, which act on $L^2(G, \mathcal{H})$ by the formulas $\tilde{\pi}(a)(f)(t) = \pi(\alpha_t^{-1}(a))(f(t))$ and $\lambda_s(f)(t) = f(s^{-1}t)$.

The *crossed product* $A \rtimes_\alpha G$ is by definition a C^*-algebra generated by a universal covariant homomorphism (i_A, i_G) of (A, G, α) into the multiplier algebra $M(A \rtimes_\alpha G)$: for every covariant representation (π, U) of (A, G, α) on $\mathcal{H}$, there is a representation $\pi \times U$ of $A \rtimes_\alpha G$ on $\mathcal{H}$, called the *integrated form of* (π, U), such that $\pi = (\pi \times U) \circ i_A$ and $U = (\pi \times U) \circ i_G$. This is made precise in [**107**], and it is proved there that there is, up to isomorphism, exactly one such triple $(A \rtimes_\alpha G, i_A, i_G)$. There is also a *reduced crossed product*, which is the image $A \rtimes_{\alpha,r} G := \tilde{\pi} \times \lambda(A \rtimes_\alpha G)$ of the crossed product under the regular representation induced from a faithful representation π of A.

Remark 6.1. (To be ignored on a first reading.) Our definition of crossed product was a little loose: one has to specify in what sense (i_A, i_G) generates $A \times_\alpha G$, because the generators $i_G(s)$ do not lie in $A \rtimes_\alpha G$ unless A has an identity and G is discrete. However, for $f \in C_c(G)$ and $a \in A$, the function $s \mapsto f(s) i_A(a) i_G(s)$ is a continuous function of compact support with values in $A \rtimes_\alpha G$, and its integral

$$i_A(a) i_G(f) := \int_G f(s) i_A(a) i_G(s)\, ds$$

with respect to Haar measure makes sense in $A \rtimes_\alpha G$. (As the notation suggests, we can also make sense of $i_G(f)$ as an element of the multiplier algebra $M(A \rtimes_\alpha G)$. For details, see the Appendix in [**107**] or [**114**, Lemma C.11].) When we say that (i_A, i_G) generates $A \rtimes_\alpha G$, we mean that the elements $i_A(a) i_G(f)$ span a dense subspace of $A \rtimes_\alpha G$. The crossed product is usually constructed as a completion of a convolution algebra based on $C_c(G, A)$, and then this assertion looks more natural.

Now suppose G is abelian. We denote by $\widehat{G}$ the *dual group* of continuous homomorphisms $\theta : G \to \mathbb{T}$, which is itself a locally compact abelian group under pointwise multiplication [**44**, §4.1]. For $s \in G$, $\varepsilon_s(\theta) := \theta(s)$ defines a continuous homomorphism $\varepsilon_s : \widehat{G} \to \mathbb{T}$, and the Pontryagin duality theorem says that $s \mapsto \varepsilon_s$ is an isomorphism of G onto the dual of $\widehat{G}$ [**44**, §4.3]; we use this map to identify G with the double dual. When α is an action of the abelian group G on A, there is a *dual system* $(A \rtimes_\alpha G, \widehat{G}, \widehat{\alpha})$ such that $\widehat{\alpha}_\theta(i_A(a)) = i_A(a)$ and $\widehat{\alpha}_\theta(i_G(s)) = \theta(s) i_G(s)$; to see this, observe that $(A \rtimes_\alpha G, i_A, \theta i_G)$ has the universal property which characterises the crossed product, and then uniqueness of the crossed product gives the desired isomorphism $\widehat{\alpha}_\theta : (A \rtimes_\alpha G, i_A, i_G) \to (A \rtimes_\alpha G, i_A, \theta i_G)$ [**107**, Proposition 5]. The Takesaki-Takai duality theorem says that the original system (A, G, α) is stably isomorphic to the double dual system $((A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}, G, \widehat{\widehat{\alpha}})$, in the sense that there is an isomorphism of $(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}$ onto $A \otimes \mathcal{K}(L^2(G))$ which carries $\widehat{\widehat{\alpha}}$ into the tensor product $\alpha \otimes (\operatorname{Ad} \rho)$ of α with conjugation by the right-regular representation ρ on $L^2(G)$ (see [**139**] or [**107**, Theorem 6]).

When the group G is non-abelian, the analogue of Takesaki-Takai duality uses a dual coaction in place of the dual action, and we can recover (A, G, α) from $A \rtimes_\alpha G$ by taking the crossed product by this dual coaction. For locally compact groups, coactions can be technically formidable: because the canonical images of A and G only exist in the multiplier algebra of the crossed product they generate, one is forced to deal with multiplier algebras at every turn, and because tensor products are involved, this can get complicated. Coactions of discrete groups are much easier to handle, and this is what arises in connection with graph algebras. So here we will only consider coactions of discrete groups. For those who are interested in the general theory, we suggest looking first at the expository article [**109**], and then the more detailed survey in [**32**, Appendix A].

Let G be a (discrete)[1] group. The *group C^*-algebra* of G is the C^*-algebra generated by a universal unitary representation u of G in the group $UC^*(G)$ of unitary elements of $C^*(G)$. Applying this universal property to the representation $u \otimes u$ of G in the tensor product[2] $C^*(G) \otimes C^*(G)$ gives a homomorphism $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ such that $\delta_G(u_s) = u_s \otimes u_s$ for $s \in G$. The homomorphism δ_G is *coassociative* in the sense that $(\delta_G \otimes \iota) \circ \delta_G = (\iota \otimes \delta_G) \circ \delta_G$, where ι denotes the identity automorphism of a C^*-algebra (here, of $C^*(G)$); δ_G is called the *comultiplication* on $C^*(G)$. A *coaction* of G on a C^*-algebra B is a homomorphism $\delta : B \to B \otimes C^*(G)$ satisfying the *coaction identity*

$$(\delta \otimes \iota) \circ \delta = (\iota \otimes \delta_G) \circ \delta : B \to B \otimes C^*(G) \otimes C^*(G).$$

EXAMPLE 6.2. Suppose α is an action of a discrete group G on A, in which case

$$A \rtimes_\alpha G = \overline{\operatorname{span}}\{i_A(a) i_G(s) : a \in A,\ s \in G\}. \tag{6.1}$$

The *dual coaction* is the integrated form

$$\widehat{\alpha} := (i_A \otimes 1) \times (i_G \otimes u) : A \rtimes_\alpha G \to (A \rtimes_\alpha G) \otimes C^*(G);$$

[1]That is, just a group. We say "discrete" to emphasise that no topology is involved.

[2]In these notes we use only the spatial tensor product of C^*-algebras, which is discussed in [**114**, Appendix B.1], for example.

strictly speaking, one has to represent $(A \rtimes_\alpha G) \otimes C^*(G)$ on Hilbert space to apply the universal property of the crossed product, but it then follows from (6.1) that the range of the integrated form lies in $(A \rtimes_\alpha G) \otimes C^*(G)$. The coaction identity holds because both sides are the integrated form of $(i_A \otimes 1 \otimes 1, i_G \otimes u \otimes u)$.

EXAMPLE 6.3. Suppose that E is a directed graph, G is a discrete group and $c : E^1 \to G$ is a function; we call c a *labelling* of the edges of E by G. Then $\{s_e \otimes u_{c(e)}, p_v \otimes 1)$ is a Cuntz-Krieger E-family in $C^*(E) \otimes C^*(G)$, and the universal property of the graph algebra gives a homomorphism $\delta_c : C^*(E) \to C^*(E) \otimes C^*(G)$ such that $\delta_c(s_e) = s_e \otimes u_{c(e)}$ and $\delta(p_v) = p_v \otimes 1$. It is easy to check on generators that δ_c is a coaction of G on $C^*(E)$.

EXAMPLE 6.4. Suppose that Γ is a compact abelian group, and α is an action of Γ on A. Then the dual $G := \widehat{\Gamma}$ is a discrete abelian group with dual Γ, and the Fourier transform for G gives an isomorphism of $C^*(G)$ onto $C(\Gamma)$ which carries δ_G into the comultiplication $\alpha_\Gamma : C(\Gamma) \to C(\Gamma) \otimes C(\Gamma) = C(\Gamma \times \Gamma)$ defined by $\alpha_\Gamma(f)(\gamma, \tau) = f(\gamma\tau)$. The map $\delta : A \to C(\Gamma, A) = A \otimes C(\Gamma)$ defined by $\delta(a)(\gamma) = \alpha_\gamma(a)$ satisfies

$$(\delta \otimes \iota)(\delta(a))(\gamma, \tau) = \alpha_\gamma(\alpha_\tau(a)) = \alpha_{\gamma\tau}(a) = (\iota \otimes \alpha_\Gamma)(\delta(a))(\gamma, \tau);$$

thus, viewed as a homomorphism of A into $A \otimes C^*(G) \cong A \otimes C(\Gamma)$, δ is a coaction of G on A.

Coactions of groups on a C^*-algebra B are easier to understand when G is discrete because we can think of them as giving *gradings* of B. Let δ be a coaction of a group G on a C^*-algebra B, and consider the *spectral subspaces*

$$B_s := \{b \in B : \delta(b) = b \otimes u_s\},$$

which because δ is a $*$-homomorphism satisfy $B_s B_t \subset B_{st}$ and $B_s^* = B_{s^{-1}}$. In the examples, it is obvious that $\bigcup_{s \in G} B_s$ spans a dense subspace of G: in Example 6.2, $i_A(a) i_G(s) \in (A \rtimes_\alpha G)_s$, and this follows from (6.1); in Example 6.3, the elements $s_\mu s_\nu^*$ belong to the spectral subspace $C^*(E)_s$ for $s := (\prod c(\mu_i))(\prod c(\nu_j))^{-1}$, and this follows from Corollary 1.16[3].

REMARK 6.5. We caution that $\{B_s : s \in G\}$ is not a grading of B in the usual algebraic sense: B is $\overline{\text{span}}(\bigcup_s B_s)$ rather than $\text{span}(\bigcup_s B_s)$, and the examples show that there is a big difference. When $s \neq t$ in G, we have $B_s \cap B_t = \emptyset$, and hence $\text{span}(\bigcup_s B_s)$ is the algebraic direct sum of the subspaces B_s. However, there is no obvious way to describe the norm on $\text{span}(\bigcup_s B_s)$ in terms of the norms on B_s, and hence no obvious Banach space direct-sum decomposition of B. The subspaces $\{B_s : s \in G\}$ form what is known as a *Fell bundle* over G, and one can often recover B as a C^*-completion of the $*$-algebra of sections of this Fell bundle. For more details and further references on this point of view, see [**105**] and [**34**, §2].

Suppose δ is a coaction of a discrete group G on a C^*-algebra B. A pair (π, μ) consisting of non-degenerate representations $\pi : B \to B(\mathcal{H})$ and $\mu : c_0(G) \to B(\mathcal{H})$ is a *covariant representation* of (B, G, δ) if

$$b \in B_s, f \in c_0(G) \Longrightarrow \pi(b)\mu(f) = \mu(\text{lt}_s(f))\pi(b). \tag{6.2}$$

[3]In fact, a theorem of Baaj and Skandalis [**5**, Corollaire 7.15] implies that for every coaction δ of a discrete group on B, $\bigcup_{s \in G} B_s$ spans a dense subspace of B; coaction afficionados would say that coactions of discrete groups are automatically non-degenerate.

(We caution that this definition only works when G is discrete; the usual definition is [**32**, Definition A.32], which is proved in [**24**, Lemma 3.1] to be equivalent to (6.2) when G is discrete.) To construct non-trivial examples, let π be a non-degenerate representation of B on $\mathcal{H}$. The *regular representation of* (B, G, δ) *induced by* π is the representation

$$((\pi \otimes \lambda) \circ \delta, 1 \otimes M) : (B, c_0(G)) \to B(\ell^2(G, \mathcal{H})) = B(\mathcal{H} \otimes \ell^2(G)), \tag{6.3}$$

where λ is the left-regular representation of G on $\ell^2(G)$ and M is the representatiuon of $c_0(G)$ by multiplication operators on $\ell^2(G)$. To see that the regular representation is covariant, note that $b \in B_s$ implies $(\pi \otimes \lambda)(\delta(b)) = \pi(b) \otimes \lambda_s$, and use the covariance of (M, λ) for the action $\mathrm{lt} : G \to \operatorname{Aut} c_0(G)$ to compute

$$\begin{aligned}(\pi \otimes \lambda)(\delta(b))(1 \otimes M(f)) &= (\pi(b) \otimes \lambda_s)(1 \otimes M(f))\\ &= \pi(b) \otimes (M(\mathrm{lt}_s(f))\lambda_s)\\ &= (1 \otimes M(\mathrm{lt}_s(f)))(\pi(b) \otimes \lambda_s)\\ &= (1 \otimes M)(\mathrm{lt}_s(f))(\pi \otimes \lambda)(\delta(b)).\end{aligned}$$

The *crossed product* $B \rtimes_\delta G$ is generated by a universal covariant representation (j_B, j_G); we write $\pi \times \mu$ for the representation of $B \rtimes_\delta G$ such that $\pi = (\pi \times \mu) \circ j_B$ and $\mu = (\pi \times \mu) \circ j_G$. Because G is discrete, we have

$$B \rtimes_\delta G = \overline{\operatorname{span}}\{j_B(b) j_G(\chi_s) : b \in B,\ s \in G\},$$

where χ_s denotes the characteristic function of the set $\{s\}$. One can prove, much as we did when we proved the existence of the universal graph algebra $C^*(E)$, that there is such a crossed product, and that it is unique up to isomorphism; the crucial observation is that the elements χ_s are projections, so $\|\pi(b)\mu(\chi_s)\| \leq \|b\|$ for every covariant representation (π, μ). From the uniqueness, we deduce that $B \rtimes_\delta G$ carries a *dual action* $\widehat{\delta}$ of G, which is characterised by

$$\widehat{\delta}_s(j_B(b) j_G(f)) = j_B(b) j_G(\mathrm{rt}_s(f)).$$

There is no reduced crossed product for coactions, because it is known that if π is faithful, then the regular representation induced by π gives a faithful representation of $B \rtimes_\delta G$ [**108**, Theorem 4.1].

EXAMPLE 6.6. Suppose Γ is a compact abelian group, α is an action of Γ on A, and δ is the coaction of the dual group $G = \widehat{\Gamma}$ constructed in Example 6.4. Then the crossed product $A \rtimes_\delta G$ is $(A \rtimes_\alpha \Gamma, j_A, j_G)$, where $j_A = i_A$ and j_G is the composition of the integrated form $i_\Gamma : C^*(\Gamma) \to M(A \rtimes_\alpha \Gamma)$ with the inverse Fourier transform of $c_0(G) = c_0(\widehat{\Gamma})$ into $C^*(\Gamma)$. One way to prove this is to check that the regular representations are the same.

The cornerstones in non-abelian duality for C^*-algebras are the duality theorems of Imai-Takai and Katayama. The *Imai-Takai duality theorem* says that, if α is an action of G on A, then the crossed product system $((A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} G, G, \widehat{\widehat{\alpha}})$ is stably isomorphic to the original system (A, G, α). The *Katayama duality theorem* says that, if δ is a coaction of G on B, then $((B \rtimes_\delta G) \rtimes_{\widehat{\delta}, r} G, G, \widehat{\widehat{\delta}})$ is stably isomorphic to the original system (B, G, δ). These theorems are proved in [**32**, Appendix A.8], where further references are given.

Crossed products by dual coactions, then, are well understood. The crossed products by the coactions on graph algebras in Example 6.3 turn out to be related to a graph-theoretic construction. Suppose that E is a directed graph and $c : E^1 \to G$ is a labelling. The *skew-product graph* $E \times_c G$ has vertex set $(E \times_c G)^0 = E^0 \times G$, edge set $(E \times_c G)^1 = E^1 \times G$, and range and source maps defined by

$$r(f,t) = (r(f), c(f)t) \quad \text{and} \quad s(f,t) = (s(f), t).$$

PROPOSITION 6.7. *Suppose that E is a directed graph and $c : E^1 \to G$ is a labelling, and let δ_c be the corresponding coaction of G on $C^*(E)$. Then there is an isomorphism of $C^*(E) \rtimes_{\delta_c} G$ onto the C^*-algebra $C^*(E \times_c G)$ of the skew product which converts the dual action of G into the action β defined on a generating Cuntz-Krieger family $\{s, p\}$ by $\beta_u(s_{(e,t)}) = s_{(e,tu^{-1})}$, $\beta_u(p_{(v,t)}) = p_{(v,tu^{-1})}$.*

To prove the theorem, consider the universal Cuntz-Krieger E-family $\{s, p\}$ in $C^*(E)$, verify that

$$t_{(f,t)} := j_{C^*(E)}(s_f) j_G(\chi_t) \quad \text{and} \quad q_{(v,t)} := j_{C^*(E)}(p_v) j_G(\chi_t)$$

is a Cuntz-Krieger $(E \times_c G)$-family in $C^*(E) \rtimes_{\delta_c} G$, and then invoke the gauge-invariant uniqueness theorem. For row-finite graphs, the details are in [**70**, Theorem 2.4]; the general result is a special case of [**24**, Theorem 3.4], which we will discuss later.

The skew product $E \times_c G$ has a natural right action by G such that $(f, s) \cdot t = (f, st)$. This action is free, and a theorem of Gross and Tucker [**53**, Theorem 2.2.2] asserts that every free action on a directed graph arise this way. To see this, suppose G acts freely on the right of a directed graph F, and take E to be the *quotient graph* F/G, in which $(F/G)^i$ are the orbit spaces F^i/G. Then we can realise F as a skew product over F/G, as follows: choose a section $\sigma : (F/G)^0 = F^0/G \to F^0$ for the quotient map, define $c : (F/G)^1 = F^1/G \to G$ by

$$c(fG) = tu^{-1} \quad \text{where } s(f) = \sigma(s(f)G)u \text{ and } r(f) = \sigma(r(f)G)t,$$

and verify that the functions $\phi^i : F^i \to (F/G)^i \times_c G$ defined by

$$\phi^0(v) = (vG, t) \quad \text{where } v = \sigma(vG)t, \text{ and}$$
$$\phi^1(f) = (fG, u) \quad \text{where } s(f) = \sigma(s(f)G)u$$

give an isomorphism of directed graphs which converts the original action into that given by $(fG, t) \cdot u = (fG, tu)$.

If G acts freely on the right of a graph F, then there is an induced action α of G on $C^*(F)$. The isomorphism ϕ of the previous paragraph induces an isomorphism $\phi_* : C^*(F) \to C^*((F/G) \times_c G)$ which converts α into the action β appearing in Proposition 6.7. From this and Proposition 6.7 we have the following isomorphisms of crossed products:

$$C^*(F) \rtimes_{\alpha,r} G \cong C^*((F/G) \times_c G) \rtimes_{\beta,r} G \cong (C^*(F/G) \rtimes_{\delta_c} G) \rtimes_{\widehat{\delta}_c,r} G.$$

Katayama's duality theorem says that the double crossed product on the right is stably isomorphic to $C^*(F/G)$, and we recover a beautiful theorem of Kumjian and Pask [**80**, Corollary 3.9]:

THEOREM 6.8. *Suppose a group G acts freely on a directed graph F, and let α denote the induced action on $C^*(F)$. Then the crossed product $C^*(F) \rtimes_{\alpha,r} G$ is stably isomorphic to the C^*-algebra $C^*(F/G)$ of the quotient graph.*

REMARK 6.9. The argument we used to prove Theorem 6.8 is that of [**70**, §1]. The reduced crossed products arise because they appear in Katayama's duality theorem. In fact $C^*(F) \rtimes_\alpha G \cong C^*(F) \rtimes_{\alpha,r} G$ for these actions; this is proved in [**70**, §3], by establishing directly a stable isomorphism of $C^*(F) \rtimes_\alpha G$ with $C^*(F/G)$ (which was the original Kumjian-Pask theorem), and comparing the two stable isomorphisms [**70**, Corollary 3.2].

REMARK 6.10. Kumjian and Pask's theorem was inspired by a theorem of Green [**51**] about actions of a locally compact group G on a locally compact space X which are *proper* in the sense that inverse images of compact sets under the map $(x, s) \mapsto (x, x \cdot s)$ are compact. Green's theorem says that if G acts freely and properly on a locally compact Hausdorff space X, then the crossed product $C_0(X) \rtimes_{\mathrm{rt}} G$ is stably isomorphic to $C_0(X/G)$. Green's theorem has a powerful generalisation called the symmetric imprimitivity theorem [**117**], which concerns a pair of commuting free and proper actions on the same space, and it too has an analogue for commuting free actions on directed graphs [**97**, Theorem 2.1]. This suggests that graph algebras are behaving like algebras of continuous functions on non-commutative spaces, and it is intriguing to wonder what other properties of proper actions on locally compact spaces might have analogues for actions on graph algebras.

The skew products $E \times_c G$ appearing in Proposition 6.7 are examples of coverings of E. In general, a *covering* of E is a directed graph F together with a surjective graph homomorphism $p : F \to E$ which is a local isomorphism, in the sense that for each vertex $v \in F^0$, p maps $s^{-1}(v)$ and $r^{-1}(v)$ bijectively onto $s^{-1}(p(v))$ and $r^{-1}(p(v))$; for the skew product, $p : (e, t) \mapsto e$ has these properties. So Proposition 6.7 identifies the C^*-algebra of the covering graph $E \times_c G$ as the crossed product of $C^*(E)$ by a coaction of G. At this point, one naturally asks to which coverings this applies, and if there are other coverings, whether we can extend this result in some way.

The usual classification of connected coverings of a topological space, as described in [**50**, Chapters 12 and 13], for example, works for a directed graph E — indeed, one could apply that classification as it stands to a drawing of E in Euclidean space, which would satisfy the hypotheses of the usual theory. One can also develop the theory combinatorially (see [**133**, Chapter 2], for example). The word "connected" in the context of coverings is not the obvious notion of connectedness for directed graphs. A *walk* in a directed graph E is a path in the underlying undirected graph; formally, we write $E^{-1} := \{e^{-1} : e \in E^1\}$, define $r(e^{-1}) = s(e)$ and $s(e^{-1}) = r(e)$, and then a walk a is a sequence $a = a_1 a_2 \cdots a_n$ with $a_i \in E^1 \cup E^{-1}$ and $s(a_i) = r(a_{i+1})$ for $1 \leq i < n$. The directed graph E is *connected* if for every two vertices $v, w \in E^0$, there is a walk $a = a_1 \cdots a_n$ with $v = s(a) := s(a_n)$ and $w = r(a) := r(a_1)$. A walk is *reduced* if $a_i \neq a_{i+1}^{-1}$ for all i (where $(e^{-1})^{-1}$ is by definition taken to be e). We can multiply reduced walks with the appropriate range and source by concatenating and cancelling. For $w \in E^0$, the *fundamental group* $\pi_1(E, w)$ *of* E *based at* w is the group of reduced walks a with $s(a) = r(a) = w$; the identity is the walk consisting of the vertex w, and $(a_1 \cdots a_n)^{-1} = a_n^{-1} \cdots a_1^{-1}$. A connected covering $p : F \to E$ induces a homomorphism $p_* : \pi_1(F, v) \to \pi_1(E, p(v))$, and the covering is *regular* if the image of p_* is a normal subgroup. The regular coverings are distinguished by their automorphism group: a connected covering is regular if and only if the automorphism group acts transitively on each fibre $p^{-1}(w)$. If so, the automorphism group is isomorphic to $\pi_1(E, p(v))/p_*\pi_1(F, v)$ (see [**50**, Section 13b]).

In a skew-product covering $p : E \times_c G \to E$, the right action of G acts as automorphisms of the covering, and it acts transitively on each fibre $p^{-1}(w) = \{(w, t) : t \in G\}$, so $p : E \times_c G \to E$ is a regular covering whenever it is connected. On the other hand, every regular covering admits a free action of its automorphism group $\pi_1(E, p(v))/p_*\pi_1(F, v)$, and hence by the Gross-Tucker theorem is isomorphic to a skew product. Thus the connected skew products of E are precisely the regular coverings of E.

Suppose that $p : F \to E$ is a connected covering. When $p : F \to E$ is not regular, the subgroup $p_*\pi_1(F, v)$ of $\pi_1(E, p(v))$ is not normal, but the fibre $p^{-1}(p(v))$ is still naturally in bijective correspondence with the coset space or *homogeneous space* $\pi_1(E, p(v))/p_*\pi_1(F, v)$ [**24**, Lemma 2.1], and there is a version of the Gross-Tucker theorem which identifies F as a *relative skew product*. Suppose E is a directed graph, $c : E^1 \to G$ is a labelling, and H is a subgroup of G. The relative skew product is the graph $E \times_c (G/H)$ with $(E \times_c (G/H))^i = E^i \times (G/H)$,

$$r(e, tH) = (r(e), c(e)tH) \quad \text{and} \quad s(e, tH) = (s(e), tH).$$

To define a suitable labelling for F, we need to fix a *spanning tree* T for E, which is a subgraph T of E with $T^0 = E^0$ and the property that, for every $v, w \in E^0$, there is exactly one reduced walk a in T with $s(a) = w$ and $r(a) = v$; a Zorn's lemma argument shows that every connected graph has a spanning tree. The following result of Pask and Rho [**98**] is given a shorter proof in [**24**, §2].

PROPOSITION 6.11. *Let $p : F \to E$ be a connected covering of directed graphs and let $v \in F^0$. Choose a spanning tree T for E, and for $w \in E^0$, let a_w be the reduced walk in T from $p(v)$ to w. Define $c : E^1 \to \pi_1(E, p(v))$ by $c(e) = a_{r(e)} e a_{s(e)}^{-1}$. Then F is isomorphic to the relative skew product $E \times_c (\pi_1(E, p(v))/p_*\pi_1(F, v))$.*

This suggests that we look for an analogue of Proposition 6.7 in which we aim to describe $C^*(F)$ as a crossed product by a coaction of the homogeneous space $\pi_1(E, p(v))/p_*\pi_1(F, v)$. What exactly we should mean by the phrase "crossed product by a coaction of a homogeneous space" is a question of some interest in non-abelian duality, and at present we do not even have a notion of "coaction of a homogeneous space". However, in Proposition 6.11 we have a labelling $c : E^1 \to \pi_1(E, p(v))$, and hence by Example 6.3 a coaction δ_c of $\pi_1(E, p(v))$ on $C^*(E)$. So we are in the situation where we have a coaction δ of a group G on a C^*-algebra B and a subgroup H of G, and we want to make sense of the crossed product of B by G/H.

Suppose $\delta : B \to B \otimes C^*(G)$ is a coaction of a discrete group G, and H is a subgroup of G. Let $\pi : B \to B(\mathcal{H})$ be a faithful representation. Then the regular representation of (6.3) gives a faithful representation $((\pi \otimes \lambda) \circ \delta) \times (1 \otimes M)$ of $B \rtimes_\delta G$ on $\ell^2(G, \mathcal{H}) = \mathcal{H} \otimes \ell^2(G)$ (see [**108**, Theorem 4.1]). Viewing functions on G/H as functions on G which are constant on cosets gives a representation $M^{G/H}$ of $c_0(G/H)$ by multiplication operators on $\ell^2(G)$, and

$$\overline{\operatorname{span}}\{(\pi \otimes \lambda) \circ \delta(b)(1 \otimes M^{G/H})(f) : b \in B,\ f \in c_0(G/H)\}$$

is a C^*-subalgebra of $B(\mathcal{H} \otimes \ell^2(G))$. In [**33**], this was called the *reduced crossed product of B by G/H*, and denoted by $B \rtimes_{\delta,r} (G/H)$.

REMARK 6.12. There are several reasons for calling this the reduced crossed product, the most obvious being to avoid ambiguity when the subgroup is normal. When N is a normal subgroup of G, the quotient map induces a homomorphism $q : C^*(G) \to C^*(G/N)$, and $(\iota \otimes q) \circ \delta$ is a coaction $\delta|$ of G/N on B called the *restriction* of δ. If π is a representation of B on $\mathcal{H}$, then $((\pi \otimes \lambda) \circ \delta, 1 \otimes M^{G/N})$ is a covariant representation of $(B, G/N, \delta|)$ on $\mathcal{H} \otimes \ell^2(G)$. When π is faithful and N is amenable, $((\pi \otimes \lambda) \circ \delta) \times (1 \otimes M^{G/N})$ is a faithful representation of $B \rtimes_{\delta|} (G/N)$; when N is not amenable, this representation need not be faithful, and it is helpful to call the image $B \rtimes_{\delta,r} (G/N)$ the reduced crossed product to remind us that it is not necessarily isomorphic to $B \rtimes_{\delta|} (G/N)$. There is also possible ambiguity for dual coactions, and the notation helps resolve this too (see [**33**, Proposition 2.8]). For further discussion of this point, see [**24**, Remark 4.1].

With this notion of crossed product by a homogeneous space, one can mimic the proof of Proposition 6.7 to see that $C^*(E \times_c (G/H))$ is isomorphic to the crossed product $C^*(E) \rtimes_{\delta_c,r} (G/H)$ [**24**, Theorem 3.4]. Combining this with the description of covering graphs as relative skew-products yields the following theorem, which is Theorem 3.2 of [**24**]:

THEOREM 6.13. *Let $p : F \to E$ be a connected covering of directed graphs and let $v \in F^0$. Choose a spanning tree T for E, and let $c : E^1 \to \pi_1(E, p(v))$ be the labelling of Proposition 6.11. Then $C^*(F)$ is isomorphic to the reduced crossed product $C^*(E) \rtimes_{\delta_c,r} (\pi_1(E, p(v))/p_*\pi_1(F, v))$.*

When we first considered crossed products by homogeneous spaces in [**33**], the main examples we had in mind involved dual coactions, and arose in connection with Green's imprimitivity theorem [**52**, Theorem 6]. Theorem 6.13 provides a whole new family of interesting examples, and led us to look again at the general theory of crossed products by homogeneous spaces, and in particular at the possibility of proving an imprimitivity theorem for them. To see why this might be interesting, we recall the original motivation for imprimitivity theorems for coactions.

Given an action α of a locally compact group G on a C^*-algebra A and a closed subgroup H of G, Green's imprimitivity theorem describes a Morita equivalence between $(A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\mathrm{lt}} G$ and $A \rtimes_\alpha H$, which can be used to identify the representations of $A \rtimes_\alpha G$ which have been induced from $A \rtimes_\alpha H$. For those who have not seen it before, *Morita equivalence* is an equivalence relation on C^*-algebras which is slightly weaker than stable isomorphism. Roughly, two C^*-algebras A and B are Morita equivalent if there is an A-B bimodule X which is simultaneously and compatibly a left Hilbert A-module and a right Hilbert B-module; X is then called an *imprimitivity bimodule*. Morita equivalence is discussed in detail in [**114**, Chapter 3], and the relationship with stable isomorphism is described in a theorem of Brown, Green and Rieffel [**114**, Theorem 5.55]. As an example, if pBp is a full corner in a C^*-algebra B, then pB is an imprimitivity bimodule which implements a Morita equivalence between pBp and B (see [**114**, Example 3.6]).

The crossed product $(A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\mathrm{lt}} G$ appearing in Green's imprimitivity theorem is universal for pairs of representations $\pi \times U$ of $A \rtimes_\alpha G$ and μ of $C_0(G/H)$ such that $(\pi \otimes \mu, U)$ is covariant for $\alpha \otimes \mathrm{lt}$. However we decide to define the crossed product $B \rtimes (G/H)$, we would want to have $(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} (G/H)$ isomorphic to $(A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\mathrm{lt}} G$. When H is normal,

$$(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} (G/H) := (A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}|} (G/H)$$

has this property [**32**, Theorem A.64]; when H is amenable,

$$(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} (G/H) := (A \rtimes_\alpha G) \rtimes_{\widehat{\alpha},r} (G/H)$$

has this property [**33**, Proposition 2.8]. In general, we want Green's imprimitivity theorem to give a Morita equivalence between $(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} (G/H)$ and $A \rtimes_\alpha H$.

The analogous imprimitivity theorem for crossed products by coactions should then give a Morita equivalence between $(B \rtimes_\delta G) \rtimes_{\widehat{\delta}} H$ and $B \rtimes_\delta (G/H)$. Mansfield proved such a theorem when H is normal and amenable, using the crossed product by $\delta|$ on the right-hand side, which he showed to be isomorphic to $B \rtimes_{\delta,r} (G/H)$ under his assumptions [**86**]. Kaliszewski and Quigg subsequently showed that the amenability hypothesis can sometimes be removed [**69**], and Echterhoff and Quigg showed that, when G is discrete, the normality hypothesis can be removed too [**35**, Theorem 3.4]. Mansfield proved his theorem by directly constructing an imprimitivity bimodule as a completion of a dense $*$-subalgebra $\mathcal{D}$ of $B \rtimes_\delta G$; in [**33**, Theorem 5.1], it was shown using duality that $(B \rtimes_\delta G) \rtimes_{\widehat{\delta}} H$ is always Morita equivalent to $B \rtimes_{\delta,r} (G/H)$, but it remained an open problem to directly construct such an equivalence. In [**62**, Remark 4.5], it was suggested that Rieffel's proper actions might be useful in extending Mansfield's theorem to non-normal subgroups H, and we digress briefly to discuss these actions.

Rieffel's theory of proper actions on non-commutative C^*-algebras [**118**] provides a systematic way of constructing imprimitivity bimodules. It concerns an action α of a locally compact group G on a C^*-algebra A for which there is a dense invariant subalgebra A_0 with suitable properties. The motivating example is the action rt on $A = C_0(X)$ induced by a proper action on X, in which case $A_0 = C_c(X)$ has the required properties. Rieffel's machine then yields a Morita equivalence between the reduced crossed product $A \rtimes_{\alpha,r} G$ and a *generalised fixed-point algebra* A^α; the equivalence is implemented by a bimodule which is a completion of A_0 [**118**, Corollary 1.7]. Graph algebras provide an interesting new family of proper actions. Indeed, suppose G acts freely on a graph F and α is the induced action on $C^*(F)$. Then

$$A_0 := \operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in F^*\}$$

is a dense α-invariant subalgebra of $C^*(F)$ which is quite easily seen to have the properties which Rieffel requires. Thus the action α on $C^*(F)$ is proper in Rieffel's sense. It is shown in [**97**, Corollary 1.5] that $C^*(F/G)$ is isomorphic to Rieffel's generalised fixed-point algebra $C^*(F)^\alpha$, so Rieffel's machine yields a new proof of the Morita equivalence of $C^*(F) \rtimes_{\alpha,r} G$ and $C^*(F/G)$ in the Kumjian-Pask theorem (Theorem 6.8). (In [**97**], this new proof was bootstrapped using a technique from [**17**] to prove an analogue of the symmetric imprimitivity theorem discussed in Remark 6.10.)

Now suppose δ_c is the coaction of G on $C^*(E)$ associated to a labelling $c : E^1 \to G$, and H is a subgroup of G. In [**24**, Corollary 3.5], Deicke, Pask and I obtained the Morita equivalence of $(C^*(E) \rtimes_{\delta_c} G) \rtimes_{\widehat{\delta_c},r} H$ and $C^*(E) \rtimes_{\delta_c,r} (G/H)$ as a by-product of our proof of Theorem 6.13. Motivated by this, we proved in [**24**, Theorem 5.1] that if δ is a coaction of G on a C^*-algebra B, then the dual action of H on $B \rtimes_\delta G$ is proper in Rieffel's sense with respect to the subalgebra

$$(B \rtimes_\delta G)_0 := \operatorname{span}\{j_B(b) j_G(\chi_t) : b \in B_s,\ t \in G\},$$

and the generalised fixed-point algebra is

$$\overline{\text{span}}\{j_B(b)j_G(\chi_t H) : b \in B_s,\ t \in G\} = B \rtimes_{\delta,r} (G/H).$$

It turned out that the resulting Morita equivalence had already been constructed by Echterhoff and Quigg using other methods [**35**], but the success of [**24**] encouraged an Huef and I to have another crack at implementing the suggestion of [**62**, Remark 4.5] in the context of coactions of locally compact groups. This time we were able to make it work [**61**]. The trick was to take for $(B \rtimes_\delta G)_0$ the dense $*$-subalgebra $\mathcal{D}$ constructed by Mansfield, and the main technical ingredient in proving that it has the required properties was an averaging technique of Olesen and Pedsersen which had previously been applied to coactions by Quigg [**105**]. The main theorem of [**61**] is an explicit Morita equivalence between $(B \rtimes_\delta G) \rtimes_{\widehat{\delta},r} H$ and $B \rtimes_{\delta,r} (G/H)$ for an arbitrary closed subgroup H of a locally compact group G. Thus the motivation provided by coactions on graph algebras enabled us to completely remove both hypotheses of amenability and normality from Mansfield's theorem.

CHAPTER 7

K-theory of graph algebras

K-theory associates to every C^*-algebra A a pair of abelian groups $K_0(A)$ and $K_1(A)$ which contain a great deal of information about A. We will review the construction of these groups, discuss a few of their properties, and then compute the groups $K_0(A)$ and $K_1(A)$ when A is a graph algebra. There are several sources where the constructions are carefully done, so we are going to be a little sloppy in the hope that we can quickly give newcomers a feel for the subject. (In particular, see Remark 7.6.) Those who know this stuff should skip the next three pages.

The group $K_0(A)$ of a C^*-algebra A is constructed from the sets $\operatorname{Proj}(M_n(A))$ of projections in the C^*-algebras $M_n(A)$ of $n \times n$ matrices over A. We define an equivalence relation on $\operatorname{Proj}(M_n(A))$ by declaring the initial projection u^*u of every partial isometry $u \in M_n(A)$ to be equivalent to its final projection uu^*. Identifying a projection p with the projection $p \oplus 0$ obtained by adding a row and column of zeros to the bottom and right of p allows us to view $\operatorname{Proj}(M_n(A))$ as a subset of $\operatorname{Proj}(M_{n+1}(A))$; passing from p to $p \oplus 0$ respects the equivalence relations, so they combine to give a well-defined equivalence relation on $\operatorname{Proj}_\infty(A) := \bigcup_{n=0}^\infty \operatorname{Proj}(M_n(A))$. The set $D(A)$ of equivalence classes $\{[p] : p \in \operatorname{Proj}_\infty(A)\}$ is an abelian semigroup, with $[p] + [q]$ defined to be the class $[p_1 + q_1]$ of the sum of representatives $p_1 \in [p]$ and $q_1 \in [q]$ satisfying $p_1 q_1 = 0$; we can always find such representatives by making the n in $M_n(A)$ larger. Then $K_0(A)$ is the group of all formal differences

$$K_0(A) = \{[p] - [q] : p, q \in \operatorname{Proj}_\infty(A)\}$$

with

$$([p] - [q]) + ([r] - [s]) = ([p] + [r]) - ([q] + [s]).$$

For the details, see [**124**, pages 21–41].

Example 7.1. Let $A = \mathbb{C}$. Two projections p and q in $M_n(\mathbb{C})$ are equivalent if and only if they have the same rank. To see this, recall that the rank of p is the dimension of the range $p\mathbb{C}^n$ of p, which is an inner-product space. Since two finite-dimensional inner-product spaces are isomorphic if and only if they have the same dimension, and since an isomorphism $u : p\mathbb{C}^n \to q\mathbb{C}^n$ is a partial isometry $u \in M_n(\mathbb{C})$ with $u^*u = p$ and $uu^* = q$, we deduce that $[p] = [q]$ if and only if $\operatorname{rank} p = \operatorname{rank} q$. Thus $D(\mathbb{C}) = \mathbb{N}$ (including 0), and $K_0(\mathbb{C}) = \mathbb{Z}$.

The same analysis works for $A = M_n(\mathbb{C})$, and there is an isomorphism d_n of $K_0(M_n(\mathbb{C}))$ onto $\mathbb{Z}$ such that $d_n([p]) = \operatorname{rank} p$ for every $p \in \operatorname{Proj}_\infty(M_n(\mathbb{C}))$.

Example 7.2. Let E be a row-finite directed graph. Then the vertex projections $\{p_v : v \in E^0\}$ define classes $[p_v]$ in $D(C^*(E))$, and the Cuntz-Krieger relations

$$p_v = \sum_{r(e)=v} s_e s_e^*$$

imply that

$$[p_v] = \sum_{r(e)=v} [s_e s_e^*] = \sum_{r(e)=v} [s_e^* s_e] = \sum_{r(e)=v} [p_{s(e)}] \tag{7.1}$$

in $K_0(C^*(E))$. Theorem 7.16 below says in part that $K_0(C^*(E))$ is generated by $\{[p_v] : v \in E^0\}$ subject only to the relations (7.1) induced by the Cuntz-Krieger relations.

The other K-group $K_1(A)$ is defined using the groups $U(M_n(A))$ of unitary elements of the matrix algebras $M_n(A)$. We say $u, v \in U(M_n(A))$ are equivalent if there is a continuous path $t \mapsto u_t : [0,1] \to U(M_n(A))$ with $u_0 = u$ and $u_1 = v$. This equivalence relation is respected by the embeddings of $M_n(A)$ in $M_{n+1}(A)$ which send u to the matrix $u \oplus 1$ obtained by adding a 1 in the bottom right-hand corner and zeros everywhere else in the right column and bottom row. Thus we have a well-defined equivalence relation on $U_\infty(A) := \bigcup_n U(M_n(A))$. The group $K_1(A)$ is the set $\{[u] : u \in U_\infty(A)\}$ of equivalence classes with multiplication given by $[u][v] = [uv]$; the identity is the class containing the identity matrices 1_n. It is true (but not obvious) that $K_1(A)$ is an abelian group (see [**124**, §8.1]).

Example 7.3. Consider $A = \mathbb{C}$. Every element u of $U(M_n(\mathbb{C}))$ is diagonalisable: there is a unitary matrix v such that v^*uv is diagonal with entries of the form $\exp(2\pi i\theta_j)$. If d_t is the diagonal matrix with entries $\exp(2\pi it\theta_j)$, then $t \mapsto vd_tv^*$ is a continuous path with $vd_0v^* = 1_n$ and $vd_1v^* = u$. Thus $K_1(\mathbb{C})$ has just the one element $[1_n]$, and we write $K_1(\mathbb{C}) = 0$. Similarly, $K_1(M_n(\mathbb{C})) = 0$ for every $n \in \mathbb{N}$.

Example 7.4. For an example of a C^*-algebra with non-zero K_1, consider $A = C(\mathbb{T})$. Then $U(A)$ consists of the continuous functions $u : \mathbb{T} \to \mathbb{T}$, and continuous paths in $U(A)$ are the same as homotopies between functions. The homotopy class of a function $u : \mathbb{T} \to \mathbb{T}$ is determined by its winding number $\deg u$, which is an integer. For example, if $u_k(z) = z^k$, then $\deg u_k = k$. The classes $[u_k]$ in $K_1(C(\mathbb{T}))$ are still distinct: a continuous path $t \mapsto u_t$ joining $u_k \oplus 1_{n-1}$ to $u_m \oplus 1_{n-1}$ in $U(M_n(A))$ would give a continuous path $t \mapsto \det u_t$ in $U(A)$ joining u_k to u_m. So $K_1(C(\mathbb{T}))$ is definitely non-trivial. It is true but not so obvious that $[u] \mapsto \deg(\det u)$ is an isomorphism of $K_1(C(\mathbb{T}))$ onto $\mathbb{Z}$ (see the discussion in [**124**, §8.3]).

A homomorphism $\phi : A \to B$ induces homomorphisms $\phi_n : M_n(A) \to M_n(B)$ such that $\phi_n((a_{ij})) = (\phi(a_{ij}))$. These homomorphisms map projections to projections and unitaries to unitaries, and hence induce homomorphisms $\phi_* : K_0(A) \to K_0(B)$ and $\phi_* : K_1(A) \to K_1(B)$ such that

$$\phi_*([p] - [q]) = [\phi_n(p)] - [\phi_n(q)] \quad \text{and} \quad \phi_*([u]) = [\phi_n(u)].$$

This process is *functorial*: the identity homomorphism induces the identity maps on K-theory, and $(\phi \circ \psi)_* = \phi_* \circ \psi_*$. Thus K_0 and K_1 become functors from the category of C^*-algebras to the category of abelian groups.

Example 7.5. Suppose $\phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is a homomorphism. Recall from Example 7.1 that there is an isomorphism $d_m : K_0(M_m(\mathbb{C})) \to \mathbb{Z}$ such that $d_m([p]) = \operatorname{rank} p$. So if p is a rank-one projection in $M_m(\mathbb{C})$, such as $p = e_{11}$, and $r = \operatorname{rank}(\phi_m(p))$, then the induced homomorphism ϕ_* on K_0 is multiplication by r in the sense that $d_n(\phi_*(c)) = rd_m(c)$ for all $c \in K_0(M_m(\mathbb{C}))$.

REMARK 7.6. We have deliberately glossed over the subtleties which arise when A does not have an identity. It is immediately clear that there is a problem with our definition of $K_1(A)$, since A cannot have unitary elements when A does not have an identity. However, it suffices to adjoin an identity to A and set $K_1(A) := K_1(A^+)$. It turns out that it is also smart to do this when defining $K_0(A)$. This time, though, we then want to ignore the class represented by the identity; formally, we define $\phi : A^+ \to \mathbb{C}$ by $\phi(a + \lambda 1) = \lambda$, and take

$$K_0(A) := \ker(\phi_* : K_0(A^+) \to \mathcal{K}_0(\mathbb{C}) = \mathbb{Z}).$$

Of course one then has to check that this process defines functors, and you can see how this might get messy.

We need two basic facts about K-theory, and one big theorem. First, we need to know that the K-groups $K_i(A \oplus B)$ of a C^*-algebraic direct sum are naturally isomorphic to $K_i(A) \oplus K_i(B)$. This is quite straightforward (see [**124**, Propositions 4.3.4 and 8.2.6]). Second, we need to know that the functors K_i are *continuous* in the sense that

$$K_i\big(\overline{\textstyle\bigcup_{n=1}^\infty A_n}\big) = \varinjlim K_i(A_n); \tag{7.2}$$

the *direct limit* appearing on the right of (7.2) is defined in [**124**, §6.2], for example. To prove (7.2), one has to approximate projections and unitaries in $A = \overline{\bigcup_{n=1}^\infty A_n}$ by projections and unitaries in some A_n. Given a projection $p \in A$, for example, we can approximate p by a self-adjoint element a in some A_n, and use the functional calculus for a to build a projection $q \in A_n$ which is near p. One then has to check that this approximation process respects the equivalence relation, and this is harder. See [**124**, Theorem 6.3.2] for K_0 and [**124**, Proposition 8.2.7] for K_1.

EXAMPLE 7.7. These two basic facts allow us to compute the K-theory of AF-algebras. Suppose $A = \overline{\bigcup_{n=1}^\infty A_n}$ and each A_n is finite-dimensional. Then each A_n is isomorphic to a C^*-algebraic direct sum $\bigoplus_{j=1}^{d_n} M_{k(n,j)}(\mathbb{C})$, and hence

$$K_i(A) = \textstyle\bigoplus_{j=1}^{d_n} K_i(M_{k(n,j)}(\mathbb{C})) = \begin{cases} \mathbb{Z}^{d_n} & \text{if } i = 0 \\ 0 & \text{if } i = 1. \end{cases}$$

Thus by continuity we have $K_1(A) = 0$ and $K_0(A) = \varinjlim \mathbb{Z}^{d_n}$. We can compute the bonding maps in the direct limit by observing that every homomorphism ϕ of $M_m(\mathbb{C})$ into a direct sum $\bigoplus_{j=1}^{J} M_{k(j)}(\mathbb{C})$ has the form $\phi(a) = (\phi_1(a), \cdots, \phi_J(a))$ for homomorphisms $\phi_j : M_m(\mathbb{C}) \to M_{k(j)}(\mathbb{C})$, and then each

$$(\phi_j)_* : \mathbb{Z} = K_0(M_n(\mathbb{C})) \to \mathbb{Z} = K_0(M_{k(j)}(\mathbb{C}))$$

is multiplication by the integer $r_j = \operatorname{rank}(\phi_j(e_{11}))$ (see Example 7.5).

One of the first successes of K-theory for C^*-algebras was Elliott's classification theorem for AF-algebras, which says that the K_0-group of an AF-algebra A determines A up to isomorphism. We are being a little glib here: there is an ordering on $K_0(A)$ with positive cone $K_0(A)_+ := \{[p] : p \in \operatorname{Proj}_\infty(A)\}$, and the theorem says that the *ordered group* $K_0(A)$ classifies A up to isomorphism. A precise statement is given in [**18**, Theorem IV.4.3]. There are now classification theorems for many different classes of C^*-algebras, all involving K-theory, and we shall mention another at the end of this chapter. These classification theorems are discussed in [**123**] and [**85**], for example.

EXAMPLE 7.8. As a more concrete example of an AF-algebra, consider the C^*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a separable Hilbert space $\mathcal{H}$. Let $\{e_n : n \in \mathbb{N}\}$ be a orthonormal basis for $\mathcal{H}$, and use it to write $\mathcal{K}(\mathcal{H}) = \overline{\bigcup_n M_n(\mathbb{C})}$, as on page 103. Then the inclusions $i_n : M_n(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ take a matrix a to $a \oplus 0$, and hence preserve rank. Thus the isomorphisms d_n of Example 7.1 combine to give an isomorphism of $K_0(\mathcal{K}(\mathcal{H})) = \varinjlim K_0(M_n(\mathbb{C}))$ onto $\mathbb{Z}$ which takes $[p]$ to $\operatorname{rank} p$. In other words, $\mathcal{K}(\mathcal{H})$ is the free abelian group generated by the class of the rank-one projections.

The big theorem which we need is the dual Pimsner-Voiculescu sequence, which allows us to compute the K-theory of a C^*-algebra A which carries an action α of the circle group $\mathbb{T}$ in terms of the K-theory of the crossed product C^*-algebra $A \rtimes_\alpha \mathbb{T}$. The dual of the abelian group $\mathbb{T}$ is naturally identified with $\mathbb{Z}$ (the character corresponding to $n \in \mathbb{Z}$ is $z \mapsto z^n$), so there is a dual action $\widehat{\alpha} : \mathbb{Z} \to \operatorname{Aut}(A \rtimes_\alpha \mathbb{T})$ which fixes the copy $i_A(A)$ of A and scales the generator $i_\mathbb{T}(z)$ by $\widehat{\alpha}_n(i_\mathbb{T}(z)) = z^n i_\mathbb{T}(z)$. We write $\widehat{\alpha}$ for the generator $\widehat{\alpha}_1$. The *dual Pimsner-Voiculescu sequence* is a six-term exact sequence

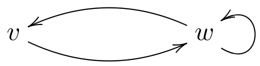
$$\begin{array}{ccccc} K_0(A \rtimes_\alpha \mathbb{T}) & \xrightarrow{\operatorname{id} - \widehat{\alpha}_*^{-1}} & K_0(A \rtimes_\alpha \mathbb{T}) & \longrightarrow & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \longleftarrow & K_1(A \rtimes_\alpha \mathbb{T}) & \xleftarrow{\operatorname{id} - \widehat{\alpha}_*^{-1}} & K_1(A \rtimes_\alpha \mathbb{T}). \end{array} \tag{7.3}$$

Here each arrow represents a homomorphism between the designated groups, and part of the assertion of the theorem is that these homomorphisms exist. Saying that the sequence is exact means that the kernel of the homomorphism represented by each arrow is the image of the homomorphism represented by the previous arrow. There are several six-term exact sequences in K-theory, and all are deep theorems. The dual Pimsner-Voiculescu sequence is, as the name suggests, the dual of another six-term sequence called the *Pimsner-Voiculescu sequence*, which describes the K-theory of crossed products by $\mathbb{Z}$. The existence of the dual sequence was first proved by Paschke [**99**], and the existence of both sequences is established in [**10**, §10.6]; there is no elementary proof of this theorem, and it seems unlikely that there ever will be.

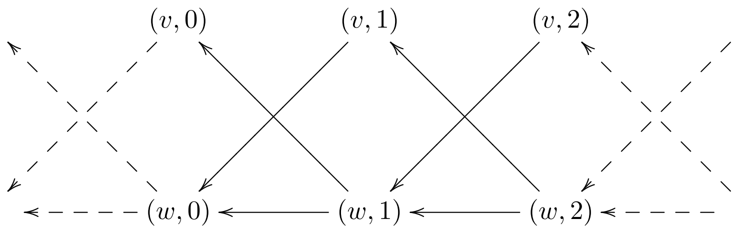
We will be applying the dual Pimsner-Voiculescu sequence to the gauge action γ of $\mathbb{T}$ on the algebra $C^*(E)$ of a row-finite graph. This is particularly effective because we can realise the crossed product $C^*(E) \rtimes_\gamma \mathbb{T}$ as the C^*-algebra of a graph F such that $C^*(F)$ is AF, and then compute its K-theory as in Example 7.7. In particular, since $K_1(C^*(E) \rtimes_\gamma \mathbb{T}) = 0$, the dual Pimsner-Voiculescu sequence collapses to a four-term sequence in which both K-groups of $C^*(E)$ appear. In Cuntz's original computations of K-theory for a Cuntz-Krieger algebra $\mathcal{O}_A$, he realised $\mathcal{O}_A \otimes \mathcal{K}$ as a crossed product by $\mathbb{Z}$ and applied the Pimsner-Voiculescu sequence [**15**]. The idea of using the dual sequence comes from [**96**], and the argument we present here is from [**113**].

Suppose E is a row-finite graph without sources. To realise $C^*(E) \rtimes_\gamma \mathbb{T}$ as a graph algebra, we define a labelling $c : E^1 \to \mathbb{Z}$ by $c(e) = -1$ for all e, and consider the skew product $E \times_1 \mathbb{Z} := E \times_c \mathbb{Z}$ in which $(E \times_1 \mathbb{Z})^i = E^i \times \mathbb{Z}$ and $s(e,n) = (s(e), n)$, $r(e,n) = (r(e), n-1)$. Recall from Proposition 6.7 that there is a natural action β of $\mathbb{Z}$ on $C^*(E \times_1 \mathbb{Z})$ such that $\beta_m(s_{(e,n)}) = s_{(e,n+m)}$.

Example 7.9. For the following directed graph E



the skew product $E \times_1 \mathbb{Z}$ looks like



Notice that the skew product has no cycles, so its C^*-algebra is AF (see Remark 4.3).

Lemma 7.10. *There is an isomorphism ϕ of $C^*(E \times_1 \mathbb{Z})$ onto $C^*(E) \rtimes_\gamma \mathbb{T}$ such that $\phi \circ \beta_m = \widehat{\gamma}_m \circ \phi$.*

Proof. We want to construct a Cuntz-Krieger $(E \times_1 \mathbb{Z})$-family in $C^*(E) \rtimes_\gamma \mathbb{T}$. We write $(i_A, i_\mathbb{T})$ for the universal covariant representation of $(C^*(E), \mathbb{T}, \gamma)$ in the crossed product. Since $\mathbb{T}$ is not discrete, we look for elements of the form

$$i_A(a) i_\mathbb{T}(f) := \int_\mathbb{T} i_A(a) f(z) i_\mathbb{T}(z)\, dz,$$

where $a \in C^*(E)$ and $f \in C_c(\mathbb{T}) = C(\mathbb{T})$ (now is the time to look at Remark 6.1). We define $f_n(z) = z^n$, and take

$$t_{(e,n)} = i_A(s_e) i_\mathbb{T}(f_n), \quad q_{(v,n)} = i_A(p_v) i_\mathbb{T}(f_n).$$

Provided one remembers that $\overline{z^n} = z^{-n}$ for $z \in \mathbb{T}$ and $\int_\mathbb{T} f(z^{-1})\, dz = \int_\mathbb{T} f(z)\, dz$, it is easy to check that each $i_\mathbb{T}(f_n)$ is self-adjoint, and the following formal calculation using the invariance of Haar measure (part (d) of Lemma 3.1) should convince you that the $i_\mathbb{T}(f_n)$ are mutually orthogonal projections:

$$\begin{aligned} i_\mathbb{T}(f_n) i_\mathbb{T}(f_m) &= \iint z^n w^m i_\mathbb{T}(zw)\, dw\, dz = \iint z^{n-m} (zw)^m i_\mathbb{T}(zw)\, dw\, dz \\ &= \iint z^{n-m} w^m i_\mathbb{T}(w)\, dw\, dz = \Big(\int z^{n-m}\, dz \Big) i_\mathbb{T}(f_m) \\ &= \begin{cases} i_\mathbb{T}(f_m) & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(This can be made precise either by multiplying everything by $i_A(a)$, or by appealing to [**114**, Lemma C.11].) The covariance of $(i_A, i_\mathbb{T})$ implies that

$$\begin{aligned} i_\mathbb{T}(f_n) i_A(s_e) &= \int z^n i_\mathbb{T}(z) i_A(s_e)\, dz = \int z^n i_A(\gamma_z(s_e)) i_\mathbb{T}(z)\, dz \\ &= \int z^n i_A(z s_e) i_\mathbb{T}(z)\, dz = \int z^{n+1} i_A(s_e) i_\mathbb{T}(z)\, dz \\ &= i_A(s_e) i_\mathbb{T}(f_{n+1}), \end{aligned}$$

and that $i_{\mathbb{T}}(f_n)$ commutes with $i_A(p_v)$. Thus the $q_{(v,n)}$ are mutually orthogonal projections. The calculations

$$\begin{aligned} t_{(e,n)}^* t_{(e,n)} &= i_{\mathbb{T}}(f_n)^* i_A(s_e)^* i_A(s_e) i_{\mathbb{T}}(f_n) \\ &= i_{\mathbb{T}}(f_n)^* i_A(p_{s(e)}) i_{\mathbb{T}}(f_n) \\ &= q_{(s(e),n)} = q_{s(e,n)} \end{aligned}$$

and

$$\begin{aligned} q_{(v,n)} = i_A(p_v) i_{\mathbb{T}}(f_n) &= \sum_{r(e)=v} i_A(s_e) i_A(s_e)^* i_{\mathbb{T}}(f_n) \\ &= \sum_{r(e)=v} i_A(s_e) i_{\mathbb{T}}(f_{n+1}) i_A(s_e)^* = \sum_{r(e)=v} t_{(e,n+1)} t_{(e,n+1)}^* \\ &= \sum_{r(e,m)=(v,n)} t_{(e,m)} t_{(e,m)}^* \end{aligned}$$

show that $\{t_{(e,n)}, q_{(v,n)}\}$ is a Cuntz-Krieger $(E \times_1 \mathbb{Z})$-family.

Now the universal property of the graph algebra gives a homomorphism ϕ of $C^*(E \times_1 \mathbb{Z})$ into $C^*(E) \rtimes_\gamma \mathbb{T}$ which carries $\{s_{(e,n)}, p_{(v,n)}\}$ into $\{t_{(e,n)}, q_{(v,n)}\}$. Since the skew product has no cycles, the Cuntz-Krieger uniqueness theorem (Theorem 2.4) implies that ϕ is an injection. The dual action $\widehat{\gamma}$ satisfies $\widehat{\gamma}_m(i_A(a)) = i_A(a)$ and $\widehat{\gamma}_m(i_{\mathbb{T}}(z)) = z^m i_{\mathbb{T}}(z)$, so $\widehat{\gamma}_m(i_{\mathbb{T}}(f_n)) = i_{\mathbb{T}}(f_{n+m})$, and we can check $\phi \circ \beta_m = \widehat{\gamma}_m \circ \phi$ on the generators $\{s_{(e,n)}, p_{(v,n)}\}$. So it remains to show that ϕ is surjective, and since it is a homomorphism of C^*-algebras, it suffices to check that it has dense range. The estimate $\left\| \int g(z)\,dz \right\| \le \int \|g(z)\|\,dz$ (part (b) of Lemma 3.1) shows that $f \mapsto i_{\mathbb{T}}(f)$ is continuous for the sup norm on $C(\mathbb{T})$. The functions f_n span a dense subalgebra of $C(\mathbb{T})$, so it follows from the density of

$$\operatorname{span}\{i_A(a) i_{\mathbb{T}}(f) : a \in C^*(E), f \in C(\mathbb{T})\}$$

in $C^*(E) \rtimes_\gamma \mathbb{T}$ (see Remark 6.1) that ϕ has dense range. □

REMARK 7.11. By Example 6.4, the gauge action γ of $\mathbb{T}$ gives rise to a coaction δ of the dual group $\mathbb{Z} = \widehat{\mathbb{T}}$, and $C^*(E) \rtimes_\gamma \mathbb{T}$ is then isomorphic to $C^*(E) \rtimes_\delta \mathbb{Z}$ (Example 6.6). So we could deduce Lemma 7.10 from Proposition 6.7 by proving that the coaction δ corresponding to the gauge action γ is the coaction δ_c arising from the cocycle $c : E^1 \to \mathbb{Z}$ defined by $c(e) = -1$.

Since the skew product has no cycles, its C^*-algebra $C^*(E \times_1 \mathbb{Z})$ is AF (see Remark 4.3). Thus $K_1(C^*(E) \rtimes_\gamma \mathbb{T}) = K_1(C^*(E \times_1 \mathbb{Z})) = 0$, and the six-term exact sequence (7.3) collapses to

$$0 \to K_1(C^*(E)) \to K_0(C^*(E) \rtimes_\gamma \mathbb{T}) \overset{\mathrm{id} - \widehat{\gamma}_*^{-1}}{\longrightarrow} K_0(C^*(E) \rtimes_\gamma \mathbb{T}) \to K_0(C^*(E)) \to 0.$$

Since the isomorphism ϕ of Lemma 7.10 induces isomorphisms on K-theory, we can replace $C^*(E) \rtimes_\gamma \mathbb{T}$ by $C^*(E \times_1 \mathbb{Z})$ and $\widehat{\gamma} = \widehat{\gamma}_1$ by the generator $\beta = \beta_1$. This gives an exact sequence

$$0 \to K_1(C^*(E)) \to K_0(C^*(E \times_1 \mathbb{Z})) \overset{\mathrm{id} - \beta_*^{-1}}{\longrightarrow} K_0(C^*(E \times_1 \mathbb{Z})) \to K_0(C^*(E)) \to 0. \tag{7.4}$$

Thus:

LEMMA 7.12. *$K_1(C^*(E))$ is the kernel of the homomorphism*

$$\mathrm{id} - \beta_*^{-1} : K_0(C^*(E \times_1 \mathbb{Z})) \longrightarrow K_0(C^*(E \times_1 \mathbb{Z})),$$

and $K_0(C^*(E))$ is the cokernel.

We will compute this kernel and cokernel by identifying a simpler homomorphism which has the same kernel and cokernel, and which we can write down by looking at the graph.

We first need to compute $K_0(C^*(E \times_1 \mathbb{Z}))$. To do this, we will write $C^*(E \times_1 \mathbb{Z})$ as a direct limit $\overline{\bigcup_n A_n}$ of subalgebras A_n whose K-theory we can compute, and then use the continuity of K_0. For each $n \in \mathbb{N}$, let F_n be the subgraph of $E \times_1 \mathbb{Z}$ consisting of the edges (e, k) and vertices (v, k) with $k \leq n$. Since E has no sources, the sources in F_n are the vertices (v, n). The subset $\{s_{(e,k)}, p_{(v,k)} : k \leq n\}$ of the canonical Cuntz-Krieger family generating $C^*(E \times_1 \mathbb{Z})$ is a Cuntz-Krieger F_n-family with all the projections non-zero, and hence the Cuntz-Krieger uniqueness theorem gives an injection of $C^*(F_n)$ into $C^*(E \times_1 \mathbb{Z})$; we use this injection to identify $C^*(F_n)$ with the C^*-subalgebra of $C^*(E \times_1 \mathbb{Z})$ generated by $\{s_{(e,k)}, p_{(v,k)} : k \leq n\}$. We then have $C^*(F_n) \subset C^*(F_{n+1})$, and

$$C^*(E \times_1 \mathbb{Z}) = \overline{\bigcup_{n \in \mathbb{N}} C^*(F_n)}.$$

LEMMA 7.13. *For each $n \in \mathbb{N}$, $K_0(C^*(F_n))$ is the free abelian group generated by $\{[p_{(v,n)}] : v \in E^0\}$.*

PROOF. We consider a typical element $s_\mu s_\nu^*$ of the spanning set for $C^*(F_n)$, so that $s(\mu) = s(\nu) = (w, k)$, say. Since every path of length $n - k$ with range (w, k) begins at a source, we can by applying the Cuntz-Krieger relations $n - k$ times write $s_\mu s_\nu^*$ as a finite sum of terms $s_\alpha s_\beta^*$ in which $s(\alpha) = s(\beta)$ is a source (v, n). In other words,

$$C^*(F_n) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu) = (v, n) \text{ for some } v \in E^0\}.$$

For each fixed (v, n), the elements $s_\mu s_\nu^*$ with $s(\mu) = s(\nu) = (v, n)$ form a family of matrix units, and hence

$$A_{(v,n)} := \overline{\operatorname{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu) = (v, n)\}$$

is a C^*-algebra which is isomorphic to the algebra of compact operators on

$$\ell^2\big(\{\mu \in F_n^* : s(\mu) = (v, n)\}\big)$$

(see Corollary A.9). When $s(\nu)$ and $s(\alpha)$ are different sources, we have $s_\nu^* s_\alpha = 0$, so the subalgebras $A_{(v,n)}$ satisfy $A_{(w,n)} A_{(v,n)} = 0$ when $w \neq v$, and Corollary A.11 implies that $C^*(F_n) = \bigoplus_{v \in E^0} A_{(v,n)}$ (or, more precisely, that $C^*(F_n)$ is isomorphic to $\bigoplus_{v \in E^0} A_{(v,n)}$ in a very concrete way).

Since $p_{(v,n)}$ is a rank-one projection in $A_{(v,n)}$, we can deduce from Example 7.8 that $K_0(A_{(v,n)})$ is the free abelian group generated by $[p_{(v,n)}]$, and hence that

$$K_0(C^*(F_n)) = K_0\big(\textstyle\bigoplus_{v \in E^0} A_{(v,n)}\big) = \bigoplus_{v \in E^0} K_0(A_{(v,n)}) = \bigoplus_{v \in E^0} \mathbb{Z}[p_{(v,n)}],$$

as required. (If you are nervous about pulling the infinite direct sum out of K_0, write the infinite direct sum as a direct limit of finite direct sums, as in the proof of Corollary A.11, and use the continuity of K_0.) □

Continuity of K_0 says that $K_0(C^*(E\times_1\mathbb{Z}))=\varinjlim_n K_0(C^*(F_n))$; to compute the direct limit, we need to compute the bonding maps, which are induced by the inclusions of $C^*(F_n)$ in $C^*(F_{n+1})$. The Cuntz-Krieger relation at (v,n) implies that

$$\begin{aligned}[p_{(v,n)}] &= \sum_{r(e,k)=(v,n)} [s_{(e,k)}s^*_{(e,k)}] = \sum_{r(e)=v} [s_{(e,n+1)}s^*_{(e,n+1)}] \\ &= \sum_{r(e)=v} [s^*_{(e,n+1)}s_{(e,n+1)}] = \sum_{r(e)=v} [p_{(s(e),n+1)}].\end{aligned}$$

In terms of the vertex matrix $E^0\times E^0$ matrix A_E defined by

$$A_E(v,w)=\#\{e\in E^1 : s(e)=w \text{ and } r(e)=v\}, \tag{7.5}$$

this becomes

$$[p_{(v,n)}]=\sum_{w\in E^0} A_E(v,w)[p_{(w,n+1)}].$$

Thus if we use Lemma 7.13 to identify every $K_0(C^*(F_n))$ with the direct sum $\mathbb{Z}^{E^0}$, the bonding map is multiplication by the transpose A_E^t. We have proved:

COROLLARY 7.14. *$K_0(E\times_1\mathbb{Z})$ is isomorphic to the direct limit $\varinjlim(\mathbb{Z}^{E_0},A_E^t)$ in which each bonding map is multiplication by the transpose of the vertex matrix A_E of E.*

Notice that, because E is row-finite, the sum $\sum_{w\in E^0}A_E(v,w)$ of each row in A_E is finite, so each column in A_E^t has finite sum, and multiplication by A_E^t gives a well-defined map on the direct sum

$$\mathbb{Z}^{E^0}:=\{\{n_v : v\in E^0\} : \text{ all but finitely many } n_v \text{ are zero}\}.$$

We now need to compute the effect of the homomorphism β_*^{-1} on

$$\varinjlim(\mathbb{Z}^{E_0},A_E^t)\cong K_0(E\times_1\mathbb{Z}).$$

Since $\beta^{-1}(p_{(v,n)})=p_{(v,n-1)}$ and $\beta^{-1}(s_{(v,n)})=s_{(v,n-1)}$, β^{-1} maps each $C^*(F_n)$ into itself; since

$$\beta^{-1}(p_{(v,n)})=p_{(v,n-1)}=\sum_{r(e,k)=(v,n-1)} s_{(e,k)}s^*_{(e,k)}=\sum_{r(e)=v} s_{(e,n)}s^*_{(e,n)},$$

and since $[s_{(e,n)}s^*_{(e,n)}]=[s^*_{(e,n)}s_{(e,n)}]=[p_{(s(e),n)}]$, the induced automorphism β_*^{-1} of $K_0(C^*(F_n))$ satisfies

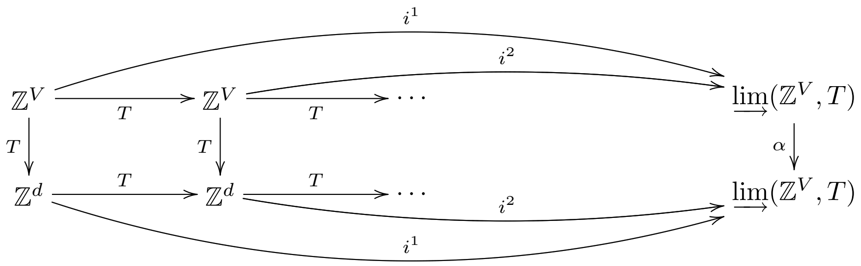
$$\beta_*^{-1}([p_{(v,n)}])=\sum_{r(e)=v}[p_{(s(e),n)}]=\sum_{w\in E^0}A_E(v,w)[p_{(w,n)}].$$

Thus the homomorphism $\beta_*^{-1}:\mathbb{Z}^{E^0}=K_0(C^*(F_n))\to\mathbb{Z}^{E^0}=K_0(C^*(F_n))$ is multiplication by A_E^t.

We are now in the situation of the following lemma, with $V=E^0$ and $T=A_E^t$.

LEMMA 7.15. *Suppose that V is a countable set and T is a column-finite $V\times V$ matrix with integer entries. For $n\in\mathbb{N}$, let $G_n=\mathbb{Z}^V$, define $i_n:G_n\to G_{n+1}$ by $i_n(a)=Ta$, and denote by i^n the canonical map of G_n into $\varinjlim(G_n,i_n)$. Let α be the endomorphism of $\varinjlim(G_n,i_n)$ such that $\alpha(i^n(a))=i^n(Ta)$. Then the map i^1 is an isomorphism of* $\ker(1-T)$ *onto* $\ker(\mathrm{id}-\alpha)$, *and induces an isomorphism of* $\operatorname{coker}(1-T)$ *onto* $\operatorname{coker}(\mathrm{id}-\alpha)$.

Proof. The following diagram may help to follow the argument:



The defining property $\alpha(i^1(a)) = i^1(Ta)$ implies that i^1 maps $\ker(1-T)$ into $\ker(\mathrm{id} - \alpha)$. Suppose $a \in \ker(1-T)$ satisfies $i^1(a) = 0$. Then $T^{m-1}a = 0$ for some m, which because $a = Ta$ implies that $a = T^{m-1}a = 0$. Thus i^1 is injective on $\ker(1-T)$. Suppose that $x = i^n(a) \in \ker(\mathrm{id} - \alpha)$, so that $i^n(a) = i^n(Ta)$. Then for some $m \geq n$, $b := T^{m-n}a \in G_m$ satisfies $x = i^m(b)$ and $b = Tb$. But $b = Tb$ says that $b \in \ker(1-T)$, and implies that $x = i^m(b) = i^m(T^{m-1}b) = i^1(b)$. Thus i^1 maps $\ker(1-T)$ onto $\ker(\mathrm{id} - \alpha)$.

The defining property $\alpha(i^1(a)) = i^1(Ta)$ also implies that i^1 maps $(1-T)\mathbb{Z}^V$ into the image of $\mathrm{id} - \alpha$, and hence induces a map j^1 on cokernels. To see that j^1 is injective, suppose $j^1(a) = 0$, or equivalently that $i^1(a)$ belongs to $\mathrm{im}(\mathrm{id} - \alpha)$. Then there exists $y = i^n(b)$ such that $i^1(a) = y - \alpha(y) = i^n(b - Tb)$. By making n larger, we can ensure that $T^{n-1}a = b - Tb$, and then

$$a = a - T^{n-1}a + T^{n-1}a = (1-T)(1 + T + \cdots T^{n-2})a + (1-T)b$$

belongs to $(1-T)\mathbb{Z}^V$. Thus j^1 is injective. To see that j^1 is surjective, suppose $y = i^n(b) \in \varinjlim G_n$. Observe that for every $c \in \mathbb{Z}^V$ we have $i^n(c) - i^n(Tc) = (\mathrm{id} - \alpha)(i^n(c))$, and hence $i^n(c)$ and $i^n(Tc)$ have the same class in $\mathrm{coker}(\mathrm{id} - \alpha)$. Applying this observation $n-1$ times shows that y has the same class in $\mathrm{coker}(\mathrm{id} - \alpha)$ as $i^n(T^{n-1}b) = i^1(b)$, and hence $y + \mathrm{im}(\mathrm{id} - \alpha) = j^1(b)$ belongs to the image of j^1. □

The results of our calculation are summarised in the following theorem.

Theorem 7.16. *Let E be a row-finite graph with no sources, and let A_E be the vertex matrix of E defined by* (7.5). *Then $K_1(C^*(E))$ is isomorphic to the kernel of $1 - A_E^t : \mathbb{Z}^{E^0} \to \mathbb{Z}^{E^0}$, and $K_0(C^*(E))$ is isomorphic to the cokernel.*

Proof. Lemma 7.12 says that K_1 and K_0 are the kernel and cokernel of the endomorphism $\mathrm{id} - \beta_*^{-1}$ of $K_0(C^*(E \times_1 \mathbb{Z}))$. We found an isomorphism of $K_0(C^*(E \times_1 \mathbb{Z}))$ onto $\varinjlim(\mathbb{Z}^{E^0}, A_E^t)$ which converts β_*^{-1} into the endomorphism of the direct limit induced by the homomorphism $A_E^t : \mathbb{Z}^{E^0} \to \mathbb{Z}^{E^0}$, or in other words the endomorphism α of Lemma 7.15 corresponding to $T = A_E^t$. Thus the result follows from Lemma 7.15. □

Remark 7.17. We have stated and proved the theorem in this generality because we wanted to illustrate the ideas as clearly as possible, and with a bit more work we could have done substantially better. Theorem 3.2 of [**113**] is a version of Theorem 7.16 for row-finite graphs with sources, and its proof is basically the one we have given here; in [**7**, §6], it was shown how to adapt this argument to handle arbitrary graphs [**7**, Theorem 6.1]. One can also compute the K-theory of $C^*(E)$ for arbitrary infinite E by reducing to Theorem 7.16 using a Drinen-Tomforde desingularisation [**30**, Theorem 3.1], or by applying the computations of K-theory for the Cuntz-Krieger algebras of infinite matrices [**41**, Theorem 4.5], [**134**, §6], [**113**, Theorem 4.1].

Theorem 7.16 implies that $K_1(C^*(E))$ is a subgroup of the free abelian group $\mathbb{Z}^{E^0}$, and hence is itself free. Szymański has shown that this is the only restriction on the K-groups of a graph algebra [**136**, Theorem 1.2]. To state his theorem, recall that a graph E is *transitive* if for every $v, w \in E^0$, we have $v \geq w$ and $w \geq v$.

Theorem 7.18. *Suppose G_0 and G_1 are countable abelian groups, G_1 is free and $c \in G_0$. Then there are a row-finite transitive graph E with infinitely many vertices and a vertex $v \in E^0$ such that $(K_0(C^*(E)), [p_v]) \cong (G_0, c)$ and $K_1(C^*(E)) \cong G_1$.*

Indeed, in the proof of [**136**, Theorem 1.2], Szymański describes explicitly how to find such a graph E.

Proposition 4.2 implies that the C^*-algebras of transitive graphs are simple. We saw in Remark 4.3 that they are also purely infinite in the sense of [**123**, §4.1], and that graph algebras always belong to the bootstrap class $\mathcal{N}$.

A major theorem of Kirchberg and Phillips says that the purely infinite simple C^*-algebras which belong to the bootstrap class $\mathcal{N}$ are classified by their K-theory. There are in fact two such theorems, one for stable algebras and one for unital algebras. The graphs arising in Szymański's theorem are transitive and have infinitely many vertices, and hence their C^*-algebras are stable by [**143**, Corollary 3.3]. The theorem of Kirchberg and Phillips says that if A and B are separable C^*-algebras which are stable, simple, purely infinite, belong to the bootstrap class $\mathcal{N}$, and have $K_i(A) \cong K_i(B)$ for $i = 0$ and $i = 1$, then A is isomorphic to B (see [**123**, Theorem 8.4.1]). Thus:

Corollary 7.19. *Suppose that A is a separable C^*-algebra which is stable, simple, purely infinite and belongs to the bootstrap class $\mathcal{N}$. If $K_1(A)$ is a free abelian group, then A is isomorphic to the C^*-algebra of some row-finite transitive graph.*

The point of the existence of the vertex v in Theorem 7.18 is to provide models for the unital purely infinite simple C^*-algebras. The Kirchberg-Phillips classification of unital algebras requires that the isomorphism on K_0 preserves the class [1] of the identity. The vertex projection p_v is an identity for the corner $p_vC^*(E)p_v$. Since $C^*(E)$ is simple, the corner $p_vC^*(E)p_v$ is full, and the inclusion of $p_vC^*(E)p_v$ in $C^*(E)$ induces an isomorphism on K-theory [**99**, Proposition 1.2]. Thus $p_vC^*(E)p_v$ is a purely infinite, simple, unital C^*-algebra in the class $\mathcal{N}$ with $(K_0(p_vC^*(E)p_v), [p_v]) \cong (G_0, c)$ and $K_1(p_vC^*(E)p_v) \cong G_1$. So graph algebras provide a rich supply of models for both stable and unital simple C^*-algebras.

CHAPTER 8

Cuntz-Pimsner algebras

Let A be a C^*-algebra and X a right A-module. An *A-valued inner product on* X is a function $(x, y) \mapsto \langle x, y\rangle_A$ which is linear in the second variable, and which satisfies

- $\langle x, x\rangle_A \geq 0$ and $\langle x, x\rangle_A = 0 \Longrightarrow x = 0$;
- $\langle x, y\rangle_A^* = \langle y, x\rangle_A$;
- $\langle x, y \cdot a\rangle_A = \langle x, y\rangle_A a$

for all $x, y \in X$ and $a \in A$. (We write $\langle x, y\rangle$ for $\langle x, y\rangle_A$ if we think it is clear which algebra A we're talking about). It then follows that $x \mapsto \langle x, y\rangle_A$ is conjugate linear, and that $\langle x \cdot a, y\rangle_A^* = a^*\langle x, y\rangle_A$.

A first lemma in the subject is a version of the Cauchy-Schwarz inequality which says that

$$\langle y, x\rangle_A\langle x, y\rangle_A \leq \|\langle x, x\rangle_A\|\langle y, y\rangle_A$$

as elements of the C^*-algebra A [**114**, Lemma 2.5]. This implies that

$$\|x\|_A := \|\langle x, x\rangle_A\|^{1/2}$$

defines a norm on X [**114**, Corollary 2.7], and that the module operations and inner product are continuous for this norm. If X is complete in this norm, we say that X is a (right) *Hilbert A-module*. We sometimes write X_A to emphasise that we are viewing X as a module with coefficient algebra A. The basic theory of Hilbert modules is discussed in [**84**] and [**114**, Chapter 2], for example.

Example 8.1. If $\mathcal{H}$ is a Hilbert space with inner product $(x, y) \mapsto (x \,|\, y)$ linear in the first variable, then $\mathcal{H}$ is a Hilbert $\mathbb{C}$-module with the action of $\mathbb{C}$ by scalar multiplication and $\langle x, y\rangle_{\mathbb{C}} := (y \,|\, x)$. (As in [**114**], we adopt the convention that round brackets denote inner products which are conjugate linear in the second variable.)

Example 8.2. For $n \in \mathbb{N}$, $A^n = \{x = (x_1, \cdots, x_n) : x_i \in A\}$ is a Hilbert A-module with $x \cdot a = (x_1 a, \cdots, x_n a)$ and $\langle x, y\rangle_A = \sum_{i=1}^n x_i^* y_i$.

Example 8.3. Let E be a directed graph, and take A to be the C^*-algebra $c_0(E^0)$ of functions $a : E^0 \to \mathbb{C}$ which vanish at infinity in the sense that they are arbitrarily small off finite sets. On the space $c_c(E^1)$ of functions of finite support, we define

$$(x \cdot a)(e) = x(e)a(s(e)) \quad\text{and}\quad \langle x, y\rangle_A(v) = \sum_{\{e\in E^1 : s(e)=v\}} \overline{x(e)}y(e).$$

Then we can complete $c_c(E^1)$ to get a Hilbert A-module $X(E)$; this involves first modding out by the vectors $\{x \in X_c(E^1) : \langle x, x\rangle_A = 0\}$, and then completing in the usual sense ([**114**, Lemma 2.16] justifies this).

We think of a Hilbert module X_A as an analogue of a Hilbert space in which the scalars are elements of the C^*-algebra A. The analogue of the bounded linear operators on Hilbert space are the *adjointable operators* on X: the A-module homomorphisms $T : X \to X$ which have an *adjoint* $T^* : X \to X$ satisfying

$$\langle Tx, y\rangle_A = \langle x, T^*y\rangle_A \quad \text{for every } x, y \in X.$$

With the usual operator norm, the adjointable operators on X_A form a C^*-algebra $\mathcal{L}(X)$ [**114**, Proposition 2.21].

EXAMPLE 8.4. For any Hilbert module X_A and any $x, y \in X$, the function $\Theta_{x,y} : z \mapsto x \cdot \langle y, z\rangle_A$ is an adjointable operator on X, with adjoint $\Theta_{y,x}$. The closed span of the operators $\Theta_{x,y}$ is an ideal $\mathcal{K}(X)$ in the C^*-algebra $\mathcal{L}(X)$ — in fact, $T\Theta_{x,y} = \Theta_{Tx,y}$ [**114**, Lemma 2.25]. Elements of $\mathcal{K}(X)$ are called *compact operators*; they are not usually compact as linear operators on the Banach space X, but this does not seem to cause confusion.

A *correspondence over a C^*-algebra A* is a right Hilbert A-module X together with a left action of A by adjointable operators on X which is given by a homomorphism $\phi : A \to \mathcal{L}(X)$; in other words, $a \cdot x$ is by definition $\phi(a)(x)$. By convention, homomorphisms between C^*-algebras preserve adjoints, so we are requiring here that $\phi(a^*)$ is an adjoint for $\phi(a)$; in module notation, this requirement becomes the familiar-looking formula $\langle a \cdot x, y\rangle_A = \langle x, a^* \cdot y\rangle_A$.

REMARK 8.5. The name "correspondence" comes from the von Neumann algebra literature, and our correspondences are sometimes called "C^*-correspondences" to emphasise this. In the literature, they are also described as "Hilbert bimodules," but this phrase has also been used to describe several other kinds of bimodules, and especially those in which there is also a left inner-product. Sticklers for the rules of hyphenation[1] find the expression "right-Hilbert A-A bimodules" more informative, and this has been extensively used in non-abelian duality (see [**32**], for example).

EXAMPLE 8.6. Let α be an endomorphism of a C^*-algebra A, and let X be the Hilbert module A_A (that is, the case $n = 1$ of Example 8.2). Then the homomorphism $\phi : A \to \mathcal{L}(X)$ defined by $\phi(a)(x) = \alpha(a)x$ makes X a correspondence over A.

EXAMPLE 8.7. If E is a directed graph, then the Hilbert $c_0(E^0)$-module $X(E)$ of Example 8.3 becomes a correspondence over $c_0(E^0)$ if we define $\phi(a)(x)(e) = a(r(e))x(e)$ for $x \in c_c(E^1) \subset X(E)$. To see that this defines an operator on the completion, one has to check that $\phi(a)$ is bounded for the norm induced by the inner product, but this is easy. One then verifies that $\phi(a^*)$ is an adjoint for $\phi(a)$. We call $X(E)$ the *graph correspondence* associated to E.

Correspondences arise in surprisingly many contexts, and in these contexts they often satisfy extra hypotheses which make their analysis easier. For example, we will find it helpful later to make assumptions which imply that $\phi : A \to \mathcal{L}(X)$ has range in $\mathcal{K}(X)$. One has to be a little careful when reading the literature, since there may be standing hypotheses which are not locally explicit. The next proposition shows that innocuous-looking hypotheses on the graph correspondences $X(E)$ can amount to significant hypotheses on the underlying graph E.

[1] See [**144**, page 171], for example.

PROPOSITION 8.8. *Let E be a directed graph and $X(E)$ the associated graph correspondence. For a vertex $v \in E^0$, let δ_v denote the point mass at v, viewed as an element of $c_0(E^0)$. Then*

(a) $\phi(\delta_v) \in \mathcal{K}(X(E)) \iff |r^{-1}(v)| < \infty$;

(b) $\phi(\delta_v) = 0 \iff v$ *is a source;*

(c) $\{\langle x, y\rangle : x, y \in X(E)\}$ *generates* $c_0(E^0) \iff E$ *has no sinks;*

(d) $X(E)$ *is* essential*:* $a \cdot x = 0$ *for all* $a \in c_0(E^0) \implies x = 0$.

PROOF. Part (a) is Proposition 4.4 of [**48**]; indeed, it is shown there that if $|r^{-1}(v)| < \infty$, then

$$\phi(\delta_v) = \sum_{\{e:\, r(e)=v\}} \Theta_{\delta_e,\delta_e}. \tag{8.1}$$

The other three parts are easy. □

Associated to each correspondence are several C^*-algebras. To describe the first, let X be a correspondence over A. A *Toeplitz representation* (ψ, π) of X in a C^*-algebra B consists of a linear map $\psi : X \to B$ and a homomorphism $\pi : A \to B$ such that

(T1) $\psi(x \cdot a) = \psi(x)\pi(a)$,

(T2) $\psi(x)^*\psi(y) = \pi(\langle x, y\rangle_A)$, and

(T3) $\psi(a \cdot x) = \pi(a)\psi(x)$

for $x, y \in X$ and $a \in A$. (In fact (T1) follows from (T2), and we have included it for emphasis.)

It follows from (T2) and the automatic continuity of the homomorphism π that the linear map ψ is continuous for the norm $\|\cdot\|_A$ on X, and that ψ is isometric if π is injective. The *Toeplitz algebra* of X is the C^*-algebra $\mathcal{T}_X$ generated by a universal Toeplitz representation (i_X, i_A); if (ψ, π) is a Toeplitz representation of X, we write $\psi \times \pi$ for the representation of $\mathcal{T}_X$ such that $(\psi \times \pi) \circ i_X = \psi$ and $(\psi \times \pi) \circ i_A = \pi$. It is shown in [**48**, Proposition 1.3] that there is such a C^*-algebra, that it is unique up to isomorphism, that i_A is injective, and that there is a continuous gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} \mathcal{T}_X$ such that $\gamma_z(i_A(a)) = i_A(a)$ for $a \in A$ and $\gamma_z(i_X(x)) = z i_X(x)$ for $x \in X$.

To see that these Toeplitz algebras behave like graph algebras, we need some notation. Suppose X and Y are correspondences over A. The vector-space tensor product $X \odot Y$ is also an A-A bimodule with $a \cdot (x \otimes y) = (a \cdot x) \otimes y$ and $(x \otimes y) \cdot a = x \otimes (y \cdot a)$. The pairing

$$\langle x \otimes y, \xi \otimes \eta\rangle_A := \langle\langle \xi, x\rangle_A \cdot y, \eta\rangle_A \tag{8.2}$$

defines an A-valued inner-product on $X \odot Y$, and completing in the associated norm gives a right Hilbert A-module $X \otimes_A Y$. (This is the *interior tensor product* discussed in [**84**, pages 38–44].) The notation $\otimes_A$ is intended to remind us that this tensor product is balanced in the sense that $(x \cdot a) \otimes y = x \otimes (a \cdot y)$ in $X \otimes_A Y$; this happens because the completing process involves modding out by vectors of length 0, and, as we can easily check, $\|(x \cdot a) \otimes y - x \otimes (a \cdot y)\|_A = 0$. The left action of A is by adjointable operators, so $X \otimes_A Y$ is another correspondence over A. If (ψ, π) and (μ, π) are Toeplitz representations of X and Y in a C^*-algebra B with the same representation π of A, then there is a Toeplitz representation $(\psi \otimes \mu, \pi)$ of $X \otimes_A Y$ in B such that $\psi \otimes \mu(x \otimes y)$ is the product $\psi(x)\mu(y)$ in B [**48**, Proposition 1.8].

Now suppose we have a Toeplitz representation (ψ, π) of a single correspondence X over A. We write $X^{\otimes n}$ for the nth tensor power $X \otimes_A X \otimes_A \cdots \otimes_A X$, and $(\psi^{\otimes n}, \pi)$ for the corresponding Toeplitz representation of $X^{\otimes n}$; it is notationally helpful to take $X^{\otimes 0} = A$ and $\psi^{\otimes 0} = \pi$. The natural isomorphism of $X^{\otimes m} \otimes_A X^{\otimes n}$ onto $X^{\otimes(m+n)}$ carries $\psi^{\otimes m} \otimes \psi^{\otimes n}$ into $\psi^{\otimes(m+n)}$. (There is a minor subtlety when $n = 0$ which we can ignore here because our modules are always essential. See [**48**, page 164].)

The key observation to understanding the Toeplitz algebra $\mathcal{T}_X$ is that any product of the form $\psi^{\otimes m}(x)^*\psi^{\otimes n}(y)$ can be simplified. To see why, suppose first that π is faithful and that $m \leq n$. We can approximate $y \in X^{\otimes n}$ by a sum of elementary tensors, and in particular by sums of elements of the form $w \otimes z$ where $w \in X^{\otimes m}$ and $z \in X^{\otimes(n-m)}$. We can now compute:

$$\begin{aligned}\psi^{\otimes m}(x)^*\psi^{\otimes n}(w \otimes z) &= \psi^{\otimes m}(x)^*\psi^{\otimes m}(w)\psi^{\otimes(n-m)}(z) \\ &= \pi(\langle x, w\rangle_A)\psi^{\otimes(n-m)}(z) \\ &= \psi^{\otimes(n-m)}\big(\langle x, w\rangle_A \cdot z\big).\end{aligned} \tag{8.3}$$

Since $\psi^{\otimes n}$ and $\psi^{\otimes(n-m)}$ are isometric, it follows from (8.3) that $\psi^{\otimes m}(x)^*\psi^{\otimes n}(y)$ is in the range of $\psi^{\otimes(n-m)}$ for every $x \in X^{\otimes m}$ and $y \in X^{\otimes n}$. When $m \geq n$, a similar argument in the x-variable shows that $\psi^{\otimes m}(x)^*\psi^{\otimes n}(y)$ has the form $\psi^{\otimes(m-n)}(w)^*$. Thus all products of generators in $\psi \times \pi(\mathcal{T}_X)$ collapse to ones of the form $\psi^{\otimes m}(x)\psi^{\otimes n}(y)^*$.

Applying this analysis to the universal Toeplitz representation (i_X, i_A) gives the following proposition.

PROPOSITION 8.9. *Let X be a correspondence over a C^*-algebra A. Then*

$$\mathcal{T}_X = \overline{\operatorname{span}}\{i_X^{\otimes m}(x)i_X^{\otimes n}(y)^* : m, n \geq 0,\ x \in X^{\otimes m},\ y \in X^{\otimes n}\}.$$

So the Toeplitz algebra has a grading over $\mathbb{Z}$ which looks formally like that of the graph algebras. Of course this is not an accident:

EXAMPLE 8.10. Suppose (ψ, π) is a Toeplitz representation of a graph correspondence $X(E)$, and set $S_e = \psi(\delta_e)$, $P_v = \pi(\delta_v)$. Then the P_v are mutually orthogonal projections because the δ_v are mutually orthogonal projections in $A = c_0(E^0)$. Equation (T2) implies that $S_e^*S_e = P_{s(e)}$ and $S_e^*S_f = 0$ for $f \neq e$, so the S_e are partial isometries with initial projection $P_{s(e)}$ and orthogonal range projections, and Equation (T3) implies that these range projections satisfy $S_eS_e^* \leq P_{r(e)}$, so that

$$P_v \geq \sum_{\{e \in E^1 : r(e) = v\}} S_eS_e^* \quad \text{for every } v \in E^0;$$

we say that $\{S_e, P_v\}$ is a *Toeplitz-Cuntz-Krieger E-family*. Every such family arises this way from a Toeplitz representation of $X(E)$ [**48**, Example 1.2], and the Toeplitz algebra $\mathcal{T}_{X(E)}$ is therefore universal for Toeplitz-Cuntz-Krieger E-families. Since the point masses δ_e span $c_c(E^1) \subset X(E)$ and $\psi^{\otimes n}(\delta_{\mu_1} \otimes \cdots \otimes \delta_{\mu_n}) = S_{\mu_1} \cdots S_{\mu_n}$ is non-zero precisely when $\mu = \mu_1 \cdots \mu_n$ is a path in E, we recover

$$\mathcal{T}_{X(E)} = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*,\ s(\mu) = s(\nu)\},$$

where $\{s_e, p_v\} = \{i_{X(E)}(\delta_e), i_A(\delta_v)\}$ is the Toeplitz-Cuntz-Krieger family associated to the universal Toeplitz representation, and we have as usual taken $s_\mu := p_v$ when.$\mu = v$ is a path of length 0.

So $C^*E)$ is in general a proper quotient of the Toeplitz algebra $\mathcal{T}_{X(E)}$ of the graph correspondence. There is an analogous quotient of $\mathcal{T}_X$ for an arbitrary correspondence X, which is universal for a more restrictive class of Toeplitz representations. To describe this class, we need the following construction[2].

PROPOSITION 8.11. *Let X be a correspondence over a C^*-algebra A, and let (ψ, π) be a Toeplitz representation of a correspondence in a C^*-algebra B. Then there is a homomorphism $(\psi, \pi)^{(1)} : \mathcal{K}(X) \to B$ such that*

$$(\psi, \pi)^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^* \quad \text{for } x, y \in X. \tag{8.4}$$

We then have

(a) $(\psi, \pi)^{(1)}(T)\psi(x) = \psi(Tx)$ *for* $T \in \mathcal{K}(X)$ *and* $x \in X$;
(b) *if ρ is a homomorphism of B into a C^*-algebra C, then* $(\rho \circ \psi, \rho \circ \pi)^{(1)} = \rho \circ (\psi, \pi)^{(1)}$;
(c) *if π is faithful, so is* $(\psi, \pi)^{(1)}$.

PROOF. To prove the existence of $(\psi, \pi)^{(1)}$, we represent B faithfully on a Hilbert space $\mathcal{H}$, and then consider the representation of $\mathcal{K}(X)$ on $X \otimes_A \mathcal{H}$ induced from $\pi : A \to B(\mathcal{H})$. The map ψ gives an isometry $x \otimes h \mapsto \psi(x)h$ of $X \otimes_B \mathcal{H}$ into $\mathcal{H}$, and we take $(\psi, \pi)^{(1)}$ to be the equivalent representation of $\mathcal{K}(X)$ on the range of this isometry. Part (c) is true because the induced representation of $\mathcal{K}(X)$ is faithful (see [**114**, Corollary 2.74]), and part (a) follows from a computation on rank-one operators (see [**48**, Proposition 1.6], for example). If $\rho : B \to C$ is a homomorphism, then $(\rho \circ \psi, \rho \circ \pi)$ is a Toeplitz representation of X, and we have

$$(\rho \circ \psi, \rho \circ \pi)^{(1)}(\Theta_{x,y}) = \rho \circ \psi(x)\rho \circ \psi(y)^* = \rho \circ (\psi, \pi)^{(1)}(\Theta_{x,y}) \qquad \text{for } x, y \in X.$$

Since both $(\rho\circ\psi, \rho\circ\pi)^{(1)}$ and $\rho\circ(\psi, \pi)^{(1)}$ are linear and continuous, this implies (b). □

If I is an ideal in a C^*-algebra A, we denote by $I^\perp$ the set of elements $a \in A$ such that $ab = 0$ for all $b \in I$. A Toeplitz representation (ψ, π) of a correspondence X over A is *Cuntz-Pimsner covariant* if for every $a \in \phi^{-1}(\mathcal{K}(X)) \cap (\ker \phi)^\perp$ we have $(\psi, \pi)^{(1)}(\phi(a)) = \pi(a)$. The *Cuntz-Pimsner algebra* $\mathcal{O}_X$ is the quotient of $\mathcal{T}_X$ by the ideal generated by

$$\{(i_X, i_A)^{(1)}(\phi(a)) - i_A(a) : a \in \phi^{-1}(\mathcal{K}(X)) \cap (\ker \phi)^\perp\}. \tag{8.5}$$

Let $q : \mathcal{T}_X \to \mathcal{O}_X$ be the quotient map. Then $(k_X, k_A) := (q \circ i_X, q \circ i_A)$ is a Cuntz-Pimsner covariant representation of X in $\mathcal{O}_X$ which is universal for Cuntz-Pimsner covariant representations of X. The representation of $\mathcal{O}_X$ corresponding to a Cuntz-Pimsner covariant pair (ψ, π) is denoted by $\psi \times \pi$, or by $\psi \times_{\mathcal{O}} \pi$ if confusion seems possible.

REMARK 8.12. When ϕ is injective, $\mathcal{O}_X$ was called the augmented C^*-algebra of X in [**104**], and this $\mathcal{O}_X$ coincides with the Cuntz-Pimsner algebra studied in [**48**] or [**47**], for example. When ϕ is injective and X is full in the sense that the range of the inner product generates A, $\mathcal{O}_X$ coincides also with Pimsner's C^*-algebra of X which was the main object of study in [**104**]. When ϕ is not injective, or more

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[2]In the literature the map $(\psi, \pi)^{(1)}$ of Proposition 8.11 is usually denoted $\pi^{(1)}$, which we think is unfortunate because it depends on ψ. In [**48**], $(\psi, \pi)^{(1)}$ was denoted $\rho^{\psi,\pi}$, but this did not catch on.

precisely when $(\ker\phi)^\perp$ is not all of A, the Cuntz-Pimsner algebra is usually taken to be the quotient by the ideal generated by

$$\{(i_X, i_A)^{(1)}(\phi(a)) - i_A(a) : a \in \phi^{-1}(\mathcal{K}(X))\}.$$

Using only the generators (8.5) associated to elements of $(\ker\phi)^\perp$ was suggested by Katsura [**72, 74, 75**], and the next example shows why Katsura's innovation has been quickly accepted (see [**91, 92**]).

EXAMPLE 8.13. For a graph correspondence $X(E)$, we saw in Proposition 8.8 that $\phi^{-1}(\mathcal{K}(X(E)))$ is spanned by the point masses δ_v associated to vertices v with $r^{-1}(v)$ finite, and that a point mass satisfies $\phi(\delta_v) = 0$ precisely when v is a source. So

$$\phi^{-1}(\mathcal{K}(X)) \cap (\ker\phi)^\perp = \overline{\operatorname{span}}\{\delta_v : 0 < |r^{-1}(v)| < \infty\}.$$

From (8.1) we deduce that a Toeplitz representation (ψ, π) of $X(E)$ is Cuntz-Pimsner covariant if and only if

$$\pi(\delta_v) = (\psi,\pi)^{(1)}(\phi(\delta_v)) = (\psi,\pi)^{(1)}\big(\textstyle\sum_{r(e)=v} \Theta_{\delta_e,\delta_e}\big) = \displaystyle\sum_{r(e)=v} \psi(\delta_e)\psi(\delta_e)^*$$

for every vertex v satisfying $0 < |r^{-1}(v)| < \infty$. That is, if and only if the Toeplitz-Cuntz-Krieger E-family $\{\psi(\delta_e), \pi(\delta_v)\}$ is a Cuntz-Krieger E-family. Thus $\mathcal{O}_{X(E)}$ is universal for Cuntz-Krieger E-families, and is isomorphic to $C^*(E)$.

At this point it is tempting to study the more general Cuntz-Pimsner algebras, and deduce what we need to know about graph algebras from the general theory. At present, this theory is not sufficiently developed to make this practicable. Nevertheless, it is certainly worthwhile to review what is available.

Suppose that X is a correspondence over a C^*-algebra A. The gauge action on $\mathcal{T}_X$ fixes the generators (8.5), and hence gives a natural gauge action on the quotient $\mathbb{Q}_X$. The analysis of the core in $\mathcal{O}_X$ has been carried out in full generality, and there is a very satisfactory gauge-invariant uniqueness theorem ([**74**, Theorem 6.4] and [**92**, Theorem 5.3]). The approach taken in [**74**] involved a direct analysis of the core, whereas the approach of [**92**] used a technique from [**91**] to reduce to the case in which ϕ is injective and [**47**, Theorem 4.1] applies. The gauge-invariant ideals in $\mathcal{O}_X$ were described in [**75**].

There is as yet no completely satisfactory Cuntz-Krieger uniqueness theorem for Cuntz-Pimsner algebras. The closest obtained so far is that of [**66**], which applies to correspondences X_A over unital algebras which are finitely generated and projective as right A-modules. The condition of (I)-freeness used in [**66**] describes what happens in $\mathcal{O}_X$ rather than identifying conditions on the correspondence X_A which imply uniqueness. Nevertheless, the ideas in [**66**] led in [**67**] to a Cuntz-Krieger uniqueness theorem for the C^*-algebras of directed graphs associated to locally finite $\{0,1\}$-matrices, under a hypothesis which turned out to be equivalent to asking that every cycle has an entry, as in [**82**] and Theorem 2.4.

Several authors have described criteria for the simplicity of Cuntz-Pimsner algebras [**66, 89, 127, 147**]. For example, we discuss a theorem of Schweizer [**127**]. An ideal I in a C^*-algebra A is *invariant* for a correspondence X_A if $\langle X, I\cdot X\rangle_A \subset I$. Schweizer's theorem concerns a full correspondence X over a unital algebra A; it says that $\mathcal{O}_X$ is simple if and only if there are no non-trivial invariant ideals and $X^{\otimes n}$ is not isomorphic to A_A for $n > 0$. From this theorem, Schweizer derived simplicity criteria for crossed products by endomorphisms [**127**, §4], compact topological quivers [**126**, Corollary 5.2], and Exel-Laca algebras [**126**, Corollary 5.4].

In his original paper [**104**], Pimsner described a six-term exact sequence for the K-theory of the Cuntz-Pimsner algebra $\mathcal{O}_X$; nowadays, we would say that he was working with correspondences for which ϕ is injective and the inner product is full, but Katsura has shown that these hypotheses are not necessary [**74**, Theorem 8.6]. Nuclearity and exactness of Cuntz-Pimsner algebras have been studied in [**31**] and [**74**, §7].

REMARK 8.14. If X is a correspondence over a C^*-algebra A, the *tensor algebra of* X is the norm-closed subalgebra

$$\mathcal{T}_+(X) := \overline{\operatorname{span}}\{i_X^{\otimes n}(x) : n \geq 0, x \in X^{\otimes n}\}$$

of the Toeplitz algebra $\mathcal{T}_X$. This non-self-adjoint algebra was first studied by Muhly and Solel [**88, 90**]. When $X = X(E)$ is a graph correspondence, $\mathcal{T}_+(X(E))$ has a natural representation on $\ell^2(E^*)$, and the weak-operator closure $\mathcal{L}_E$ of $\mathcal{T}_+(X(E))$ in this representation is known as the *free semigroupoid algebra* of E [**78**]. Many algebras of current interest to non-self-adjoint operator algebraists can be realised as $\mathcal{L}_E$ for suitable E.

Solel [**130**] and Katsoulis and Kribs [**71**] have (independently) shown that the tensor algebra $\mathcal{T}_+(X(E))$ completely determines the graph E. For example, if E and F are directed graphs and there is a Banach algebra isomorphism ϕ of $\mathcal{T}_+(X(E))$ onto $\mathcal{T}_+(X(F))$, then E is isomorphic to F [**71**, Theorem 2.11]. Indeed, under the assumption that there are either no sources or no sinks, it suffices that ϕ is an algebra isomorphism [**71**, Corollary 2.15]. Thus the tensor algebra is a finer invariant of a graph than the graph algebra $C^*(E)$: the dual-graph construction of Corollary 2.6, for example, shows that it is easy to find non-isomorphic graphs with isomorphic C^*-algebras. The free semigroupoid algebras $\mathcal{L}_E$ are also finer invariants (see [**78, 71, 130**]).

CHAPTER 9

Topological graphs

We will illustrate how correspondences provide a powerful framework for studying graph algebras and their generalisations by proving a uniqueness theorem of Katsura [**73**]. In keeping with our desire to present the main ideas as clearly as possible, we will prove Katsura's theorem for the continuous analogue of row-finite graphs; his theorem is much more general, and has already been substantially generalised by Muhly and Tomforde [**92**]. Our treatment leans heavily on both [**73**] and [**92**].

A *topological graph* consists of locally compact spaces E^0 and E^1, a continuous map $r : E^1 \to E^0$, and a local homeomorphism $s : E^1 \to E^0$. As the notation suggests, we want to think of points in E^0 as vertices, and a point $e \in E^1$ as an edge from $s(e)$ to $r(e)$. Katsura's route to topological-graph algebras is through Cuntz-Pimsner algebras.

As a point of notation, when T is a locally compact space, we write $C_0(T)$ for the C^*-algebra of continuous functions $g : T \to \mathbb{C}$ which vanish at infinity, and $C_c(T)$ for the subspace consisting of the functions $g \in C_0(T)$ whose support

$$\operatorname{supp} g := \overline{\{t \in T : g(t) \neq 0\}}$$

is compact. Those not familar with locally compact spaces can mentally assume that they are all compact, or refer to [**103**], where there is a succinct treatment of locally compact spaces which covers exactly what operator algebraists need to know.

Suppose that $E = (E^0, E^1, r, s)$ is a topological graph. For $x, y \in C_c(E^1)$ and $a \in A := C_0(E^0)$, we define

$$(x \cdot a)(e) := x(e)a(s(e)) \quad \text{and} \quad \langle x, y\rangle_A(v) := \sum_{\{e\in E^1 :\, s(e)=v\}} \overline{x(e)}y(e).$$

Since s is a local homeomorphism, we can cover $\operatorname{supp} x$ by finitely many open sets U_i such that $s(U_i)$ is open and $s|_{U_i}$ has a continuous inverse, and then near any given vertex $v \in E^0$,

$$\langle x, y\rangle_A = \sum_{\{i :\, v\in s(U_i)\}} \overline{(x \circ (s|_{U_i})^{-1})}(y \circ (s|_{U_i})^{-1})$$

is a finite sum of continuous functions. Thus $\langle x, y\rangle_A \in C_c(E^0) \subset A$. It is easy to check that $\langle \cdot, \cdot\rangle_A$ is an A-valued inner product on $C_c(E^1)$, and hence the completion $X(E)$ is a Hilbert A-module. The formula

$$(a \cdot x)(e) := a(r(e))x(e)$$

defines an action of A by adjointable operators on $X(E)$, so that $X(E)$ becomes a correspondence over $A = C_0(E^0)$, which we call the *graph correspondence* associated to E. The C^*-algebra of E is by definition the Cuntz-Pimsner algebra $\mathcal{O}_{X(E)}$; we denote by (k_X, k_A) the canonical Cuntz-Pimsner covariant representation of $X(E)$ in $\mathcal{O}_{X(E)}$.

We are going to simplify our arguments by restricting attention to topological graphs in which the range map r is proper, in the sense that the inverse images of compact sets are compact. Comparing the next proposition with Proposition 8.8(a) shows why we view these topological graphs as the analogue of row-finite graphs. If E^1 and E^0 are compact, then properness is automatic, and we have the analogue of finite graphs. Descriptions of $\phi^{-1}(\mathcal{K}(X(E)))$ for general topological graphs are given in [**73**, Proposition 1.17] and [**92**, Corollary 3.12].

PROPOSITION 9.1. *Suppose that E is a topological graph, and that r is proper. Then the left action of $A = C_0(E^0)$ on the graph correspondence $X(E)$ is by compact operators. Indeed, for every $g \in C_c(E^0)$, there exist $\xi_j, \eta_j \in C_c(E^1)$ such that*

$$\phi(g) = \sum_{j=1}^{J} \Theta_{\xi_j,\eta_j}. \tag{9.1}$$

PROOF. Because r is proper, $K := \operatorname{supp}(g \circ r)$ is compact. We can therefore find finitely many open sets U_j such that s is injective on U_j and $K \subset \bigcup_{j=1}^{J} U_j$. Choose a partition of unity $\{w_j\}$ for K with $\operatorname{supp} w_j \subset U_j$, and take

$$\xi_j := (g \circ r)\sqrt{w_j}, \quad \eta_j := \sqrt{w_j}.$$

To see that these functions have the required properties, we first note that for $e \in E^1$, we have

$$g(r(e)) = g(r(e)) \sum_{j=1}^{J} w_k(e) = \sum_{j=1}^{J} \xi_j(e)\overline{\eta_j(e)} \tag{9.2}$$

(both sides vanish if $e \notin K$). Because $\operatorname{supp} w_j \subset U_j$ and s is injective on U_j, we also have

$$\xi_j(e)\overline{\eta_j(f)} = 0 \text{ whenever } s(e) = s(f) \text{ and } e \neq f. \tag{9.3}$$

For $x \in C_c(E^1)$, we compute using (9.3):

$$\begin{aligned}\Big(\sum_{j=1}^{J} \Theta_{\xi_j,\eta_j}(x)\Big)(e) &= \sum_{j=1}^{J} \xi_j(e)\langle \eta_j, x\rangle_A(s(e)) \\ &= \sum_{j=1}^{J} \xi_j(e)\Big(\sum_{s(f)=s(e)} \overline{\eta_j(f)}x(f)\Big) \\ &= \sum_{j=1}^{J} \xi_j(e)\overline{\eta_j(e)}x(e),\end{aligned}$$

which by (9.2) is precisely $g(r(e))x(e) = (g \cdot x)(e) = \phi(g)(e)$. Thus both sides of (9.1) agree on the dense subset $C_c(E^1)$ of $X(E)$, and hence on all of $X(E)$. Now the continuity of the C^*-algebra homomorphism ϕ implies that $\phi(A) \subset \mathcal{K}(X(E))$. □

As we mentioned in the previous section, there are satisfactory gauge-invariant uniqueness theorems for Cuntz-Pimsner algebras, and hence in particular for the C^*-algebras of topological graphs. Katsura's remarkable achievement is a uniqueness theorem of Cuntz-Krieger type for the C^*-algebras of topological graphs. His theorem has an intuitively attractive graph-theoretic hypothesis, and uses the path structure of the graph in its proof. His hypothesis is a continuous analogue of the hypothesis in Theorem 2.4 that every cycle has an entry, which was called Condition (L) in [**82**].

Let $E = (E^0, E^1, r, s)$ be a topological graph. A *path* in E is a sequence $\mu = \mu_1\mu_2 \cdots \mu_n$ of points in E^1 such that $s(\mu_i) = r(\mu_{i+1})$ for all i; we write $|\mu| := n$ for the length of μ, and we view points v of E^0 as paths with $|v| = 0$. A path μ with $|\mu| \geq 1$ is a *cycle* if $r(\mu_1) = s(\mu_n)$ and $s(\mu_i) \neq s(\mu_j)$ for $i \neq j$. An edge e is an *entry* to a path μ if there exists i such that $r(e) = r(\mu_i)$ and $e \neq \mu_i$. Following Muhly and Tomforde, we say that E *satisfies Condition* (L) if the set

$$\{v \in E^0 : \text{every cycle through } v \text{ has an entry}\}$$

is dense in E^0. (Katsura says that E is *topologically free*.)

We are now ready to state the version of Katsura's uniqueness theorem which we will prove. In addition to assuming that r is proper, we assume that r has dense range, which is the topological analogue of assuming that E has no sources. In fact, since the range of any proper map is closed, this is equivalent to asking that r be proper and surjective. Neither assumption is necessary, but they do make the proof a lot more transparent.

THEOREM 9.2 (Katsura). *Suppose that E is a topological graph which satisfies Condition* (L), *and that r is proper and surjective. Suppose that (ψ, π) is a Cuntz-Pimsner covariant representation of the graph correspondence $X(E)$. If π is injective, then the representation $\psi \times_{\mathcal{O}} \pi$ of $\mathcal{O}_{X(E)}$ is injective.*

We begin as in Chapter 3 by analysing the core, which is the fixed-point algebra $\mathcal{O}_{X(E)}^{\gamma}$ for the gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} \mathcal{O}_{X(E)}$. To simplify the notation, we often write A for $C_0(E^0)$ and X for $X(E)$. Since we know from Proposition 8.9 that

$$\mathcal{O}_X = \overline{\operatorname{span}}\{k_X^{\otimes m}(x)k_X^{\otimes n}(y)^* : m, n \geq 0,\ x \in X^{\otimes m},\ y \in X^{\otimes n}\},$$

and since the expectation $\Phi : b \mapsto \int_{\mathbb{T}} \gamma_z(b)\, dz$ onto the fixed-point algebra is continuous and satisfies

$$\Phi(k_X^{\otimes m}(x)k_X^{\otimes n}(y)^*) = \begin{cases} k_X^{\otimes n}(x)k_X^{\otimes n}(y)^* & \text{if } m = n \\ 0 & \text{if } m \neq n, \end{cases}$$

we have

$$\mathcal{O}_X^{\gamma} = \overline{\operatorname{span}}\{k_X^{\otimes n}(x)k_X^{\otimes n}(y)^* : n \geq 0,\ x, y \in X^{\otimes n}\}.$$

For fixed $n \geq 0$, we set

$$\mathcal{F}_n := \overline{\operatorname{span}}\{k_X^{\otimes n}(x)k_X^{\otimes n}(y)^* : x, y \in X^{\otimes n}\}.$$

LEMMA 9.3. *Suppose that E is a topological graph whose range map r is proper. Then $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for every $n \geq 0$.*

PROOF. Let x, y be elements of the algebraic tensor product $C_c(E^1)^{\otimes n} \subset X^{\otimes n}$, and choose $g \in C_c(E^0)$ such that $g = 1$ on $s(\operatorname{supp} x_n)$, so that $x = x \cdot g$. Then the Cuntz-Pimsner covariance of (k_X, k_A) implies that

$$\begin{aligned} k_X^{\otimes n}(x)k_X^{\otimes n}(y)^* &= k_X^{\otimes n}(x \cdot g)k_X^{\otimes n}(y)^* \\ &= k_X^{\otimes n}(x)k_A(g)k_X^{\otimes n}(y)^* \\ &= k_X^{\otimes n}(x)(k_X, k_A)^{(1)}(\phi(g))k_X^{\otimes n}(y)^*. \end{aligned} \tag{9.4}$$

Choose $\xi_j, \eta_j \in C_c(E^1)$ such that $\phi(g) = \sum_{j=1}^J \Theta_{\xi_j,\eta_j}$, as in Proposition 9.1. Then

$$(k_X, k_A)^{(1)}(\phi(g)) = \sum_{j=1}^J k_X(\xi_j)k_X(\eta_j)^*,$$

and plugging this into the middle of (9.4) shows that

$$k_X^{\otimes n}(x)k_X^{\otimes n}(y)^* = \sum_{j=1}^J k_X^{\otimes(n+1)}(x \otimes \xi_j)k_X^{\otimes(n+1)}(y \otimes \eta_j)^*$$

belongs to $\mathcal{F}_{n+1}$. The result follows. □

COROLLARY 9.4. *Suppose that E is a topological graph such that the range map r is proper. Then $\mathcal{O}_{X(E)}^\gamma = \overline{\bigcup_{n\geq 0} \mathcal{F}_n}$.*

REMARK 9.5. We can carry out a similar analysis of the core in $\mathcal{T}_X$ and $\mathcal{O}_X$ for any correspondence X_A in which A acts on the left by compact operators, so that $\phi(A) \subset \mathcal{K}(X)$. When $\phi(A)$ is not contained in $\mathcal{K}(X)$, $\mathcal{F}_n$ is contained in the multiplier algebra $M(\mathcal{F}_{n+1})$ rather than $\mathcal{F}_{n+1}$, and the analysis of the core is more subtle (see [**47**, Section 4]).

The next step in the strategy of Chapter 3 is to show that representations which are faithful on $A = C_0(E^0)$ are faithful on the core. With the inductive limit description of the core in Corollary 9.4, this is easy.

PROPOSITION 9.6. *Suppose that E is a topological graph such that the range map r is proper, and (ψ, π) is a Cuntz-Pimsner covariant representation of the graph correspondence $X(E)$ such that π is injective. Then the representation $\psi \times_{\mathcal{O}} \pi$ of $\mathcal{O}_{X(E)}$ is injective on the core $\mathcal{O}_{X(E)}^\gamma$.*

PROOF. The subalgebra $\mathcal{F}_n$ is the image of $\mathcal{K}(X^{\otimes n})$ under $(k_X^{\otimes n}, k_A)^{(1)}$. We know from Proposition 8.11(c) that $(\psi^{\otimes n}, \pi)^{(1)}$ is injective, and from Proposition 8.11(b) that

$$(\psi^{\otimes n}, \pi)^{(1)} = \big((\psi \times_{\mathcal{O}} \pi) \circ k_X^{\otimes n}, (\psi \times_{\mathcal{O}} \pi) \circ k_A\big)^{(1)} = (\psi \times_{\mathcal{O}} \pi) \circ (k_X^{\otimes n}, k_A)^{(1)}.$$

Thus $\psi \times_{\mathcal{O}} \pi$ is injective on each $\mathcal{F}_n$. (Notice for future use that this argument also shows that $(k_X^{\otimes n}, k_A)^{(1)}$ is injective on $\mathcal{F}_n$.) This implies that $\psi \times_{\mathcal{O}} \pi$ is isometric on every $\mathcal{F}_n$, and hence also on $\mathcal{O}_X^\gamma = \overline{\bigcup_{n\geq 0} \mathcal{F}_n}$. Thus $\psi \times_{\mathcal{O}} \pi$ is injective on $\mathcal{O}_X^\gamma$. □

To follow the strategy of Chapter 3, we need an estimate

$$\|\psi \times_{\mathcal{O}} \pi(\Phi(d))\| \leq \|\psi \times_{\mathcal{O}} \pi(d)\| \quad \text{for } d \in \mathcal{O}_{X(E)}. \tag{9.5}$$

One key observation in [**73**] is that the tensor correspondences $X^{\otimes n}$ have a natural interpretation in terms of the path spaces of the topological graph. We can then use our graph-theoretic hypotheses on cycles in E to prove the estimate (9.5).

Let E be a topological graph. Because the maps r and s are continuous, the set E^n of paths of length n is a closed subset of the product space $E^1 \times \cdots \times E^1$, and is therefore locally compact in the subspace topology. The range map $r : \mu = \mu_1 \cdots \mu_n \mapsto r(\mu_1)$ is continuous, and the source map $s : \mu = \mu_1 \cdots \mu_n \mapsto s(\mu_n)$ is a local homeomorphism: indeed, if U_i is an open neighbourhood of μ_i in E^1 such that s is a homeomorphism of U_i onto a open set in E^0, and we inductively shrink U_i to ensure that $r(U_{i+1}) \subset s(U_i)$, then $U := (U_1 \times \cdots U_n) \cap E^n$ is an open neighbourhood of μ on which s is a homeomorphism, and $s(U) = s(U_n)$ is open. Thus (E^0, E^n, r, s) is itself a topological graph. When there might be doubt about which range and source maps we are using, we write r^n and s^n for the ones defined on E^n.

PROPOSITION 9.7. *Suppose that E is a topological graph and $n \in \mathbb{N}$. Then the map ρ of the algebraic tensor product $C_c(E^1)^{\otimes n}$ into $C_c(E^n)$ defined by*

$$\rho(x_1 \otimes \cdots \otimes x_n)(\mu) = x_1(\mu_1)x_2(\mu_2)\cdots x_n(\mu_n)$$

extends to an isomorphism of $X(E)^{\otimes n}$ onto $X(E^n)$ (by which we mean a $C_0(E^0)$-module isomorphism which preserves the $C_0(E^0)$-valued inner products).

PROOF. To simplify the notation, we suppose $n = 2$. It is easy to see that $\rho(x \cdot a) = \rho(x) \cdot a$ for $a \in C_0(E^0)$. To see that ρ preserves the inner products, we fix $x, y, \xi, \eta \in C_c(E^1)$, $v \in E^0$, and compute, using the formula (8.2) for the inner product on the tensor product:

$$\begin{aligned}
\langle x \otimes \xi, y \otimes \eta \rangle(v) &= \langle \langle y, x \rangle \cdot \xi, \eta \rangle(v) \\
&= \sum_{s(e)=v} \overline{\langle y, x \rangle (r(e)) \xi(e)} \eta(e) \\
&= \sum_{s(e)=v} \Big(\overline{\sum_{s(f)=r(e)} \overline{y(f)} x(f)} \Big) \overline{\xi(e)} \eta(e) \\
&= \sum_{s(fe)=v} \overline{x(f)\xi(e)} y(f)\eta(e) \\
&= \langle \rho(x \otimes \xi), \rho(y \otimes \eta) \rangle(v).
\end{aligned}$$

This implies in particular that ρ is isometric, and hence extends to an isometry on the completion; the continuity of the module action and inner product imply that this extension of ρ has the right algebraic properties. So it remains to see that the range of ρ is dense.

Using a partition of unity, we can write any function in $C_c(E^2)$ as a sum of functions with support in open sets of the form $U = (U_1 \times U_2) \cap E^2$, where s is a homeomorphism on U_i with open range and $r(U_2) \subset s(U_1)$. So suppose g has support in such a set U. For each $e \in U_2$ there is exactly one $f = f(e)$ in U_1 such that $s(f(e)) = r(e)$, and then $f(e)e$ is the only path in U with first edge e. Define $y \in C_c(E^1)$ by

$$y(e) = \begin{cases} g(f(e)e) & \text{if } e \in U_2 \\ 0 & \text{otherwise,} \end{cases}$$

and note that $\operatorname{supp} y$ is a compact subset of U_2. Choose $x \in C_c(E^1)$ such that $x = 1$ on $s^{-1}(r(\operatorname{supp} y)) \cap U_1$ and $\operatorname{supp} x \subset U_1$. Then $\rho(x \otimes y)(fe) = x(f)y(e) = 0 = g(fe)$ unless $fe \in U$ and $e \in \operatorname{supp} y$; if $fe \in U$ and $e \in \operatorname{supp} y$, then $f \in s^{-1}(r(\operatorname{supp} y)) \cap U_1$, $f = f(e)$, and

$$\rho(x \otimes y)(fe) = x(f)y(e) = y(e) = g(f(e)e) = g(fe).$$

Thus g belongs to the range of ρ, and the range of ρ is dense. □

We now investigate consequences of Condition (L). We say that a path $\mu \in E^n$ is *non-returning* if $\mu_k \neq \mu_n$ for every $k < n$.

PROPOSITION 9.8. *Suppose that E is a topological graph such that the range map r has dense range, and suppose that E satisfies Condition* (L). *Let V be an open set in E^0 and $n \in \mathbb{N}$. Then there exists $m \geq n$ and a non-returning path $\mu \in E^m$ such that $r(\mu) \in V$.*

Suppose to the contrary that every path in every $(r^m)^{-1}(V)$ returns. Then the hypotheses of the following lemma are satisfied.

LEMMA 9.9. *Suppose that E is a topological graph, and V is an open subset of E^0. Suppose further that there exists n such that every path μ with $r(\mu) \in V$ and $|\mu| \geq n$ returns. Let $\nu \in (r^n)^{-1}(V)$. Then there exists k such that $\nu_k \cdots \nu_n$ is a closed path with no entries.*

PROOF. Since ν returns, there exists $k \leq n$ such that $\nu_{k-1} = \nu_n$. We suppose that $\nu_k \cdots \nu_n$ has an entry, and look for a contradiction. Then there exist l and $e \in E^1$ such that $k \leq l \leq n$, $r(e) = r(\nu_l)$ and $e \neq \nu_l$. Take the shortest subpath $\nu_1 \cdots \nu_j$ of ν with $s(\nu_j) = r(e)$ (in other words, j is the smallest integer such that $s(\nu_j) = r(e)$, and j could be $l-1$). There is a cycle γ based at $r(e)$ to which e is an entry: just take a subcycle of the closed path $\nu_k \cdots \nu_n$ containing ν_l, and relabel it so it starts and ends at $s(\nu_j) = r(e)$. Then $\mu := \nu_1 \cdots \nu_j \gamma\gamma \cdots \gamma e$ is a non-returning path with $r(\mu) \in V$, no matter how many copies of γ we stick in. We can certainly stick enough in to ensure that $|\mu| \geq n$, and then we have contradicted the hypothesis. So $\nu_k \cdots \nu_n$ cannot have an entry. □

PROOF OF PROPOSITION 9.8. As promised, suppose to the contrary that every path in every $(r^m)^{-1}(V)$ returns. Then Lemma 9.9 implies that the source $s^n(\nu)$ of every path $\nu \in (r^n)^{-1}(V)$ is the base of a cycle with no entries. In other words,

$$(s^n)((r^n)^{-1}(V)) \subset E^0 \setminus \{v \in E^0 : \text{every cycle through } v \text{ has an entry}\}. \tag{9.6}$$

Because r has dense range and is continuous, $r^{-1}(V)$ is a nonempty open subset of E^1, $s(r^{-1}(V))$ is a nonempty open subset of E^0, and

$$(r^2)^{-1}(V) = \big(r^{-1}(V) \times r^{-1}(s(r^{-1}(V)))\big) \cap E^2$$

is a nonempty open subset of E^2; proceeding by induction, we can deduce that $(s^n)((r^n)^{-1}(V))$ is a nonempty open subset of E^0. Thus (9.6) contradicts the assertion that E satisfies Condition (L). □

COROLLARY 9.10. *Suppose that E is a topological graph such that the range map r has dense range, and suppose that E satisfies Condition* (L). *Let V be an open set in E^0 and $n \in \mathbb{N}$. Then there exist $m \geq n$ and an open set U contained in $(r^m)^{-1}(V)$ such that*

$$\mu, \nu \in U \Longrightarrow \mu_k \neq \nu_m \text{ for } 1 \leq k < m.$$

PROOF. By Proposition 9.8, there exists $m \geq n$ such that there is a non-returning path $\mu \in (r^m)^{-1}(V)$. We then have $\mu_k \neq \mu_m$ for $1 \leq k < m$. Choose open neighbourhoods U_k of μ_k such that $U_k \cap U_m = \emptyset$ for $1 \leq k < m$, and then the open neighbourhood $U = (U_1 \times \cdots \times U_m) \cap (r^m)^{-1}(V)$ of μ has the required property. □

The key step in verifying the estimate (9.5) is the following technical result, which is a special case of [**73**, Proposition 5.10] and [**92**, Proposition 6.14].

PROPOSITION 9.11. *Suppose that E is a topological graph which satisfies Condition* (L)*, and that r is proper and surjective. Suppose that (ψ, π) is a Cuntz-Pimsner covariant representation of the graph correspondence $X(E)$ in a C^*-algebra B such that π is injective, and let*

$$c = \sum_{i=1}^{I} k_X^{\otimes m_i}(x_i) k_X^{\otimes n_i}(y_i)^* \quad \textit{for some } x_i \in C_c(E^1)^{\otimes m_i},\ y_i \in C_c(E^1)^{\otimes n_i}. \tag{9.7}$$

Then for each $\epsilon > 0$ there exist $a, b \in B$ with $\|a\| \leq 1$, $\|b\| \leq 1$, and

$$\big|\, \|a^*(\psi \times \pi(c))b\| - \|\Phi(c)\| \,\big| < \epsilon. \tag{9.8}$$

PROOF. Let $n = \max\{m_i, n_i\}$. Then it follows from Lemma 9.3 that

$$\Phi(c) = \sum_{\{i:\, m_i = n_i\}} k_X^{\otimes m_i}(x_i) k_X^{\otimes n_i}(y_i)^*$$

belongs to $\mathcal{F}_n = (k_X^{\otimes n}, k_A)^{(1)}(\mathcal{K}(X^{\otimes n}))$; say $\Phi(x) = (k_X^{\otimes n}, k_A)^{(1)}(T)$. We know from the proof of Proposition 9.6 that $(k_X^{\otimes n}, k_A)^{(1)}$ is faithful on $\mathcal{F}_n$, so we have $\|\Phi(c)\| = \|T\|$, and we can find unit vectors $\xi, \eta \in C_c(E^1)^{\otimes n} \subset X^{\otimes n}$ such that

$$\|\Phi(c)\| \geq \|\langle \xi, T\eta \rangle\| > \|\Phi(x)\| - \epsilon.$$

We can further assume by multiplying ξ by a function in $C_c(E^0)$ that $\langle \xi, T\eta \rangle \geq 0$.

The elements a and b of the proposition will have the form $a = \psi^{\otimes n}(\xi)\psi^{\otimes m}(\zeta)$ and $b = \psi^{\otimes n}(\eta)\psi^{\otimes m}(\zeta)$, so we next compute the effect of multiplying $\psi \times \pi(c)$ by $\psi^{\otimes n}(\xi)^*$ on the left and $\psi^{\otimes n}(\eta)$ on the right.

For the first calculation, we use Proposition 8.11 freely:

$$\begin{aligned}
\psi^{\otimes n}(\xi)^* \psi \times \pi(\Phi(c)) \psi^{\otimes n}(\eta) &= \psi^{\otimes n}(\xi)^* \psi \times \pi((k_X^{\otimes n}, k_A)^{(1)}(T)) \psi^{\otimes n}(\eta) \\
&= \psi^{\otimes n}(\xi)^* (\psi^{\otimes n}, \pi)^{(1)}(T) \psi^{\otimes n}(\eta) \\
&= \psi^{\otimes n}(\xi)^* \psi^{\otimes n}(T\eta) \\
&= \pi(\langle \xi, T\eta \rangle).
\end{aligned}$$

The summands in $c - \Phi(c)$ all have either $n_i > m_i$ or $n_i < m_i$. Suppose that $n_i > m_i$, and that ξ and η are elementary tensors

$$\begin{gathered}
\xi = \xi' \otimes \xi'' \otimes \xi''' \in C_c(E^1)^{\otimes m_i} \otimes C_c(E^1)^{\otimes (n - n_i)} \otimes C_c(E^1)^{\otimes (n_i - m_i)}, \\
\eta = \eta' \otimes \eta'' \in C_c(E^1)^{\otimes n_i} \otimes C_c(E^1)^{\otimes (n - n_i)}.
\end{gathered}$$

Then

$$\begin{aligned}
&\psi^{\otimes n}(\xi)^* \psi^{\otimes m_i}(x_i) \psi^{\otimes n_i}(y_i)^* \psi^{\otimes n}(\eta) \\
&\qquad = \psi^{\otimes (n - m_i)}(\xi'' \otimes \xi''')^* \pi\big(\langle \xi', x_i \rangle \langle y_i, \eta' \rangle\big) \psi^{\otimes (n - n_i)}(\eta''),
\end{aligned}$$

which has the form

$$\begin{aligned}
&\psi^{\otimes (n_i - m_i)}(\xi''')^* \psi^{\otimes (n - n_i)}(\xi'')^* \pi(d) \psi^{\otimes (n - n_i)}(\eta'') \\
&\qquad = \psi^{\otimes (n_i - m_i)}(\xi''')^* \psi^{\otimes (n - n_i)}(\xi'')^* \psi^{\otimes (n - n_i)}(d \cdot \eta'') \\
&\qquad = \psi^{\otimes (n_i - m_i)}(\xi''')^* \pi(\langle \xi'', d \cdot \eta'' \rangle) \\
&\qquad = \psi^{\otimes (n_i - m_i)}(\langle d \cdot \eta'', \xi'' \rangle \cdot \xi''')^*.
\end{aligned}$$

When ξ and η are sums of elementary tensors, we get a sum of such terms, which we can write in the form $\psi^{\otimes(n_i-m_i)}(\xi_i)$. We can perform a similar computation when $n_i > m_i$. Thus we may as well assume by reindexing that

$$\psi^{\otimes n}(\xi)^*\psi \times \pi(\Phi(c))\psi^{\otimes n}(\eta) = \pi(\langle \xi, T\eta\rangle) + \sum_i \psi^{\otimes m_i}(\xi_i)^* + \sum_j \psi^{\otimes n_j}(\eta_j), \tag{9.9}$$

where $n \geq m_i > 0$ and $n \geq n_j > 0$.

The purpose of the second factor $\psi^{\otimes m}(\zeta)$ in the definitions of a and b will be to kill the two sums in (9.9) without changing the norm of $\pi(\langle \xi, T\eta\rangle)$ very much. We now fix $\epsilon > 0$ and choose an open set V in E^0 such that

$$\langle \xi, T\eta\rangle(w) > \|\Phi(c)\| - \epsilon \quad \text{for all } w \in V. \tag{9.10}$$

By Corollary 9.10, there exist $m > n$ and an open set $U \subset (r^m)^{-1}(V)$ such that

$$\mu, \nu \in U \Longrightarrow \mu_k \neq \nu_m \text{ for } 1 \leq k < m. \tag{9.11}$$

By shrinking U, we may assume that s^m is injective on U, and that U is the intersection with $E^m \subset (E^1)^m$ of a product set $\prod_i U_i$. Choose $\mu \in U$, $\zeta_i \in C_c(E^1)$ such that $0 \leq \zeta_i \leq 1$, $\operatorname{supp} \zeta_i \subset U_i$, and $\zeta_i(\mu_i) = 1$, and take

$$\zeta = \zeta_1 \otimes \cdots \otimes \zeta_m \in C_c(E^1)^{\otimes m} \subset C_c(E^m).$$

Notice that because s^m is injective on $\operatorname{supp}\zeta$, we have

$$\|\zeta\|^2 = \sup_{v \in E^0} |\langle \zeta, \zeta\rangle(v)| = \sup_{\alpha \in \operatorname{supp}\zeta} |\zeta(\alpha)|^2 = |\zeta(\mu)|^2 = 1.$$

We next verify that conjugating by $\psi^{\otimes m}(\zeta)^*$ kills the sums in (9.9). We choose a summand $\psi^{\otimes m_i}(\xi_i)^*$. Since ζ is an elementary tensor, we can factor it as $\zeta' \otimes \zeta''$ in $C_c(E^1)^{\otimes(m-m_i)} \otimes C_c(E^1)^{\otimes m_i}$. Then

$$\begin{aligned}\psi^{\otimes m}(\zeta)^*\psi^{\otimes m_i}(\xi_i)^*\psi^{\otimes m}(\zeta) &= \psi^{\otimes m_i}(\zeta'')^*\psi^{\otimes m}(\xi_i \otimes \zeta')^*\psi^{\otimes m}(\zeta)\\ &= \psi^{\otimes m_i}(\langle \zeta, \xi_i \otimes \zeta'\rangle \cdot \zeta'')^*.\end{aligned}$$

To simplify notation, we write $\beta = \beta'\beta''$ for the factorisation of $\beta \in E^m$ in which $|\beta'| = m_i$ and $|\beta''| = m - m_i$. Then for $\gamma \in E^{m_i}$ we have

$$\begin{aligned}\big(\langle \zeta, \xi_i \otimes \zeta'\rangle \cdot \zeta''\big)(\gamma) &= \langle \zeta, \xi_i \otimes \zeta'\rangle(r(\gamma))\zeta''(\gamma)\\ &= \sum_{s^m(\beta)=r(\gamma)} \overline{\zeta(\beta)}\xi_i(\beta')\zeta'(\beta'')\zeta''(\gamma)\\ &= \sum_{s^m(\beta)=r(\gamma)} \overline{\zeta(\beta)}\xi_i(\beta')\zeta(\beta''\gamma),\end{aligned} \tag{9.12}$$

which is zero unless $\beta \in U$ and $\beta''\gamma \in U$. Since $(\beta''\gamma)_{m-m_i} = \beta''_{m-m_i} = \beta_m$, (9.11) says there are no such β, and (9.12) vanishes. Thus conjugating by $\psi^{\otimes m}(\zeta)^*$ kills the first sum in (9.9); since the adjoint of the second sum looks like the first, conjugating by $\psi^{\otimes m}(\zeta)^*$ also kills the second sum.

As promised, we take $a = \psi^{\otimes n}(\xi)\psi^{\otimes m}(\zeta)$ and $b = \psi^{\otimes n}(\eta)\psi^{\otimes m}(\zeta)$. Then

$$\begin{aligned}\|a\|^2 &= \|\psi^{\otimes(n+m)}(\xi \otimes \zeta)\|^2 = \|\psi^{\otimes(n+m)}(\xi \otimes \zeta)^*\psi^{\otimes(n+m)}(\xi \otimes \zeta)\|\\ &= \|\pi(\langle \xi \otimes \zeta, \xi \otimes \zeta\rangle)\| = \|\langle\langle \xi, \xi\rangle \cdot \zeta, \zeta\rangle\|\\ &\leq \|\xi\|^2\|\zeta\|^2 = 1.\end{aligned}$$

We similarly have $\|b\| \leq 1$. To see that a and b satisfy (9.8), we note that the calculation in the previous paragraph gives

$$a^*(\psi \times \pi(c))b = \psi^{\otimes m}(\zeta)^*\pi(\langle \xi, T\eta\rangle)\psi^{\otimes m}(\zeta) = \pi(\langle \zeta, \langle \xi, T\eta\rangle \cdot \zeta\rangle).$$

Thus, since π is isometric,

$$\|a^*(\psi \times \pi(x))b\| = \|\langle \zeta, \langle \xi, T\eta\rangle \cdot \zeta\rangle\| = \sup_{v\in E^0} \sum_{s^m(\beta)=v} \overline{\zeta(\beta)}\langle \xi, T\eta\rangle(r(\beta))\zeta(\beta).$$

Since s^m is injective on $\operatorname{supp}\zeta \subset U$, for each v there is at most one $\beta = \beta(v)$ giving a non-zero summand; this β must belong to U, and then $U \subset (r^m)^{-1}(V)$ forces $r(\beta(v)) \in V$. Thus (9.10) implies that

$$\begin{aligned}\|a^*(\psi \times \pi(x))b\| &= \sup_{v\in E^0} |\zeta(\beta(v))|^2 \langle \xi, T\eta\rangle(r(\beta(v))) \\ &> \sup_{v\in E^0} |\zeta(\beta(v))|^2 (\|\Phi(c)\| - \epsilon) \\ &= \|\Phi(c)\| - \epsilon.\end{aligned}$$

Since we trivially have $\|\langle \zeta, \langle \xi, T\eta\rangle \cdot \zeta\rangle\| \leq \|T\| = \|\Phi(c)\|$, this proves (9.8). □

Proof of Theorem 9.2. Suppose that (ψ, π) is a Cuntz-Pimsner covariant representation of $X(E)$ in a C^*-algebra B. Fix $d \in \mathcal{O}_{X(E)}$ and $\epsilon > 0$, and choose c of the form (9.7) such that $\|d-c\| < \epsilon/3$. By Proposition 9.11, we can find elements a and b in B with norm at most 1 such that $\big|\, \|a^*(\psi \times_{\mathcal{O}} \pi(c))b\| - \|\Phi(c)\|\, \big| < \epsilon/3$. Since Φ is norm decreasing, an $\epsilon/3$ argument shows that

$$\big|\, \|a^*(\psi \times_{\mathcal{O}} \pi(d))b\| - \|\Phi(d)\|\, \big| < \epsilon,$$

and hence

$$\begin{aligned}\|\Phi(d)\| &< \|a^*(\psi \times_{\mathcal{O}} \pi(d))b\| + \epsilon \\ &\leq \|a\|\,\|b\|\,\|\psi \times_{\mathcal{O}} \pi(d)\| + \epsilon \\ &\leq \|\psi \times_{\mathcal{O}} \pi(d)\| + \epsilon.\end{aligned}$$

Since this is true for every $\epsilon > 0$, we have now proved the estimate (9.5).

Now suppose that $\psi \times \pi(d) = 0$. Then $\psi \times_{\mathcal{O}} \pi(d^*d) = 0$, the estimate (9.5) implies that $\Phi(d^*d) = 0$, Proposition 3.2 implies that $d^*d = 0$, and hence $d = 0$. □

Remark 9.12. Topological graphs have been studied in different contexts for many years. The idea originates in a measure-theoretic context in work of Vershik and Arzumanian around 1980 [**146, 4**], and was studied in the topological context in the 1990s by Deaconu [**19, 20**], Arzumanian-Renault [**3, 116**] and Schweizer [**126**]. The introduction to [**92**] contains a review of the history.

The main theorem of this chapter is part of Katsura's extensive investigation of the C^*-algebras of topological graphs [**73, 76, 77**]. In his most recent paper [**77**], for example, he gives criteria for the simplicity of the C^*-algebras of topological graphs, and a thorough analysis of their gauge-invariant ideals. Generalisations of topological graphs involving branched coverings have been investigated in [**21**] and [**22**], and a further generalisation to topological quivers is discussed in [**90**] and [**92**]. In a topological quiver the map r is not necessarily a local homeomorphism, so its fibres are not discrete, and the construction of the C^*-algebra depends on a choice of measures on the fibres of r.

CHAPTER 10

Higher-rank graphs

Higher-rank graphs are, as the name suggests, higher-dimensional analogues of directed graphs. They were introduced by Kumjian and Pask in [**81**], and have recently been attracting a good deal of attention. Here we will discuss the elementary properties of these graphs and their C^*-algebras, focusing on the more tractable row-finite higher-rank graphs, and then we will survey what is known about the C^*-algebras of these and other classes of higher-rank graphs.

Higher-rank graphs are defined using the language of category theory. For our purposes, a *category*[1] $\mathcal{C}$ consists of two sets $\mathcal{C}^0$ and $\mathcal{C}^*$, two functions $r, s : \mathcal{C}^* \to \mathcal{C}^0$, a partially defined product $(f, g) \mapsto fg$ from

$$\{(f, g) \in \mathcal{C}^* \times \mathcal{C}^* : s(f) = r(g)\}$$

to $\mathcal{C}^*$, and distinguished elements $\{\iota_v \in \mathcal{C}^* : v \in \mathcal{C}^0\}$, satisfying

- $r(fg) = r(f)$ and $s(fg) = s(g)$;
- $(fg)h = f(gh)$ when $s(f) = r(g)$ and $s(g) = r(h)$;
- $r(\iota_v) = v = s(\iota_v)$ and $\iota_v f = f$, $g\iota_v = g$ when $r(f) = v$ and $s(g) = v$.

The elements of $\mathcal{C}^0$ are called the *objects* of the category, the elements of $\mathcal{C}^*$ are called the *morphisms*, $s(f)$ is the *domain of* f, $r(f)$ is the *codomain of* f, the operation $(f, g) \mapsto fg$ is called *composition*, and ι_v is called the *identity morphism on the object* v. If $\mathcal{C}$ and $\mathcal{D}$ are categories, a *functor* $F : \mathcal{C} \to \mathcal{D}$ is a pair of maps $F^0 : \mathcal{C}^0 \to \mathcal{D}^0$ and $F^* : \mathcal{C}^* \to \mathcal{D}^*$ which respect the domain and codomain maps and composition, and which satisfy $F^*(\iota_v) = \iota_{F^0(v)}$.

In the original applications of category theory, the objects would share some kind of common mathematical structure, such as that of a group, and the morphisms would be an appropriate family of structure-preserving functions between objects, such as group homomorphisms. For our purposes, the next example is more instructive.

Example 10.1. Let $E = (E^0, E^1, r, s)$ be a directed graph. In the *path category* $\mathcal{P}(E)$ of E, $\mathcal{P}(E)^0$ is the set E^0 of vertices, $\mathcal{P}(E)^*$ is the set E^* of finite paths μ in E, $\mu \in E^*$ has domain $s(\mu)$ and codomain $r(\mu)$, the composition of μ and ν is the product $\mu\nu = \mu_1 \cdots \mu_{|\mu|}\nu_1 \cdots \nu_{|\nu|}$ we have been using all along, and the identity morphism on $v \in E^0$ is the path v of length 0.

To define higher-rank graphs, we first have to agree to view $\mathbb{N}^k := \{0, 1, 2 \dots\}^k$ as the morphisms in a category with one object and composition given by addition; we denote this category by $\mathbb{N}^k$. As a point of notation, we write $\{e_i : 1 \le i \le k\}$

[1]Strictly speaking, we are defining a *small* category in which $\mathcal{C}^0$ and $\mathcal{C}^*$ are assumed to be sets. In all our examples they will be countable sets.

for the usual generators of $\mathbb{N}^k$ and n_i for the ith entry in $n \in \mathbb{N}^k$. A *graph of rank* k is

- a countable category Λ, together with
- a functor $d : \Lambda \to \mathbb{N}^k$, called the *degree map*,

with the following *factorisation property*: for every morphism λ and every decomposition $d(\lambda) = m + n$ with $m, n \in \mathbb{N}^k$, there exist unique morphisms μ and ν such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$. We also call (Λ, d) a *k-graph* or just a *higher-rank graph*, and we usually abbreviate (Λ, d) to Λ.

Example 10.2. With $d : E^* \to \mathbb{N}$ defined by $d(\mu) = |\mu|$, the path category $\mathcal{P}(E)$ of a directed graph becomes a 1-graph. Indeed, we can view any 1-graph Λ as the path category of the directed graph $(\Lambda^0, d^{-1}(1), r, s)$.

In view of Example 10.2, we define Λ^n to be the set $d^{-1}(n)$ of morphisms of degree n, and refer to the elements of Λ^n as *paths of degree* n. This is consistent notation when $n = 0$ provided we are willing to identify the objects $v \in \Lambda^0$ with the identity morphism ι_v on v, which we do from now on.

Example 10.3. We can define a k graph Ω_k by setting $\Omega_k^0 = \mathbb{N}^k$,

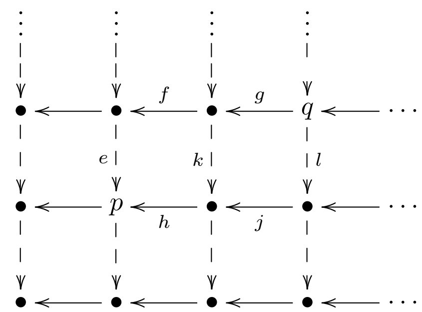
$$\Omega_k^* = \{(p, q) : p, q \in \mathbb{N}^k \text{ and } p \le q \text{ in the sense that } p_i \le q_i \text{ for all } i\},$$

$r(p, q) = p$, $s(p, q) = q$ and $d(p, q) = q - p$, and defining composition by $(p, q)(q, r) = (p, r)$.

To visualise a k-graph, we draw its 1-*skeleton*, which is the coloured directed graph $(\Lambda^0, \bigcup_{i=1}^k \Lambda^{e_i}, r, s)$ with the edges in each Λ^{e_i} drawn in a different colour. In the pictures in these notes, the dashed edges have degree $e_2 = (0, 1)$ and are described as "red".

Example 10.4. The 1-skeleton of the 2-graph Ω_2 looks like

(10.1)

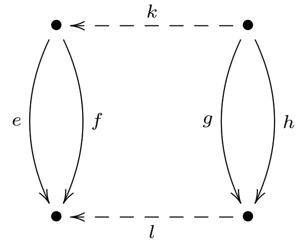


where the edges in $\Omega_2^{e_2}$ are red and those in $\Omega_2^{e_1}$ are black. In the picture, the path (p, q) has degree $(2, 1)$, so it has three factorisations $(p, q) = efg = hkg = hjl$. We think of the path (p, q) as the rectangle with vertices at p and q. Composition of morphisms then involves taking the convex hull of the corresponding rectangles. As a result, an edge in the 1-skeleton can arise in a factorisation of $\lambda\mu$ even if it doesn't arise in any factorisation of λ or μ.

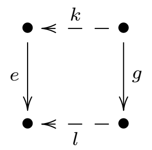
The path (p, q) is a prototypical example of a path of degree $(2, 1)$: in other k-graphs, we think of a path of degree $(2, 1)$ as a copy of this rectangle wrapped around the 1-skeleton in a colour-preserving way.

The 1-skeleton alone need not determine the k-graph.

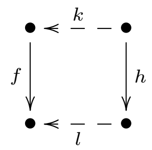
EXAMPLE 10.5. If the segment



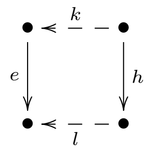
occurs in the 1-skeleton of a 2-graph, we need to know how the black-red paths ek and fk factor as red-black paths. There are two choices: the paths of degree $(1, 1)$ could be



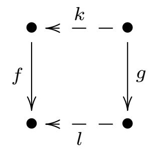
and



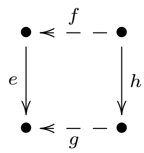
or



and



To make a 2-coloured graph into a 2-graph, it suffices to find a collection of squares



in which each red-black path gh and each black-red path ef occur exactly once. For a k-coloured graph to determine a k-graph, we need a collection of squares in which every bi-coloured path with two edges occurs exactly once, and which are associative in the sense that converting a red-green-black path into a black-green-red path via the routes $RGB \to RBG \to BRG \to BGR$ and $RGB \to GRB \to GBR \to BGR$ always gives the same answer. See Theorem 2.1 and Remark 2.3 in [**49**].

A k-graph Λ is *row-finite* if $r^{-1}(v) \cap \Lambda^n$ is finite for every $v \in \Lambda^0$ and every $n \in \mathbb{N}^k$; equivalently, Λ is row-finite if its 1-skeleton is row-finite. We say that Λ *has no sources* if for every $v \in \Lambda^0$ and every $n \in \mathbb{N}^k$, there is a path λ with $r(\lambda) = v$ and $d(\lambda) = n$; equivalently, Λ has no sources if every vertex in the 1-skeleton receives edges of every colour.

Suppose that Λ is row-finite and has no sources. Then a *Cuntz-Krieger* Λ*-family* $S = \{S_\lambda : \lambda \in \Lambda^*\}$ is a collection of partial isometries satisfying:

- $\{S_v : v \in \Lambda^0\}$ are mutually orthogonal projections;
- $S_\lambda S_\mu = S_{\lambda\mu}$ when $s(\lambda) = r(\mu)$;
- $S_\lambda^* S_\lambda = S_{s(\lambda)}$ for every $\lambda \in \Lambda^*$;

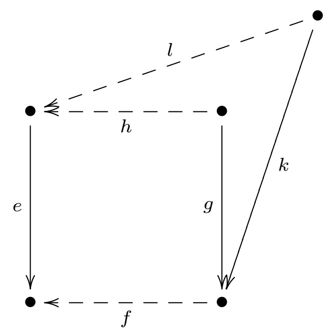
- $S_v = \sum_{\{\lambda\in\Lambda^n:\, r(\lambda)=v\}} S_\lambda S_\lambda^*$ for every $v \in \Lambda^0$ and every $n \in \mathbb{N}^k$.

Let S be a Cuntz-Krieger Λ-family. The first and last axioms imply that the partial isometries asociated to paths of the same degree have orthogonal ranges, so that $\{S_\lambda S_\lambda^* : d(\lambda) = n\}$ is a mutually orthogonal family of projections for each $n \in \mathbb{N}^k$. We also have the following reassuring lemma. As a point of notation, for $m, n \in \mathbb{N}^k$ we denote by $m \vee n$ the k-tuple whose ith entry is $\max\{m_i, n_i\}$.

LEMMA 10.6. *Let Λ be a row-finite k-graph with no sources, and let $S = \{S_\lambda\}$ be a Cuntz-Krieger Λ-family. Then for every $\lambda, \mu \in \Lambda^*$ and $q \geq d(\lambda) \vee d(\mu)$, we have*

$$S_\lambda^* S_\mu = \sum_{\{\alpha\in\Lambda^{q-d(\lambda)},\, \beta\in\Lambda^{q-d(\mu)}:\, \lambda\alpha=\mu\beta\}} S_\alpha S_\beta^*. \tag{10.2}$$

EXAMPLE 10.7. Consider a 2-graph whose 1-skeleton contains



and take $\lambda = f$, $\mu = e$. Then $d(\lambda) \vee d(\mu) = (0,1) \vee (1,0) = (1,1)$, and there are two pairs (α, β) of minimal degree such that $\lambda\alpha = \mu\beta$, namely $(\alpha,\beta) = (h,g)$ and $(\alpha,\beta) = (l,k)$. Thus for $q = (1,1)$, Equation (10.2) is $S_f^* S_e = S_g S_h^* + S_k S_l^*$.

PROOF OF LEMMA 10.6. From the Cuntz-Krieger relations at $s(\lambda)$ with degree $q - d(\lambda)$ and at $s(\mu)$ with degree $q - d(\mu)$, we have

$$\begin{aligned} S_\lambda^* S_\mu &= S_{s(\lambda)} S_\lambda^* S_\mu S_{s(\mu)} \\ &= \Big(\sum_{\{\alpha\in\Lambda^{q-d(\lambda)}:\ r(\alpha)=s(\lambda)\}} S_\alpha S_\alpha^* S_\lambda^*\Big)\Big(\sum_{\{\beta\in\Lambda^{q-d(\mu)}:\ r(\beta)=s(\mu)\}} S_\mu S_\beta S_\beta^*\Big). \qquad (10.3)\end{aligned}$$

Since each $d(\lambda\alpha) = d(\mu\beta) = q$ and the projections $\{S_\gamma S_\gamma^* : d(\gamma) = q\}$ are mutually orthogonal, we have

$$S_\alpha S_\alpha^* S_\lambda^* S_\mu S_\beta S_\beta^* = S_\alpha S_{\lambda\alpha}^* S_{\mu\beta} S_\beta^* = \begin{cases} 0 & \text{unless } \lambda\alpha = \mu\beta \\ S_\alpha S_{s(\lambda\alpha)} S_\beta^* = S_\alpha S_\beta^* & \text{if } \lambda\alpha = \mu\beta. \end{cases}$$

So the double sum which we get by multiplying out (10.3) collapses to the right-hand side of (10.2). □

COROLLARY 10.8. *Let Λ be a row-finite k-graph with no sources, and let $S = \{S_\lambda\}$ be a Cuntz-Krieger Λ-family. Then*

$$C^*(S) := C^*(\{S_\lambda\}) = \overline{\operatorname{span}}\{S_\lambda S_\mu^* : \lambda, \mu \in \Lambda^*\}.$$

Using this corollary, we can now follow the construction of the graph algebra in Proposition 1.21. We obtain:

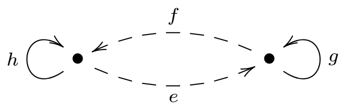
Proposition 10.9. *Suppose Λ is a row-finite k-graph with no sources. Then there is a C^*-algebra $C^*(\Lambda)$ generated by a Cuntz-Krieger E-family $\{s_\lambda\}$ such that for every Cuntz-Krieger Λ-family $t = \{t_\lambda\}$ in a C^*-algebra B, there is a homomorphism $\pi_t : C^*(\Lambda) \to B$ satisfying $\pi_t(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda^*$. The C^*-algebra $C^*(\Lambda)$ is called the* C^*-algebra of Λ.

For $k > 1$, it is not obvious that there exist non-trivial Cuntz-Krieger families, so it is not immediately clear that the generating elements s_λ of $C^*(\Lambda)$ are non-zero. We shall show that there are non-trivial families using an analogue of the infinite path space.

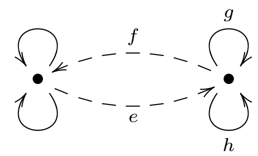
Let $v \in \Lambda^0$. An *infinite path with range* v is a graph morphism x (that is, a degree-preserving functor) from Ω_k to Λ such that $x(0) = v$. When $k = 2$, for example, we think of x as pasting the plane (10.1) round Λ in a colour-preserving way. An infinite path x is determined by the final segments $x(0, q)$ for large q and the factorisation property; we can define infinite paths inductively by choosing a cofinal sequence in $\mathbb{N}^k$ (see [**81**, Remarks 2.2]). We write Λ^∞ for the set of infinite paths in Λ.

Example 10.10. In the graph Ω_k, every finite path is uniquely determined by its degree and its range. So for each $n \in \mathbb{N}^k = \Omega_k^0$, there is exactly one infinite path with range n, and $x \mapsto r(x)$ is a bijection of Ω_k^∞ onto $\mathbb{N}^k$.

Example 10.11. There is a unique 2-graph with 1-skeleton



and in this 2-graph infinite paths are again uniquely determined by their ranges. There are two 2-graphs with 1-skeleton



determined by the factorisations of fg and he. In either 2-graph Λ, every infinite word μ in g and h determines two infinite paths x_μ and y_μ such that

$$x_\mu((0,0),(m,2n)) = (ef)^n \mu_1 \cdots \mu_m \quad \text{and}$$
$$y_\mu((0,0),(m,2n)) = f(ef)^{n-1}\mu_1 \cdots \mu_m e,$$

and a bit of thought shows that every infinite path has the form x_μ or y_μ for some μ.

Example 10.12 (The infinite-path representation). For $\lambda \in \Lambda^*$ and $x \in \Lambda^\infty$ satisfying $s(\lambda) = r(x)$, there is another infinite path λx such that

$$(\lambda x)(0,q) = \lambda x(0, q - d(\lambda)) \quad \text{for } q \geq d(\lambda).$$

If we denote by $\{e_x : x \in \Lambda^\infty\}$ the usual basis for $\ell^2(\Lambda^\infty)$, then we can define a Cuntz-Krieger Λ-family on $\ell^2(\Lambda^\infty)$ by

$$S_\lambda e_x = \begin{cases} e_{\lambda x} & \text{if } s(\lambda) = r(x) \\ 0 & \text{otherwise.} \end{cases}$$

Provided Λ has no sources, there are infinite paths ending at every vertex, and the S_λ form a Cuntz-Krieger family in which every S_λ is non-zero.

COROLLARY 10.13. *Let Λ be a row-finite k-graph with no sources. Then every partial isometry in the universal Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda^*\} \subset C^*(\Lambda)$ is non-zero.*

In the remainder of this chapter we shall survey what is known about the C^*-algebras of higher-rank graphs.

General theory. Suppose that Λ is a row-finite k-graph with no sources. The universal property implies that the pair $(C^*(\Lambda), \{s_\lambda\})$ is unique up to isomorphism, and hence there is a *gauge action*γ of the k-torus $\mathbb{T}^k$ characterised by $\gamma_z(s_\lambda) = z^{d(\lambda)}s_\lambda$ (where we are using multiindex notation: if $z = (z_1, \dots, z_k) \in \mathbb{T}^k$ and $n = (n_1, \dots, n_k) \in \mathbb{N}^k$, then $z^n := \prod_i z_i^{n_i}$). There is a very satisfactory gauge-invariant uniqueness theorem which says that if $S = \{S_\lambda\}$ is a Cuntz-Krieger Λ-family with $S_v \neq 0$ for every $v \in \Lambda^0$, and if $C^*(S)$ carries an action β of $\mathbb{T}^k$ such that $\beta_z(S_\lambda) = z^{d(\lambda)}S_\lambda$ for every λ, then the corresponding homomorphism $\pi_S : C^*(\Lambda) \to B$ is an injection [**81**, Theorem 3.4]. This uniqueness theorem is used in [**81**] to identify $C^*(\Lambda)$ as the C^*-algebra of a locally compact groupoid with unit space Λ^∞.

A major outstanding problem in the area is to find a satisfactory version of the Cuntz-Krieger uniqueness theorem for k-graphs. The first theorem of this type was proved in [**81**, Theorem 4.6] under an *aperiodicity Condition* (A) on Λ, which is equivalent to asking that the groupoid model used in [**81**] is essentially free. Direct methods like those we used in Chapter 3 yield a Cuntz-Krieger uniqueness theorem for k-graphs satisfying a slightly different hypothesis (B) [**111**, Theorem 4.3]; Condition (A) certainly implies Condition (B), but we do not know whether they are equivalent. That we can't decide whether these two hypotheses are equivalent illustrates the problem: neither is easy to visualise. What is needed is an easily verified condition on Λ which ensures that, if F is a finite collection of paths with common source v and common degree n, and G is a finite collection of paths with source v and degree larger than n, then there is a path λ with range v such that, for every $\alpha \in G$, $\alpha\lambda$ does not factor as $\beta\lambda\mu$ for some $\beta \in F$. For $k = 1$, the condition "every cycle has an entry" is such a condition. It would be very helpful to have a similarly verifiable condition for $k > 1$.

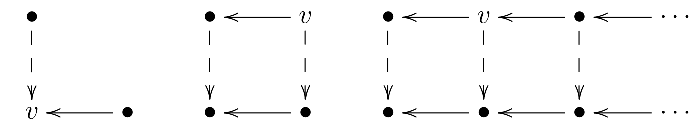
The gauge-invariant ideals in $C^*(\Lambda)$ have been classified using the methods of Chapter 4 [**111**, Theorem 5.2]. Provided Λ has a property which ensures that a Cuntz-Krieger uniqueness theorem applies to every quotient, the same analysis gives a description of all the ideals.

Higher-rank Cuntz-Krieger algebras. Just as in the rank-one case (see Remark 2.8), the first k-graph algebras which appeared were associated to $\{0, 1\}$-matrices rather than graphs. Spielberg had shown that if Γ is a free group and Ω is the boundary of its Cayley tree, then the crossed product $C(\Omega) \rtimes \Gamma$ is a Cuntz-Krieger algebra [**131**]. Robertson and Steger were studying analogous actions on the boundaries of triangle buildings, which are rank-2 analogues of trees, and noticed that the crossed products by these actions were generated by interacting pairs of Cuntz-Krieger families [**120**]. They were thus led to introduce higher-rank Cuntz-Krieger algebras $\mathcal{A}_M$ associated to certain families $M = (M_1, \cdots, M_k)$ of commuting $\{0, 1\}$-matrices [**121**].

Higher-rank graphs and their C^*-algebras were introduced by Kumjian and Pask as generalisations of the higher-rank Cuntz-Krieger algebras. They showed how every family $M = (M_1, \cdots, M_k)$ considered by Robertson and Steger gives a k-graph W_M [**81**, Example 1.7], and deduced from the gauge-invariant uniqueness theorem that $C^*(W_M)$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{A}_M$ [**81**, Corollary 3.5]. More recently, Allen, Pask and Sims have described a dual-graph construction for row-finite k-graphs, and identified the (necessarily finite) 2-graphs for which the dual has the form W_M [**2**, Proposition 4.3].

Sources. Substantial difficulties arise when we try to adapt the theory to accomodate k-graphs with sources. One problem is that there are many different kinds of sources. For example, the vertices v in the following 2-graphs are all sources in the sense that they do not receive paths of arbitrary degrees:

(10.4)



The two graphs on the right of (10.4) are examples of the locally convex k-graphs studied in [**111**]. A row-finite graph Λ is *locally convex* if the existence of $\lambda \in \Lambda^{e_i}$ and $\mu \in \Lambda^{e_j}$ with $i \neq j$ and $r(\lambda) = r(\mu)$ implies the existence of $\nu \in \Lambda^{e_j}$ with $r(\nu) = s(\lambda)$; this is automatic if Λ has no sources. For these graphs, we can adjust the Cuntz-Krieger relation at a vertex v to say

$$S_v = \textstyle\sum_{\{\lambda \in \Lambda^{\leq n} :\, r(\lambda) = v\}} S_\lambda S_\lambda^* \quad \text{for every } n \in \mathbb{N}^k,$$

where $\Lambda^{\leq n}$ consists of the paths of degree n and the paths λ with $d(\lambda) \leq n$ which cannot be non-trivially extended to paths $\lambda\mu$ with $d(\lambda\mu) \leq n$. The uniqueness theorems and the classification of gauge-invariant ideals are proved for row-finite locally convex k-graphs in [**111**].

The need to accommodate all the different kinds of sources has made it hard to formulate a procedure like adding heads which will reduce problems about k-graphs with sources to k-graphs with no sources. The problem is that every time we add a red edge e with $r(e) = v$, say, the factorisation property may require that we also add a black edge f with $s(f) = s(e)$ to ensure that every black-red path ge has a red-black factorisation; there are then similar considerations at $r(e)$, and the process can get complicated. As I finish these notes, however, Farthing seems to have found a solution to this problem for row-finite graphs [**42**].

Infinite higher-rank graphs. For graphs which are not row-finite, there are new problems. To get algebras which behave like Cuntz-Krieger algebras, we need a formula like (10.2) in which the sum on the right-hand side is finite when $q = d(\lambda) \vee d(\mu)$. (This was not a problem in a 1-graph because the analogue of (10.2) has a single term on the right-hand side when $q = \max(d(\mu), d(\lambda))$.) We say that a k-graph is *finitely aligned* if for each pair $\lambda, \mu \in \Lambda^*$ with $r(\lambda) = r(\mu)$, there are finitely many paths ν with $d(\nu) = d(\lambda) \vee d(\mu)$ which factor as $\lambda\alpha$ and as $\mu\beta$. These finitely aligned k-graphs were introduced and studied in [**110**]. In [**112**], a new family of Cuntz-Krieger relations was proposed for finitely aligned graphs, and the corresponding family of k-graph algebras analysed. In [**112**, Appendix A], there is an extensive discussion of these new relations and why they seem to be the right ones. In [**112**, Appendix B], it is shown that the new relations reduce to the usual ones when Λ is row-finite and locally convex, and for arbitrary 1-graphs. In [**102**], it is shown that the finitely aligned condition is exactly what is needed to construct a locally compact path space.

Sims has since conasidered a class of relative k-graph algebras, in which some of the Cuntz-Krieger relations are not imposed [**128**]. His motivation is that, when one tries to realise a quotient $C^*(\Lambda)/I$ as a graph algebra, a vertex which is an infinite receiver in Λ can become a finite receiver in the natural candidate for the quotient graph, but the Cuntz-Krieger relations at that vertex will not be satisfied in $C^*(\Lambda)/I$. For 1-graphs, we can either construct a graph whose C^*-algebra is $C^*(\Lambda)/I$ by adding sources to the natural candidate [**7**, Corollary 3.5], or view $C^*(\Lambda)/I$ as a relative graph algebra of the natural candidate [**91**]. For k-graphs, it is hard to add sources, for the same reasons it is hard to add heads, so to prove uniqueness theorems which apply to the quotients of k-graph algebras, Sims was forced to extend the analysis of [**112**] to relative graph algebras [**128**]. He then used his analysis to list the gauge-invariant ideals in the C^*-algebra of a finitely aligned k-graph [**129**].

Alternative approaches. One feature of the theory of graph algebras is that there are many ways to approach the subject: one can view the C^*-algebra of a 1-graph as the C^*-algebra of a locally compact groupoid [**83, 101**], as the Cuntz-Pimsner algebra of a correspondence (as in Example 8.13), or as a partial crossed product [**40**]. All these methods have yielded important insight into the structure of graph algebras, and conversely graph algebras provide an important class of examples for each of these general theories. Thus it is worth examining to what extent similar methods are available for higher-rank graphs.

As for 1-graphs, the C^*-algebras of row-finite higher-rank graphs were first studied using a groupoid model [**81**], and a direct analysis like that of Chapters 3 and 4 was subsequently carried out in [**111**]. The analysis of finitely aligned graphs in [**112**] also uses direct methods. However, the need to impose the finitely-aligned hypothesis was discovered in [**110**] by viewing the Toeplitz algebra of a k-graph as the Toeplitz algebra of a product system of correspondences over $\mathbb{N}^k$, as studied by Fowler [**45**], and there should be an analogue of the Cuntz-Pimsner algebra for product systems of correspondences which will for the product systems studied in [**110**] yield the C^*-algebras of higher-rank graphs. A groupoid model for the C^*-algebra of a finitely aligned k-graph has been developed by Farthing, Muhly and Yeend [**43**]. It is an intriguing and possibly difficult open problem to find an analogue of the Drinen-Tomforde desingularisation which can be used to reduce the study of finitely aligned k-graphs to the row-finite case; this is related to the problem of adding heads to sources.

Covering graphs. In Chapter 6 we saw that the theory of covering spaces has an attractive implementation for coverings of 1-graphs, in which the elements of the fundamental group based at a vertex v are the reduced walks in the graph which begin and end at v. An analogous theory for coverings of k-graphs has been worked out in [**95**]. When $k \geq 2$, the set of closed walks at v no longer embeds in the fundamental group [**94**, §7], and this causes a few complications, but otherwise the theory carries over in a very satisfactory way. In particular, there is a version of the Gross-Tucker theorem which realises coverings as relative skew products, and there is an extension of Theorem 6.13 which realises the C^*-algebra of a covering as a crossed product by a homogeneous space [**95**, Corollary 7.2].

***K*-theory.** The first computations of K-theory for the C^*-algebras of higher-rank graphs were carried out in the context of rank-2 Cuntz-Krieger algebras by Robertson and Steger [**122**]. Their results were converted to statments about the K-theory of finite 2-graphs by Allen, Pask and Sims, using their dual-graph construction [**2**, Theorem 4.1]. Independently, (Gwion) Evans showed that the methods used by Robertson and Steger could be applied to more general k-graphs [**39**]. Evans' results include a complete description of $K_*(C^*(\Lambda))$ for row-finite 2-graphs without sources [**39**, Proposition 2], as well as partial results for k-graphs with $k \geq 3$ (for example, [**39**, Proposition 3]).

Examples. In their original paper, Kumjian and Pask provide a variety of examples and constructions of higher-rank graph algebras. They observe that $C^*(\Omega_k)$ is isomorphic to $\mathcal{K}(\mathcal{H})$, and that the k-graph obtained by viewing the identity map on $\mathbb{N}^k$ as a degree functor has C^*-algebra isomorphic to $C(\mathbb{T}^k)$. They show that the direct product $\Lambda \times \Omega$ of a k-graph Λ and an l-graph Ω is naturally a $(k+l)$-graph, and that $C^*(\Lambda \times \Omega)$ is isomorphic to the tensor product $C^*(\Lambda) \otimes C^*(\Omega)$ (see [**81**, Corollary 3.5] and [**112**, Corollary 4.4]); this shows in particular that the tensor product of two 1-graph algebras is a 2-graph algebra. Since Kumjian and Pask also proved that every higher-rank Cuntz-Krieger algebra is a higher-rank graph algebra, the crossed products by boundary actions which motivated Robertson and Steger are also k-graph algebras.

Very recently, Pask, Sims and I have found a class of 2-graphs whose C^*-algebras are simple AT-algebras. This is interesting for several reasons. First, it shows that the dichotomy of [**82**] breaks down for higher-rank graphs. Second, it increases the set of models for simple C^*-algebras provided by graph algebras. And third, it shows that many other very interesting C^*-algebras can be realised as the C^*-algebras of higher-rank graphs. We think we have found 2-graphs, for example, whose C^*-algebras contain as full corners the Bunce-Deddens algebras and the irrational rotation algebras.

APPENDIX A

Background material

A.1. Projections and partial isometries

If M is a closed subspace of a Hilbert space $\mathcal{H}$, the *orthogonal projection* of $\mathcal{H}$ on M is a bounded linear operator $P : \mathcal{H} \to \mathcal{H}$ which is characterised by the property that $Ph \in M$ and $h - Ph$ is orthogonal to M for every $h \in \mathcal{H}$. The orthogonal projections are themselves characterised by the relations $P^2 = P = P^*$: every bounded operator $T \in B(\mathcal{H})$ such that $T^2 = T = T^*$ is the orthogonal projection of $\mathcal{H}$ onto the closed subspace $T\mathcal{H}$. We therefore say that an element p of a C^*-algebra A is a *projection* if $p^2 = p = p^*$; whenever $\pi : A \to B(\mathcal{H})$ is a representation of A on $\mathcal{H}$, $\pi(p)$ is then the orthogonal projection of $\mathcal{H}$ on $\pi(p)\mathcal{H}$.

The next two propositions show how geometric properties of closed subspaces of Hilbert space can be encoded using the $*$-algebraic structure of $B(\mathcal{H})$, and hence can be interpreted in an abstract C^*-algebra.

PROPOSITION A.1. *Suppose that P and Q are orthogonal projections onto closed subspaces of a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(a) $P\mathcal{H} \subset Q\mathcal{H}$;
(b) $QP = P = PQ$;
(c) $Q - P$ *is a projection;*
(d) $P \leq Q$ *(in the sense that $(Ph\,|\,h) \leq (Qh\,|\,h)$ for all $h \in \mathcal{H}$).*

PROOF. (a) $\Longrightarrow$ (b). For $h \in \mathcal{H}$, we have $Ph \in P\mathcal{H} \subset Q\mathcal{H}$, so $Q(Ph) = Ph$. Thus $QP = P$. Taking adjoints gives $PQ = P$.

(b) $\Longrightarrow$ (c). We just need to check that

$$(Q - P)^2 = Q^2 - QP - PQ + P^2 = Q - P - P + P = Q - P$$

and $(Q - P)^* = Q^* - P^* = Q - P$.

(c) $\Longrightarrow$ (d). Every projection is a positive operator, and hence if $Q - P$ is a projection, then $Q = P + (Q - P) \geq P$.

(d) $\Longrightarrow$ (a). Suppose $h \in P\mathcal{H}$, so that $h = Ph$. Then $P \leq Q$ implies

$$\|Qh\|^2 = (Qh\,|\,Qh) = (Qh\,|\,h) \geq (Ph\,|\,h) = \|h\|^2. \tag{A.1}$$

Since $\|h\|^2 = \|Qh\|^2 + \|(1 - Q)h\|^2$, (A.1) implies that $\|(1 - Q)h\| = 0$ and $h = Qh \in Q\mathcal{H}$. □

PROPOSITION A.2. *Suppose that P and Q are orthogonal projections onto closed subspaces of a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(a) $P\mathcal{H} \perp Q\mathcal{H}$;
(b) $QP = 0 = PQ$;
(c) $P + Q$ *is a projection.*

Proof. (a) $\Longrightarrow$ (b). For $h \in \mathcal{H}$, $Qh \in Q\mathcal{H}$ is orthogonal to $P\mathcal{H}$, so $PQh = 0$. Taking adjoints gives $QP = 0$.

(b) $\Longrightarrow$ (c). We just need to check that

$$(P+Q)^2 = P^2 + QP + PQ + Q^2 = P + 0 + 0 + Q = P + Q,$$

and that $P + Q$ is self-adjoint.

(c) $\Longrightarrow$ (b). Suppose $P + Q$ is a projection. Then $(P+Q)^2 = P + Q$ implies that $PQ = -QP$. This implies, first, that $PQPQ = (PQ)^2 = (QP)^2 = QPQP$, and, second, that

$$-PQ = P(-PQ)Q = P(QP)Q = QPQP = Q(-QP)P = -QP = PQ,$$

which in turn implies that $PQ = 0$.

(b) $\Longrightarrow$ (a). We just compute $(Ph \,|\, Qk) = (QPh \,|\, k) = 0$ for every $h, k \in \mathcal{H}$. $\square$

All except the first statements (a) in Propositions A.1 and A.2 make sense in any C^*-algebra A, and these C^*-algebraic statements are preserved by any representation of A as bounded operators on Hilbert space. So the equivalence of (b), (c) and (d) in Proposition A.1 and the equivalence of (b) and (c) in Proposition A.2 are valid in a C^*-algebra. Thus, for example:

Corollary A.3. *Suppose that $\{p_i : 1 \le i \le n\}$ are projections in a C^*-algebra A. Then $\sum_{i=1}^n p_i$ is a projection if and only if $p_i p_j = 0$ for $i \ne j$, in which case we say that the projections are* mutually orthogonal.

Proof. If the projections p_i are mutually orthogonal, a straightforward calculation shows that $p := \sum_{i=1}^n p_i$ satisfies $p^2 = p$, and p is clearly self-adjoint. We prove the converse by induction. It is trivially true when $n = 1$. Suppose that the converse is true for $n = k$, and that $\sum_{i=1}^{k+1} p_i$ is a projection. Since each $p_i = p_i^* p_i$ is a positive element of the C^*-algebra A, we have $\sum_{i=1}^{k+1} p_i \ge p_{k+1}$ in A, and hence by Proposition A.1,

$$\sum_{i=1}^{k} p_i = \Big(\sum_{i=1}^{k+1} p_i\Big) - p_{k+1}$$

is a projection in A. Now the inductive hypothesis implies that the projections $\{p_i : 1 \le i \le k\}$ are mutually orthogonal, and Proposition A.2 implies that p_{k+1} and $\sum_{i=1}^k p_i$ are mutually orthogonal. Thus for $i \le k$ we have

$$0 \le p_{k+1} p_i p_{k+1} \le p_{k+1}\Big(\sum_{i=1}^{k} p_i\Big)p_{k+1} = 0,$$

which forces $p_{k+1} p_i = 0$. Thus $\{p_i : 1 \le i \le k+1\}$ is mutually orthogonal. $\square$

An operator S on Hilbert space is a *partial isometry* if the restriction of S to $(\ker S)^\perp$ is an isometry.

Proposition A.4. *Let S be a bounded linear operator on a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(a) *S is a partial isometry;*

(b) *S^*S is a projection;*

(c) *$SS^*S = S$;*

(d) *SS^* is a projection;*

(e) $S^*SS^* = S^*$.

If so, S^*S is the projection on $(\ker S)^\perp$ and SS^* is the projection on the range of S.

PROOF. Taking adjoints in (c) gives (e), and applying (b) $\Longleftrightarrow$ (c) to S^* gives (d) $\Longleftrightarrow$ (e). It therefore suffices to prove that (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (a).

(a) $\Longrightarrow$ (b). We shall show that S^*S is the orthogonal projection P onto $(\ker S)^\perp$. The polarisation identity

$$4(Th \,|\, k) = \sum_{n=0}^{3} i^n \big(T(h + i^n k) \,|\, h + i^n k\big)$$

shows that it suffices to prove

$$\big((S^*S - P)h \,|\, h\big) = 0 \quad \text{for all } h \in \mathcal{H}. \tag{A.2}$$

If $h \in (\ker S)^\perp$, then

$$(S^*Sh \,|\, h) = \|Sh\|^2 = \|h\|^2 = (h \,|\, h) = (Ph \,|\, h),$$

which gives (A.2) for $h \in (\ker S)^\perp$. But both S^*S and P have range in $(\ker S)^\perp$ and are zero on $\ker S$, so for every $h \in \mathcal{H}$ we have

$$\big((S^*S - P)h \,|\, h\big) = \big(P(S^*S - P)Ph \,|\, h\big) = \big((S^*S - P)Ph \,|\, Ph\big)$$

so we have (A.2) for all $h \in \mathcal{H}$.

(b) $\Longrightarrow$ (c). Suppose S^*S is a projection. Then

$$\|S - SS^*S\|^2 = \|(S - SS^*S)^*(S - SS^*S)\| = \|S^*S - 2(S^*S)^2 + (S^*S)^3\| = 0,$$

which gives (c).

(c) $\Longrightarrow$ (a). We first note that that S^*S must be the projection onto $(\ker S)^\perp$. To see this, recall that the projection P on a closed subspace M is the unique operator such that $Ph \in M$ and $h - Ph \perp M$ for all $h \in \mathcal{H}$. Let $h \in \mathcal{H}$. For $k \in \ker S$, we have $(S^*Sh \,|\, k) = (Sh \,|\, Sk) = 0$, so $S^*Sh \in (\ker S)^\perp$. The relation $(S - SS^*S)h = 0$ implies that $h - S^*Sh \in \ker S$, and hence that $h - S^*Sh$ is orthogonal to $(\ker S)^\perp$. So S^*S is the projection onto $(\ker S)^\perp$. Now for $h \in (\ker S)^\perp$ we have

$$\|Sh\|^2 = (Sh \,|\, Sh) = (S^*Sh \,|\, h) = (h \,|\, h) = \|h\|^2,$$

which says that S is a partial isometry.

That S^*S is the projection on $(\ker S)^\perp$ we established in the course of proving (c) $\Longrightarrow$ (a). We trivially have $SS^*\mathcal{H} \subset S\mathcal{H}$, and (c) implies that $S\mathcal{H} = SS^*S\mathcal{H} \subset SS^*\mathcal{H}$. □

Statements (b), (c), (d) and (e) make sense in an abstract C^*-algebra, and are still equivalent there because we can always represent the C^*-algebra faithfully on Hilbert space. An element s of a C^*-algebra A which satisfies $s = ss^*s$ or one of the equivalent conditions is called a *partial isometry in A*. If s is a partial isometry in A, we call s^*s the *initial projection of s*, and ss^* the *final projection of s*.

A.2. Matrix algebras and direct sums

By viewing the set $M_n(\mathbb{C})$ of $n \times n$ complex matrices as the (automatically bounded) linear operators on the Hilbert space $\mathbb{C}^n$, it becomes a C^*-algebra. For $1 \le i, j \le n$, we define $E_{ij} \in M_n(\mathbb{C})$ by

$$(E_{ij})_{kl} = \begin{cases} 1 & \text{if } k = i \text{ and } l = j \\ 0 & \text{otherwise.} \end{cases}$$

Every $a = (a_{ij})$ can be written as $a = \sum_{i,j} a_{ij} E_{ij}$, so $\{E_{ij}\}$ is a vector space basis for $M_n(\mathbb{C})$, and calculations show that the E_{ij} satisfy

$$E_{ij}^* = E_{ji} \quad \text{and} \quad E_{ij}E_{kl} = \begin{cases} E_{il} & \text{if } j = k \\ 0 & \text{if } j \ne k. \end{cases} \tag{A.3}$$

We can recognise copies of the matrix algebra $M_n(\mathbb{C})$ by finding a family of *matrix units* $\{e_{ij}\}$ satisfying similar relations. (In practice, we usually write e_{ij} instead of E_{ij}, but in the following discussion it will be helpful to have separate notation.)

Proposition A.5. *Suppose B is a C^*-algebra and $\{e_{ij} : 1 \le i, j \le n\}$ is a set of elements of B such that*

$$e_{ij}^* = e_{ji} \quad \textit{and} \quad e_{ij}e_{kl} = \begin{cases} e_{il} & \textit{if } j = k, \\ 0 & \textit{if } j \ne k. \end{cases} \tag{A.4}$$

If one of the e_{ij} is non-zero, then they all are, and there is an injective homomorphism ϕ of $M_n(\mathbb{C})$ into B such that $\phi(E_{ij}) = e_{ij}$ for all i, j.

For the proof of the proposition, we need to know that $M_n(\mathbb{C})$ is a simple C^*-algebra: the only proper closed two-sided ideal is $\{0\}$. In fact, $M_n(\mathbb{C})$ has no non-trivial two-sided ideals, closed or not:

Lemma A.6. *$M_n(\mathbb{C})$ is simple as an algebra over $\mathbb{C}$.*

Proof. Suppose I is an ideal in $M_n(\mathbb{C})$ and I contains a non-zero element $a = (a_{ij})$. Then $a_{ij} \ne 0$ for some i and j. But then $E_{ij} = a_{ij}^{-1} E_{ii} a E_{jj}$ belongs to I, $E_{kl} = E_{ki}E_{ij}E_{jl}$ belongs to I for all k and l, and $M_n(\mathbb{C}) = \operatorname{span}\{E_{kl}\} \subset I$. □

Proof of Proposition A.5. The first assertion is clear: if $e_{ij} \ne 0$, then the relation $e_{ij} = e_{ik}e_{kl}e_{lj}$ forces $e_{kl} \ne 0$ for all k and l. Since the E_{ij} are linearly independent, we can define $\phi((a_{ij})) := \sum_{i,j} a_{ij} e_{ij}$, and then the equations (A.3) and (A.4) imply that ϕ is a $*$-homomorphism. Since there are no ideals in $M_n(\mathbb{C})$ and ϕ is non-zero, ϕ must be injective. □

If A and B are C^*-algebras, then the *direct sum* $A \oplus B$ is the C^*-algebra obtained by endowing the vector-space direct sum $A \oplus B := \{(a, b) : a \in A,\ b \in B\}$ with the operations

$$(a_1, b_1)(a_2, b_2) := (a_1a_2, b_1b_2), \quad (a, b)^* := (a^*, b^*), \quad \|(a, b)\| := \max\{\|a\|, \|b\|\}.$$

We think of $A \oplus B$ as containing copies $\{(a, 0)\}$ of A and $\{(0, b)\}$ of B which are orthogonal: $(a, 0)(0, b) = 0$ for all a, b. This allows us to recognise direct sums:

Proposition A.7. (a) *Suppose A and B are C^*-subalgebras of a C^*-algebra C such that $ab = 0$ for all $a \in A$ and $b \in B$. Then* $\operatorname{span}\{A \cup B\}$ *is a C^*-subalgebra of C, and the map $(a, b) \mapsto a + b$ is an isomorphism of $A \oplus B$ onto* $\operatorname{span}\{A \cup B\}$.

(b) *If ψ is a homomorphism of a C^*-algebraic direct sum $A \oplus B$ into a C^*-algebra C such that $a \mapsto \psi(a, 0)$ and $b \mapsto \psi(0, b)$ are injective, then ψ is injective.*

PROOF. We also have $ba = (a^*b^*)^* = 0$, and straightforward calculations using this show that $\phi : (a, b) \mapsto a + b$ is a homomorphism. This implies in particular that its range is a C^*-algebra. That ϕ is injective follows from (b). To establish (b), we compute:

$$\begin{aligned}
\psi(a,b) = 0 &\iff \psi(a,b)^*\psi(a,b) = 0 \\
&\iff \psi(a^*a, b^*b) = 0 \\
&\iff \psi(a^*a, 0) = 0 \text{ and } \psi(0, b^*b) = 0 \\
&\qquad\qquad\qquad (\text{because } 0 \leq \psi(a^*a, 0) \leq \psi(a^*a, b^*b)) \\
&\iff a^*a = 0 \text{ and } b^*b = 0 \\
&\iff a = 0 \text{ and } b = 0,
\end{aligned}$$

so ψ is injective. □

We now want to extend Proposition A.5 to the algebra $\mathcal{K}(\mathcal{H})$ of compact operators and Proposition A.7 to infinite direct sums. The key to both extensions is the following observation.

PROPOSITION A.8. *Suppose A is a C^*-algebra and there are C^*-subalgebras A_n of A such that $A_n \subset A_{n+1}$ and $A = \overline{\bigcup_{n=1}^\infty A_n}$. (One says that A is the* direct limit *of the subalgebras A_n.) If we have injective homomorphisms ϕ_n of each A_n into the same C^*-algebra B such that $\phi_{n+1}|_{A_n} = \phi_n$ for all n, then there is an injective homomorphism $\phi : A \to B$ such that $\phi|_{A_n} = \phi_n$ for all n.*

PROOF. The hypothesis implies that there is a well-defined map $\psi : \bigcup_{n=1}^\infty A_n \to B$ such that $\psi|_{A_n} = \phi_n$, and ψ is a homomorphism because each ϕ_n is. Since $\psi|_{A_n}$ is an injective homomorphism between C^*-algebras, it is isometric. Thus ψ is isometric on the dense subspace $\bigcup_{n=1}^\infty A_n$ of A, and hence extends to an isometry ϕ of $A = \overline{\bigcup_{n=1}^\infty A_n}$ into B. Since the C^*-algebra operations on A are continuous, this extension is a homomorphism of C^*-algebras, and since ϕ is isometric, it is injective. □

Now consider the C^*-algebra $\mathcal{K}(\ell^2)$ of compact operators, and let $\{e_n : n \geq 1\}$ be the usual orthonormal basis for ℓ^2. For h, k in a Hilbert space $\mathcal{H}$, we denote by $h \otimes \overline{k}$ the rank-one operator defined by $h \otimes \overline{k}(g) = (g \,|\, k)h$. Then for each $n \in \mathbb{N}$,

$$\{e_{ij} := e_i \otimes \overline{e_j} : 1 \leq i, j \leq n\}$$

is a family of non-zero matrix units in $\mathcal{K}(\ell^2)$, and hence by Proposition A.5 there is an injection ρ^n of $M_n(\mathbb{C})$ into $\mathcal{K}(\ell^2)$ such that $\rho^n(E_{ij}) = e_{ij}$. The image $A_n := \rho^n(M_n(\mathbb{C}))$ is a C^*-subalgebra of $\mathcal{K}(\ell^2)$, and $\bigcup_n A_n$ is dense in $\mathcal{K}(\ell^2)$ (see, for example, [**114**, Chapter 1]). Thus $\mathcal{K}(\ell^2) = \overline{\bigcup_n A_n}$. We usually use the map ρ^n to identify A_n with $M_n(\mathbb{C})$, and write $\mathcal{K}(\ell^2) = \overline{\bigcup_n M_n(\mathbb{C})}$.

COROLLARY A.9. *Suppose B is a C^*-algebra and $\{e_{ij} : i, j \in \mathbb{N}\}$ is a set of* matrix units *in B:*

$$e_{ij}^* = e_{ji} \quad \text{and} \quad e_{ij}e_{kl} = \begin{cases} e_{il} & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases} \tag{A.5}$$

If one of the e_{ij} is non-zero, then they all are, and there is an injective homomorphism $\phi : \mathcal{K}(\ell^2) \to B$ such that $\phi(e_i \otimes \overline{e}_j) = e_{ij}$ for all i, j.

Proof. If one e_{ij} is non-zero, the equation $e_{ij} = e_{ik}e_{kl}e_{lj}$ implies that every e_{kl} is non-zero. For each n, $\{e_{ij} : 1 \leq i, j \leq n\}$ is a set of non-zero matrix units satisfying (A.4), and hence there is an injective homomorphism $\psi_n : M_n(\mathbb{C}) \to B$ such that $\psi_n(E_{ij}) = e_{ij}$. The family $\{\psi_n\}$ is compatible with the inclusion of $M_n(\mathbb{C}) = \rho^n(M_n(\mathbb{C}))$ in $M_{n+1}(\mathbb{C}) = \rho^{n+1}(M_{n+1}(\mathbb{C}))$ described above, in the sense that $\psi_{n+1}|_{M_n(\mathbb{C})} = \psi_n$. Proposition A.8 now gives the result. □

Remark A.10. When the matrix units $\{e_{st} : s, t \in S\}$ are parametrised by some other countable set S, we can list $S = \{s_1, s_2, \cdots\}$, define $e_{ij} = e_{s_i s_j}$, and apply Corollary A.9. It is often convenient to skip the labelling step, and view Corollary A.9 as saying that any C^*-algebra spanned by a family $\{e_{st} : s, t \in S\}$ of matrix units parametrised by S is isomorphic to $\mathcal{K}(\ell^2(S))$.

We now extend Proposition A.7 to infinite direct sums. Suppose B_n is a sequence of C^*-algebras. Then the *direct sum* is the set

$$\bigoplus B_n = \big\{\{b_n\} \in \textstyle\prod_n B_n : \|b_n\| \to 0 \text{ as } n \to \infty\big\}$$

with pointwise operations and $\|\{b_n\}\| := \sup \|b_n\|$. To motivate the next result, note that if we define $i_n : B_n \to \bigoplus B_n$ by $i_n(b) := (0, \ldots, 0, b, 0, \ldots)$, then the subalgebras $i_n(B_n)$ satisfy $i_m(B_m)i_n(B_n) = 0$.

Corollary A.11. *Suppose B is a C^*-algebra and B_n is a sequence of C^*-subalgebras such that $B_m B_n = 0$ for $m \neq n$. Then there is an isomorphism ϕ of $\bigoplus B_n$ onto $\overline{\operatorname{span}}\{B_n\}$ such that $\phi(\{b_n\}) = \sum_n b_n$ when $b_n = 0$ for all but finitely many n.*

Proof. Let $A_n := \bigoplus_{m=1}^n B_m$. Then the map $\{b_m\} \mapsto (b_1, \ldots, b_n, 0, \ldots)$ embeds A_n as a C^*-subalgebra of $\bigoplus B_m$, $A_n \subset A_{n+1}$, and $\bigoplus B_m = \overline{\bigcup_n A_n}$. The maps $\phi_n : A_n \to B$ defined by $\phi_n(b_1, \ldots, b_n) = \sum_{m=1}^n b_m$ are injections by Proposition A.7, and satisfy $\phi_{n+1}|_{A_n} = \phi_n$. Thus Proposition A.8 gives the required homomorphism ϕ. □

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