C*-ALGEBRAS

1. Basic C^*-algebra theory

The following is a short version of Section 2.1 in Murphy's book. References like 2.1.1 are to Murphy, while references like 1.1 is internal.

Definition 1.1. *A Banach *-algebra A is called a C^*-algebra if for all $a \in A$ we have*

$$\|a^* a\| = \|a\|^2.$$

We shall always assume $A \neq \{0\}$. A may have an identity 1, if so it follows that $\|1\| = 1$.

Problem 1.2. *Show that in a C^*-algebra one has*

$$\|a\| = \sup_{\|b\|=1} \|ab\|.$$

Lemma. *If $u \in A$ is unitary ($u^* u = uu^* = 1$), then $\sigma(u) \subset \mathbb{T}$.*

Theorem. 2.1.1. *If $a = a^* \in A$, then $\|a\| = r(a)$.*

Proof. See Murphy. □

Problem 1.3. *Show that if $a \in A$ is normal ($a^* a = aa^*$) then also $\|a\| = r(a)$.*

Theorem. 2.1.6. *If $1 \notin A$, then $\tilde{A}$ is a C^*-algebra with norm*

$$\|(a, \lambda)\| = \sup_{\|b\|=1} \|ab + \lambda b\| = \sup_{\|b\|=1} \|ba + \lambda b\|.$$

and adjoint

$$(a, \lambda)^* = (a^*, \overline{\lambda}).$$

Proof. (The statement and proof is slightly different from Murphy's.) The equality in the definition follws from *-properties.

(i) $\|(a, 0)\| = \|a\|$. If $\lambda \neq 0$, then $(a, \lambda) = 0 \implies ab + \lambda b = 0$ for all $b \in A \implies -\frac{1}{\lambda} a$ is an identity, contradiction. So we have a norm.

(ii) $\tilde{A}$ is complete. (Proof?)

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(iii) Submultiplicative:

$$\begin{aligned}
\|(a,\lambda)(b,\mu)\| &= \|(ab+\lambda b+\mu a, \lambda\mu)\| \\
&= \sup_{\|c\|=1} \|abc+\lambda bc+\mu ac+\lambda\mu c\| \\
&= \sup_{\|c\|=\|d\|=1} \|dabc+\lambda dbc+\mu dac+\lambda\mu dc\| \\
&= \sup_{\|c\|=\|d\|=1} \|(da+\lambda d)(bc+\mu c)\| \\
&\leq \sup_{\|\|d\|=1} \|(da+\lambda d)\| \cdot \| \sup_{\|c\|=1} \|(bc+\mu c)\| \\
&= \|(a,\lambda)\| \cdot \|(b,\mu)\|
\end{aligned}$$

(iv) C^*-property:

$$\begin{aligned}
\|(a,\lambda)\|^2 &= \sup_{\|b\|=1} \|ab+\lambda b\|^2 \\
&= \sup_{\|b\|=1} \|b^*a^*ab+\lambda b^*a^*b+\overline{\lambda}b^*ab+|\lambda|^2 b^*b\| \\
&\leq \sup_{\|b\|=1} \|a^*ab+\lambda a^*b+\overline{\lambda}ab+|\lambda|^2 b\| \\
&= \|(a^*a+\lambda a^*+\overline{\lambda}a, |\lambda|^2)\| \\
&= \|(a,\lambda)^*(a,\lambda)\|^2 \\
&\leq \|(a,\lambda)\|^2.
\end{aligned}$$

So equality holds everywhere and we have a C^*-norm. □

For the nest results we refer to Murphy for proofs.

Theorem. 2.1.7. *A $*$-homomorphism $\varphi : A \mapsto B$ from a Banach *-algebra A to a C^*-algebra B is necessarily norm-decreasing.*

Theorem. 2.1.8. *If $a = a^*$ in a C^*-algebra A, then $\sigma(a) \subset \mathbb{R}$.*

Definition. Recall that $\Omega(A)$ is the set of non-zero multiplicative linear functionals on A.

Theorem. 2.1.9. *If $\tau \in \Omega(A)$ then $\tau(a^*) = \overline{\tau(a)}$.*

Lemma. 2.1.10. *(Gelfand) The Gelfand representation*

$$\varphi : A \mapsto C_0(\Omega(A)) \text{ given by } \varphi(a) = \widehat{a},$$

is an isometric *-isomorphism.

Lemma 1.4. *Let $a = a^* \in A$, and suppose $A = C^*(1,a)$. Then $\Omega(A) \cong \sigma(a)$ and the Gelfand representation φ maps a to the function $f(t) = t$.*

Proof. From Thm.1.3.7 we have that $\widehat{a} : \Omega(A) \mapsto \sigma_A(a)$ given by $\widehat{a}(\tau) = \tau(a)$ is a homeomorphism.

$\varphi : A \mapsto C_0(\Omega(A))$ is given by $\varphi(b) = \widehat{b}$. So $\psi : A \mapsto C_0(\sigma(A))$ is given by for $b \in A$ by

$$\psi(b)(t) = \varphi(b)[\widehat{a}^{-1}(t)] = \widehat{b}(\widehat{a}^{-1}(t)).$$

So for $b = a$ we get $\psi(a)(t) = t$. □

Lemma 1.5. *Let B be a closed *-subalgebra of a C^*-algebra A with $1 \in A \cap B$ and $b = b^* \in B$. Then $\sigma_A(b) = \sigma_B(b)$.*

Proof. Suppose $b = b^* \in B$, clearly $\sigma_A(b) \subset \sigma_B(b)$. Suppose $\lambda \in \sigma_B(b)$, by replacing b with $b - \lambda 1$ we can assume $\lambda = 0$. So we want to show that if b is invertible in A, then b is also invertible in B.

Suppose we have $a \in A$ with $ab = ba = 1$. We may assume $B = C^*(1, b)$ and $A = C^*(1, a, b)$.

From Thm.1.3.7 we have $\Omega(B) = \sigma_B(b)$. From Thm.1.3.4 the map

$$\widehat{b} : \Omega(A) \mapsto \sigma_A(b)$$

is onto. Claim that $\widehat{b}$ is 1-1. Using that $\tau(ab) = 1$ we get

$$\widehat{b}(\tau_1) = \widehat{b}(\tau_2) \implies \tau_1(b) = \tau_2(b) \implies \tau_1(a) = \tau_2(a) \implies \tau_1 = \tau_2 \text{ on } A.$$

So $\Omega(A) = \sigma_A(b)$.

The inclusion $\iota : B \mapsto A$ gives the map $\iota^* : \Omega(A) :\mapsto \Omega(B)$ by $\iota^*(\tau) = \tau \circ \iota$. We then have

$$\sigma_A(b) \xrightarrow{\widehat{b}^{-1}} \Omega(A) \xrightarrow{\iota^*} \Omega(B) \xrightarrow{\widehat{b}} \sigma_B(b) \ .$$

From Gelfand's theorem we have

$$C(\sigma_B(b)) \cong B \xrightarrow{\iota} A \cong C(\sigma_A(b))$$

This gives a map $\varphi : C(\sigma_B(b)) \mapsto C(\sigma_A(b))$. We observe that $\varphi(\widehat{b}) = \widehat{b}|_{\sigma_A(b)}$ and therefore $\varphi(f) = f|_{\sigma_A(b)}$ for all $f \in C(\sigma_B(b)$.

Now if $\sigma_A(b) \subsetneq \sigma_B(b)$ there is a non-zero $f \in C(\sigma_B(b))$ with $f(\sigma_A(b)) = \{0\}$, so $\varphi(f) = 0$. This contradicts that φ is 1-1. □

Theorem. 2.1.11. *Let B be a closed *-subalgebra of a C^*-algebra A with $1 \in A \cap B$ and $b \in B$. Then $\sigma_A(b) = \sigma_B(b)$.*

Proof. We just proved it for $b = b^*$. For the general case, see Murphy. □

In the above proof we used a special case of the following:

Problem 1.6. *Suppose we have a continuous map* $\theta : \Omega \mapsto \Omega'$ *between compact spaces. Then* $\theta^t(f) = f \circ \theta$ *defines a* **-homomorphism* $\theta^t : C(\Omega') \mapsto C(\Omega)$.

(i) Show that θ *is onto* $\iff$ θ^t *is 1-1.*

(ii) Show that θ *is 1-1* $\iff$ θ^t *has dense image.*

Theorem. 2.1.13. *Let* a *be a normal element of a unital* C^**-algebra* A *and suppose* z *is the inclusion map* $z : \sigma(a) \mapsto \mathbb{C}$. *Then the inverse Gelfand transform* φ *is the unique* **-homomorphism* $C(\sigma(a)) \mapsto A$ *with* $\varphi(z) = a$. φ *is is isometric with image equal* $C^*(1, a)$.

Theorem. 2.1.14. (Spectral mapping). *Let* a *be a a normal element of a unital* C^**-algebra* A, *and let* $f \in C(\sigma(a))$. *Then*

$$\sigma(f(a)) = f(\sigma(a)).$$

Moreover, if $g \in C(\sigma(f(a)))$, *then*

$$(g \circ f)(a) = g(f(a)).$$

Proof. See Murphy. □

Theorem. 2.1.15. *Let* X *be a compact Hausdorff space. Then we have* $\Omega(C(X)) \cong X$.

Proof. See Murphy. □