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HERMITIAN AND SYMMETRIC BANACH ALGEBRAS

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Abstract

In this thesis we prove some essential results about the spectrum and the spectral radius of an element in a Banach algebra. The thesis will among others contain the proof of the theorem normally known as the Spectral Mapping Theorem and proofs of two different spectral radius formulas. Further, we show the existence of the square root of an element in a Banach algebra. All these results will lead to the proof of Shirali-Ford's Theorem. The thesis ends with an example of a non-symmetric *-algebra.

Resumé

I dette projekt beviser vi nogle essentielle resultater om spektret og spektralradiusen af et element i en Banach algebra. Projektet vil blandt andet indeholde beviset for sætningen, som er kendt under navnet "the Spectral Mapping Theorem", samt beviser for to forskellige spectralradius formler. Endvidere, viser vi eksistensen af kvadratroden af et element i en Banach algebra. Alle disse resultater leder op til beviset for Shirali-Fords sætning. Projektet slutter med et eksempel på en ikke symmetrisk *-algebra.

Contents

Introduction

The main point of this thesis is to prove Shirali-Ford's Theorem, which states that a Banach *-algebra is Hermitian if and only if it is symmetric. The original proof of the theorem can be found in [9], and like Satish Shirali and James W.M. Ford we will prove the theorem by a series of lemmas.

In the first section we define Banach algebras and Banach *-algebras, and we will lay out some ground rules for the algebras. To prove Shirali-Ford's Theorem we need some knowledge about the spectrum of an element, which implies that we need some knowledge about the invertible elements in Banach algebras. Thus, section one contains some essential results about invertible elements and the spectrum. However, it is not always possible to determine the invertible or, more relevant in our context, the non-invertible elements, therefore it is convenient that we can compute the spectral radius. Hence, the first section contains two different formulas of the spectral radius. One of the techniques in the proof of Shirali-Ford's Theorem is that we can take the square root of an element in a Banach algebra, so we end first section by showing the existence of square roots in Banach algebras.

Section two contains Shirali-Ford's Theorem. We start by defining a Hermitian algebra, and thereafter all the results in section two are parts of the proof of Shirali-Ford's Theorem.

The point of section three is to consider the Banach *-algebra $\ell^1(\mathbb{F}_n)$, where $\mathbb{F}_n$ is a free group and show that $\ell^1(\mathbb{F}_n)$ contains a non-symmetric Banach *-algebra. The section starts by showing that for a group, G, $\ell^1(G)$ is in fact a Banach *-algebra. After that we will construct a bounded linear functional whose values will define a recurrence relation. The section ends by showing that a Banach *-algebra of $\ell^1(\mathbb{F}_n)$ is not Hermitian and thereby not symmetric.

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1 Banach Algebras

We start this section by giving some basic knowledge about Banach algebras. The section is primarily based on [7].

The definition of an algebra is the following: $\mathscr{A}$ is an algebra if $\mathscr{A}$ is a vector space together with a bilinear map

$$\mathscr{A} \times \mathscr{A} \to \mathscr{A}, \qquad (a,b) \mapsto ab$$

such that

$$a(bc) = (ab)c \quad a,b,c \in \mathscr{A}.$$

The definition of a subalgebra is the following: $\mathscr{B}$ is a subalgebra of $\mathscr{A}$ if $\mathscr{B}$ is a subspace of $\mathscr{A}$ such that $bb' \in \mathscr{B}$ for each $b, b' \in \mathscr{B}$.
The pair $(\mathscr{A}, \|\cdot\|)$, where $\mathscr{A}$ is an algebra and $\|\cdot\|$ is a norm on $\mathscr{A}$, is a normed algebra provided the norm is submultiplicative, i.e.,

$$\|ab\| \leq \|a\|\|b\|, \quad a,b \in \mathscr{A}.$$

The unit $\mathbb{1}$ is in $\mathscr{A}$ and $a \cdot \mathbb{1} = \mathbb{1} \cdot a = a$ for all $a \in \mathscr{A}$ and $\|\mathbb{1}\| = 1$. It needs to be said that not all algebras contain a unit, but in this thesis we only consider algebras with a unit.

In a normed algebra the multiplication operation is continuous. It follows from the inequality

$$\begin{aligned}\|ab - a'b'\| &= \|ab - ab' + ab' - a'b'\| \\ &\leq \|ab - ab'\| + \|ab' - a'b'\| \\ &= \|a(b-b')\| + \|(a-a')b'\| \\ &\leq \|a\|\|b-b'\| + \|a-a'\|\|b'\|.\end{aligned}$$

Definition 1.1. A *Banach algebra* is a normed algebra which is complete with respect to the norm.

Definition 1.2. If $\mathscr{A}$ is a Banach algebra, then an involution on $\mathscr{A}$, is a map $x \to x^*$ on $\mathscr{A}$ which satisfies,

1. $(x^*)^* = x$,
2. $(\lambda x + \mu y)^* = \bar{\lambda} x^* + \bar{\mu} y^*$,
3. $(xy)^* = y^* x^*$,

for all $x, y \in \mathscr{A}$ and $\lambda, \mu \in \mathbb{C}$. A Banach algebra with an involution is called a *Banach *-algebra.*

Definition 1.3. An element $x \in \mathscr{A}$ is *self-adjoint* with respect to the involution * if $x^* = x$. The set of all self-adjoint elements of $\mathscr{A}$ is denoted by $\mathrm{Sym}(\mathscr{A})$.

A subalgebra of a normed algebra is itself a normed algebra. The closure of a subalgebra is a subalgebra and a closed subalgebra of a Banach algebra is itself a Banach algebra.

If $(\mathscr{B}_\lambda)_{\lambda\in\Lambda}$ is a family of subalgebras of an algebra $\mathscr{A}$, then $\bigcap_{\lambda\in\Lambda}\mathscr{B}_\lambda$ is a subalgebra. Hence, for any subset S of $\mathscr{A}$ there is a smallest subalgebra $\mathscr{B}$ of $\mathscr{A}$ containing S that is the intersection of all the subalgebras containing S. We call this algebra the subalgebra of $\mathscr{A}$ generated by S.

If $\mathscr{A}$ is a normed algebra, then the closed algebra $\mathscr{C}$ generated by a set S, is the smallest closed subalgebra containing S. If $\mathscr{B}$ is the subalgebra generated by S, then $\mathscr{C}$ is the closure of $\mathscr{B}$.

A homomorphism from an algebra $\mathscr{A}$ to an algebra $\mathscr{B}$ is a linear map $\varphi\colon \mathscr{A}\to\mathscr{B}$ such that

$$\varphi(ab)=\varphi(a)\varphi(b), \qquad \forall a,b\in\mathscr{A}.$$

If $\mathscr{A}$ and $\mathscr{B}$ are algebras and $\varphi(\mathbb{1})=\mathbb{1}$, then φ is said to be a unital.

1.1 Invertible elements

An element of $\mathscr{A}$ is invertible if there exists $b\in\mathscr{A}$ such that $ab=ba=\mathbb{1}$, and in this case b is unique and we write a^{-1} instead of b. We denote the set of invertible elements by $\mathrm{Inv}(\mathscr{A})$.

Lemma 1.4. *Let $\mathscr{A}$ be an algebra. If $x_1,x_2,...,x_n\in\mathscr{A}$ commute and $x_1x_2\cdots x_n$ is invertible, then x_i is invertible for all $i=1,2,...,n$.*

Proof. We prove the statement by induction. Clearly it holds for $n=1$, since in that case the statement is vacuous. Assume that the statement is true for some $n\geq 1$ and let us prove that it is true for $n+1$. Let $x\in\mathrm{Inv}(\mathscr{A})$ be given by

$$x=x_1x_2\cdots x_{n+1}.$$

Define $y=x^{-1}(x_1x_2\cdots x_n)$. Since x_{n+1} commutes with x_i for all $i=1,2,...,n+1$ and therefore x_{n+1} also commutes with x,x^{-1} and y. Thus

$$x_{n+1}y=yx_{n+1}=x^{-1}(x_1x_2\cdots x_n)x_{n+1}=x^{-1}(x_1x_2\cdots x_{n+1})=x^{-1}x=\mathbb{1}.$$

This shows that x_{n+1} is invertible with inverse y. We conclude that

$$x_1x_2\cdots x_n=x(x_{n+1})^{-1}$$

is also invertible, and so we get by induction hypothesis that x_i is invertible for all $i=1,2,...,n$. This proves the statement. □

Theorem 1.5. *Let $\mathscr{A}$ be a Banach algebra and let $a\in\mathscr{A}$ such that $\|a\|<1$. Then $\mathbb{1}-a\in\mathrm{Inv}(\mathscr{A})$ and*

$$(\mathbb{1}-a)^{-1}=\sum_{n=0}^{\infty}a^n.$$

Proof. Since $\mathscr{A}$ is a Banach algebra and $\|a\| < 1$ then

$$\sum_{n=0}^{\infty} \|a^n\| \leq \sum_{n=0}^{\infty} \|a\|^n = \frac{1}{1-\|a\|} < \infty.$$

Hence, the series $\sum_{n=0}^{\infty} a^n$ converges. Say that $\sum_{n=0}^{\infty} a^n$ converges to b. We have $(\mathbb{1} - a)(\mathbb{1} + a + ... + a^n) = \mathbb{1} - a^{n+1}$ converges to $\mathbb{1}$ for $n \to \infty$, since $\|a\| < 1$ and therefore

$$\|a^{n+1}\| \leq \|a\|^{n+1} \to 0, \quad \text{for } n \to \infty.$$

Further,

$$\lim_{n\to\infty} (\mathbb{1} - a) \sum_{k=0}^{n} a^k = (\mathbb{1} - a)b.$$

Hence, $(\mathbb{1} - a)b = b(\mathbb{1} - a) = 1$, i.e., $(\mathbb{1} - a)$ is invertible and $b = \sum_{n=0}^{\infty} a^n$ is the inverse to $(\mathbb{1} - a)$. □

Corollary 1.6. *Let $\mathscr{A}$ be a Banach algebra and assume $\|\mathbb{1} - a\| < 1$ for $a \in \mathscr{A}$. Then $a \in \mathrm{Inv}(\mathscr{A})$ and*

$$\|a^{-1}\| \leq \frac{1}{1 - \|\mathbb{1} - a\|}.$$

Proof. Since $\|\mathbb{1} - a\| < 1$ then by Theorem 1.5, $\mathbb{1} - (\mathbb{1} - a) = a \in \mathrm{Inv}(\mathscr{A})$. Again using Theorem 1.5 we have

$$a^{-1} = \sum_{n=0}^{\infty} (\mathbb{1} - 1)^n.$$

Further,

$$\|a^{-1}\| = \Big\| \sum_{n=0}^{\infty} (\mathbb{1} - a)^n \Big\| \leq \sum_{n=0}^{\infty} \|(\mathbb{1} - a)\|^n = \frac{1}{1 - \|\mathbb{1} - a\|}.$$

□

Theorem 1.7. *If $\mathscr{A}$ is a Banach algebra, then $\mathrm{Inv}(\mathscr{A})$ is open in $\mathscr{A}$, and the map*

$$\mathrm{Inv}(\mathscr{A}) \to \mathscr{A}, \qquad a \mapsto a^{-1},$$

is continuous.

Proof. Suppose $a \in \mathrm{Inv}(\mathscr{A})$ and $b \in \mathscr{A}$ with $\|b - a\| < \|a^{-1}\|^{-1}$. Then

$$\|ba^{-1} - \mathbb{1}\| = \|ba^{-1} - aa^{-1}\| = \|(b - a)a^{-1}\| \leq \|b - a\|\|a^{-1}\| < 1.$$

Thus $\|ba^{-1} - \mathbb{1}\| < 1$. So by Theorem 1.5 we have $ba^{-1} \in \mathrm{Inv}(\mathscr{A})$, and therefore $b \in \mathrm{Inv}(\mathscr{A})$. Hence, for every $a \in \mathrm{Inv}(\mathscr{A})$ we can find an open ball around a such that the ball only contains elements of $\mathrm{Inv}(\mathscr{A})$.

Now we show that the map $\mathrm{Inv}(\mathscr{A}) \to \mathscr{A}$ is continuous. Let $\|a - b\| < \frac{1}{2}\|a^{-1}\|^{-1}$, then

$$\|\mathbb{1} - a^{-1}b\| = \|a^{-1}(a - b)\| \leq \|a^{-1}\|\|a - b\| < \|a^{-1}\|\tfrac{1}{2}\|a^{-1}\|^{-1} < \tfrac{1}{2}.$$

Further, we have

$$\|b^{-1}\| = \|b^{-1}(aa^{-1})\| \leq \|b^{-1}a\|\|a^{-1}\| = \|(a^{-1}b)^{-1}\|\|a^{-1}\|.$$

From Corollary 1.6 we have

$$\|(a^{-1}b)^{-1}\| \leq \frac{1}{1-\|\mathbb{1}-a^{-1}b\|} \leq \frac{1}{1-\frac{1}{2}} = 2.$$

Combining the two above inequalities we get $\|b^{-1}\| \leq 2\|a^{-1}\|$. Let $\epsilon > 0$ be given and let $\delta = \min\{\frac{1}{2}\|a^{-1}\|^{-1}, \frac{1}{2}\|a^{-1}\|^{-2}\epsilon\}$. For $\|a-b\| < \delta$ then

$$\begin{aligned}\|a^{-1}-b^{-1}\| = \|a^{-1}(b-a)b^{-1}\| &\leq \|a^{-1}\|\|b-a\|\|b^{-1}\| \\ &\leq \|a^{-1}\|\|b-a\|2\|a^{-1}\| \\ &< 2\|a^{-1}\|^2\delta \\ &\leq 2\|a^{-1}\|^2\tfrac{1}{2}\|a^{-1}\|^{-2}\epsilon = \epsilon\end{aligned}$$

Hence, the map $\mathrm{Inv}(\mathscr{A}) \to \mathscr{A}$ is continuous. □

1.2 The spectrum

Let $\mathbb{C}[z]$ denote the algebra of all polynomials in an indeterminate z with complex coefficients. If a is an element of an algebra $\mathscr{A}$ and $p(z) \in \mathbb{C}[z]$ is the polynomial

$$p(z) = \lambda_0 + \lambda_1 z^1 + ... + \lambda_n z^n,$$

we set

$$p(a) = \lambda_0\mathbb{1} + \lambda_1 a^1 + ... + \lambda_n a^n,$$

where $a^1 = a$ and $a^n = a^{n-1}a$. The map $\mathbb{C}[z] \to \mathscr{A}$ given by $p(z) \mapsto p(a)$ is a unital homomorphism.

The spectrum of an element $a \in \mathscr{A}$ is defined by the set

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda\mathbb{1} - a \notin \mathrm{Inv}(\mathscr{A})\}.$$

To simplify the notation we will in the following write $\lambda\mathbb{1}$ as λ. We call $\rho(a) = \mathbb{C} \setminus \sigma(a)$ the resolvent of a.

Theorem 1.8. *Let a be an element of a Banach algebra $\mathscr{A}$. If $\sigma(a)$ is non-empty and $p(z) \in \mathbb{C}[z]$, then*

$$\sigma(p(a)) = p(\sigma(a)).$$

Proof. We suppose that $p(z)$ is not constant. By factorization of $p(z)$ we have for $\mu \in \mathbb{C}$ and $\lambda_0, ..., \lambda_n \in \mathbb{C}$, where $\lambda_0 \neq 0$

$$p(z) - \mu = \lambda_0(z-\lambda_1)\cdots(z-\lambda_n),$$

and therefore,

$$p(a) - \mu = \lambda_0(a-\lambda_1)\cdots(a-\lambda_n).$$

Clearly, if $(a-\lambda_1), ..., (a-\lambda_n)$ are invertible then $p(a)-\mu$ is invertible. On the other hand, if $p(a)-\mu$ is invertible then by Lemma 1.4 $(a-\lambda_1), ..., (a-\lambda_n)$ are invertible. Thus, $p(a)-\mu$ is invertible if and only if $(a-\lambda_i)$ is invertible, for all $i = 1, 2, ..., n$. This implies that $\mu \in \sigma(p(a))$ if and only if $\lambda_i \in \sigma(a)$ therefore $p(\lambda_i) \in p(\sigma(a))$ for some $p(\lambda_i) = \mu$ and $i = 1, 2, ..., n$. Hence, $\sigma(p(a)) = p(\sigma(a))$. □

Proposition 1.9. *Let $\mathscr{A}$ be a Banach algebra. If $a \in \mathrm{Inv}(\mathscr{A})$ then $\sigma(a^{-1}) = \sigma(a)^{-1}$.*

Proof. Let $\lambda \in \mathbb{C}$. We prove that $\lambda^{-1} \notin \sigma(a^{-1})$ if and only if $\lambda \notin \sigma(a)$. Assume $\lambda^{-1} - a^{-1} \in \mathrm{Inv}(\mathscr{A})$. Since $\lambda^{-1} - a^{-1} = a^{-1}(a - \lambda)\lambda^{-1}$ and a is invertible by assumption, we conclude that $a - \lambda \in \mathrm{Inv}(\mathscr{A})$. So if $\lambda^{-1} \notin \sigma(a^{-1})$ then $\lambda \notin \sigma(a)$. Now assume $\lambda - a \in \mathrm{Inv}(\mathscr{A})$. Since $\lambda - a = \lambda(a^{-1} - \lambda^{-1})a$ and a is invertible, we conclude that $a^{-1} - \lambda^{-1} \in \mathrm{Inv}(\mathscr{A})$. So if $\lambda \notin \sigma(a)$ then $\lambda^{-1} \notin \sigma(a^{-1})$. Hence, $\sigma(a^{-1}) = \sigma(a)^{-1}$. □

Proposition 1.10. *Let $\mathscr{A}$ be a Banach *-algebra. Then $x \in \mathrm{Inv}(\mathscr{A})$ if and only if $x^* \in \mathrm{Inv}(\mathscr{A})$ with $(x^*)^{-1} = (x^{-1})^*$. Moreover, $\sigma(x^*) = \overline{\sigma(x)}$.*

Proof. We assume $x \in \mathrm{Inv}(\mathscr{A})$. Then we have $x^*(x^{-1})^* = (x^{-1}x)^* = \mathbb{1}^* = \mathbb{1}$ and $(x^{-1})^*x^* = (xx^{-1})^* = \mathbb{1}^* = \mathbb{1}$. Hence, for all $x \in \mathscr{A}$, if $x \in \mathrm{Inv}(\mathscr{A})$ then $x^* \in \mathrm{Inv}(\mathscr{A})$. In particular, it is true for x^*. Thus, $x^* \in \mathrm{Inv}(\mathscr{A})$ implies $(x^*)^* = x \in \mathrm{Inv}(\mathscr{A})$. This proves that $x \in \mathrm{Inv}(\mathscr{A})$ if and only if $x^* \in \mathrm{Inv}(\mathscr{A})$ with $(x^*)^{-1} = (x^{-1})^*$. Now we show that $\sigma(x^*) = \overline{\sigma(a)}$. Assume $\lambda \notin \sigma(x^*)$ then $(\overline{\lambda} - x)^* = \lambda - x^* \in \mathrm{Inv}(\mathscr{A})$. So by above $\overline{\lambda} - x \in \mathrm{Inv}(\mathscr{A})$, and thereby we conclude $\lambda \notin \overline{\sigma(x)}$. Similar argument shows that if $\lambda \notin \overline{\sigma(x)}$ then $\lambda \notin \sigma(x^*)$. Hence, the statement is proved. □

Lemma 1.11. *Let $\mathscr{A}$ be a Banach algebra and let $a \in \mathscr{A}$. The spectrum $\sigma(a)$ is a closed subset of the disc in the plane with centre in origin and radius $\|a\|$, and the map*

$$\rho(a) \to \mathscr{A}, \qquad \lambda \mapsto (a - \lambda)^{-1},$$

is continuous.

Proof. If $\|a\| < |\lambda|$ then $\|\lambda^{-1}a\| < |\lambda|^{-1}|\lambda| = 1$. By Theorem 1.5 it follows $\mathbb{1} - \lambda^{-1}a$ is invertible. Thus, $\lambda - a$ is invertible since $\mathbb{1} - \lambda^{-1}a = \lambda^{-1}(\lambda - a)$ and therefore $\lambda \notin \sigma(a)$. So we conclude that if $\lambda \in \sigma(a)$ then $|\lambda| \leq \|a\|$. To prove that $\sigma(a)$ is closed, we fix $a \in \mathscr{A}$. Define $g\colon \mathbb{C} \to \mathscr{A}$ given by $g(\lambda) = \lambda - a$. We have

$$\|g(\lambda) - g(\mu)\| = \|(\lambda - a) - (\mu - a)\| = \|\lambda - \mu\|.$$

Hence, g is continuous. By definition of the resolvent, we get

$$\rho(a) = \mathbb{C} \setminus \sigma(a) = \{\lambda \in \mathbb{C} \mid g(\lambda) = \lambda - a \in \mathrm{Inv}(\mathscr{A})\} = g^{-1}(\mathrm{Inv}(\mathscr{A})).$$

By Theorem 1.7 $\mathrm{Inv}(\mathscr{A})$ is open and since g is continuous then $\rho(a)$ is open. Hence, $\sigma(a)$ is closed. To prove the continuity we see that the map

$$\rho(a) \to \mathscr{A}, \qquad \lambda \mapsto (a - \lambda)^{-1}$$

is composed by the two maps,

$$\rho(a) \to \mathscr{A}, \quad \lambda \mapsto a - \lambda \quad \text{and} \quad \mathrm{Inv}(\mathscr{A}) \to \mathscr{A}, \quad a \to a^{-1}.$$

By above the first map is continuous and by Theorem 1.7 we know the second map is continuous. Hence, the composition is continuous, so $\rho(a) \to \mathscr{A}$ is continuous. □

Let $\mathscr{A}^*$ consists of linear and continuous maps $\mathscr{A} \to \mathbb{C}$.

Lemma 1.12. *Let $\mathscr{A}$ be a Banach algebra. For $a \in \mathscr{A}$ and $\tau \in \mathscr{A}^*$ then the functions mapping*

$$\lambda \mapsto \tau((a-\lambda)^{-1})) \qquad \text{and} \qquad \lambda \mapsto \tau((\mathbb{1}-\lambda a)^{-1})$$

are holomorphic on $\rho(a)$ and $\{\lambda \in \mathbb{C} \mid (\mathbb{1}-\lambda a) \in \operatorname{Inv}(\mathscr{A})\}$, respectively.

Proof. We only show $\lambda \mapsto \tau((a-\lambda)^{-1})$ is holomorphic. Let $\lambda, \mu \in \mathbb{C}$ and $\lambda \neq \mu$. Then we get

$$\begin{aligned}\frac{\tau((a-\lambda)^{-1})-\tau((a-\mu)^{-1})}{\lambda-\mu} &= \tau\Big(\tfrac{(a-\lambda)^{-1}-(a-\mu)^{-1}}{\lambda-\mu}\Big)\\ &= \tau\Big(\tfrac{(a-\lambda)^{-1}((a-\mu)-(a-\lambda))(a-\mu)^{-1}}{\lambda-\mu}\Big)\\ &= \tau\Big(\tfrac{(\lambda-\mu)(a-\lambda)^{-1}(a-\mu)^{-1}}{\lambda-\mu}\Big)\\ &= \tau((a-\lambda)^{-1}(a-\mu)^{-1})\end{aligned}$$

and $\tau((a-\lambda)^{-1}(a-\mu)^{-1}) \to \tau((a-\lambda)^{-1}(a-\lambda)^{-1})$ for $\mu \to \lambda$. By Lemma 1.11 $\lambda \mapsto (a-\lambda)^{-1}$ is continuous on $\rho(a)$ so the limit $\tau((a-\lambda)^{-1}(a-\lambda)^{-1})$ exists. Hence, the map is holomorphic. □

Remark 1.13. For all $\tau \in \mathscr{A}^*$, if $\tau(a) = 0$ then $a = 0$ [6, Corollary 1.6.2]. So we derive that if $\tau(a) = \tau(b)$ and therefore $\tau(a-b) = \tau(a) - \tau(b) = 0$, then $a - b = 0$ or equivalent $a = b$.

Theorem 1.14. *If a is an element of a Banach algebra $\mathscr{A}$, then the spectrum of a is non-empty.*

Proof. We suppose toward a contradiction that $\sigma(a) = \emptyset$. If $|\lambda| > 2\|a\|$ then

$$\|\lambda^{-1}a\| = |\lambda^{-1}|\ \|a\| < |\lambda^{-1}|\tfrac{1}{2}|\lambda| = \tfrac{1}{2},$$

so $1 - \|\lambda^{-1}a\| > \frac{1}{2}$. Using Theorem 1.5, we get

$$\begin{aligned}\|(\mathbb{1}-\lambda^{-1}a)^{-1}-\mathbb{1}\| = \Big\|\sum_{n=1}^{\infty}(\lambda^{-1}a)^n\Big\| \leq \sum_{n=1}^{\infty}\|(\lambda^{-1}a)\|^n &= \|(\lambda^{-1}a)\|\sum_{n=0}^{\infty}\|(\lambda^{-1}a)\|^n\\ &= \frac{\|\lambda^{-1}a\|}{1-\|\lambda^{-1}a\|} \leq 2\|\lambda^{-1}a\| < 1.\end{aligned}$$

Hereby, we obtain $\|(\mathbb{1}-\lambda^{-1}a)^{-1}\| < 2$, and therefore

$$\begin{aligned}\|(a-\lambda)^{-1}\| &= \|(\lambda(\lambda^{-1}a-\mathbb{1}))^{-1}\| = \|\lambda^{-1}(\lambda^{-1}a-\mathbb{1})^{-1}\|\\ &= |\lambda^{-1}|\ \|(\lambda^{-1}a-\mathbb{1})^{-1}\| < |\lambda|^{-1}2 < \|a\|^{-1}.\end{aligned}$$

Note, $\|a\|^{-1}$ is well-defined since we suppose $\sigma(a) = \emptyset$ so $a \neq 0$.

Now consider the map $f\colon \mathbb{C} \to \mathscr{A}$ given by $\lambda \mapsto (a-\lambda)^{-1}$. By Lemma 1.11 f is continuous and since the disc $2\|a\|\mathbb{D} = \{\lambda \in \mathbb{C}\colon |\lambda| \leq 2\|a\|\}$ is compact in $\mathbb{C}$, then f is bounded on the disc $2\|a\|\mathbb{D}$, i.e., there exists $0 < N \in \mathbb{N}$ such that $\|f(\lambda)\| \leq N$ for all $\lambda \in 2\|a\|\mathbb{D}$. For $\lambda \in \mathbb{C} \setminus 2\|a\|\mathbb{D}$ we have

$$\|f(\lambda)\| = \|(a-\lambda)^{-1}\| \leq \|a\|^{-1}.$$

Thus, $\|f(\lambda)\| \leq \{N, \|a\|^{-1}\}$ for all $\lambda \in \mathbb{C}$, so f is bounded on all of $\mathbb{C}$.

If $\tau \in A^*$, then by Lemma 1.12 the function $\lambda \mapsto \tau((a-\lambda)^{-1})$ is entire. Further, the function is bounded, since

$$|\tau((a-\lambda)^{-1})| \leq \|\tau\| \, \|(a-\lambda)^{-1}\| \leq \|\tau\| \max\{N, \|a\|^{-1}\}.$$

Hence, by Liouville's Theorem [8, Theorem 10.23] the function is constant. So if $g\colon \mathbb{C} \to \mathbb{C}$ given by $g(\lambda) = \tau((a-\lambda)^{-1})$ and g is constant, then

$$\tau(a^{-1}) = g(0) = g(1) = \tau((a-1)^{-1}).$$

Since this is true for all $\tau \in \mathscr{A}^*$, then by Remark 1.13 $a^{-1} = (a-1)^{-1}$ so $a = a-1$, which is a contradiction. Hence, $\sigma(a) \neq \emptyset$. □

Proposition 1.15. *If $\mathscr{A}$ is a Banach algebra in which every non-zero element is invertible, then $\mathscr{A} = \mathbb{C}1$.*

Proof. We have $\mathrm{Inv}(\mathscr{A}) = \{a \in \mathscr{A} \mid a \neq 0\}$ and by Theorem 1.14 $\sigma(a) \neq \emptyset$, so

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin \mathrm{Inv}(\mathscr{A})\} = \{\lambda \in \mathbb{C} \mid \lambda - a = 0\}.$$

Thus $\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda = a\}$. Hence, $\mathscr{A} = \mathbb{C}1$. □

Lemma 1.16. *Let $\mathscr{A}$ be a Banach algebra and let $a, b \in \mathscr{A}$. Then*

$$\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}.$$

Proof. It suffices to show that $\mathbb{1} - ab$ is invertible if and only if $\mathbb{1} - ba$ is invertible. Since for $\lambda \notin \sigma(ab)$ and $\lambda \neq 0$ then $\lambda - ab$ is invertible or equivalently $\mathbb{1} - (\lambda^{-1}a)b$ is invertible. Hence, per assumption $\mathbb{1} - b(\lambda^{-1}a)$ is invertible or equivalent $\lambda - ba$ is invertible. Thus, $\lambda \notin \sigma(ba)$. Similarly, we get that if $\lambda \notin \sigma(ba)$ and $\lambda \neq 0$ then $\lambda \notin \sigma(ab)$.

Now we prove $\mathbb{1} - ab$ is invertible if and only if $\mathbb{1} - ba$ is invertible. Assume $\mathbb{1} - ab$ is invertible. Then there exists $c \in \mathscr{A}$ such that $(\mathbb{1} - ab)c = c(\mathbb{1} - ab) = \mathbb{1}$. We consider $(\mathbb{1} - ba)(\mathbb{1} + bca)$ and $(\mathbb{1} + bca)(\mathbb{1} - ba)$,

$$(\mathbb{1} - ba)(\mathbb{1} + bca) = \mathbb{1} + b(c - abc)a - ba = \mathbb{1} + ba - ba = \mathbb{1},$$
$$(\mathbb{1} + bca)(\mathbb{1} - ba) = \mathbb{1} - ba + b(c - cab)a = \mathbb{1} - ba + ba = \mathbb{1}.$$

Thus $\mathbb{1} - ba$ has an inverse, namely, $\mathbb{1} + bca$ whenever $\mathbb{1} - ab$ is invertible. Similarly, we get that $\mathbb{1} - ab$ is invertible when $\mathbb{1} - ba$ is invertible. Thus, $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$. □

1.3 The spectral radius formula

Now we define the spectral radius. If a is an element of a Banach algebra $\mathscr{A}$, then the spectral radius is defined by

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|.$$

The definition of the spectral radius makes sense since the spectrum is compact.

Lemma 1.17. *Let $\mathscr{A}$ be a Banach algebra and let $a \in \mathscr{A}$. Then $r(a) < 1$ if and only if*

$$\sigma(a) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}.$$

Proof. Assume $r(a) = \sup_{\lambda\in\sigma(a)} |\lambda| = 1$. Then there exists $x_n \in \sigma(a)$ for each $n \in \mathbb{N}$ such that $|x_n - 1| < \frac{1}{n}$. This implies $x_n \to 1$ for $n \to \infty$. By Lemma 1.11 $\sigma(a)$ is compact, therefore $1 \in \sigma(a)$. This is a contradiction. □

Theorem 1.18. *Let $\mathscr{A}$ be a Banach algebra and let $a \in \mathscr{A}$. Then*

$$r(a) = \inf_{n\geq 1} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^n\|^{1/n}$$

Proof. If $\lambda \in \sigma(a)$ then by Theorem 1.8 $\lambda^n \in \sigma(a^n)$, so by Lemma 1.11 $|\lambda^n| \leq \|a^n\|$. Thus,

$$r(a) = r(a^n)^{1/n} = \sup_{\lambda\in\sigma(a)} |\lambda^n|^{\frac{1}{n}} \leq \inf_{n\geq 1} \|a^n\|^{\frac{1}{n}} \leq \liminf_{n\to\infty} \|a^n\|^{\frac{1}{n}}.$$

Now we show $\lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq r(a)$. Let $B(0, \frac{1}{r(a)})$ denote the open disc in $\mathbb{C}$ with centre 0 and radius $\frac{1}{r(a)}$. If $\lambda \in B(0, \frac{1}{r(a)})$ and $\lambda = 0$ then $\mathbb{1} - \lambda a = \mathbb{1} \in \mathrm{Inv}(\mathscr{A})$. On the other hand if $\lambda \neq 0$ then $\frac{1}{|\lambda|} > r(a)$ since $|\lambda| < \frac{1}{r(a)}$. This implies $\frac{1}{|\lambda|} \notin \sigma(a)$ so

$$\lambda^{-1} - a \in \mathrm{Inv}(\mathscr{A}).$$

Hence, $\lambda(\lambda^{-1} - a) = \mathbb{1} - \lambda a \in \mathrm{Inv}(\mathscr{A})$. If $\tau \in \mathscr{A}^*$, we consider the map

$$f\colon B(0, \tfrac{1}{r(a)}) \to \mathbb{C}, \qquad \lambda \mapsto \tau((1 - \lambda a)^{-1}).$$

By Lemma 1.12 f is holomorphic, therefore there exists unique complex numbers λ_n [8, Theorem 10.16] such that

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda_n \lambda^n, \quad \lambda \in B(0, \tfrac{1}{r(a)}).$$

However, $r(a) \leq \|a\|$ so if $|\lambda| < \frac{1}{r(a)}$ then $|\lambda| \leq \frac{1}{\|a\|}$ and therefore $\|\lambda a\| < 1$. By Theorem 1.5 we get

$$(1 - \lambda a)^{-1} = \sum_{n=0}^{\infty} (\lambda a)^n.$$

and therefore

$$\sum_{n=0}^{\infty} \lambda^n \lambda_n = f(\lambda) = \tau((1 - \lambda a)^{-1}) = \tau\Big(\sum_{n=0}^{\infty} \lambda^n a^n\Big) = \sum_{n=0}^{\infty} \lambda^n \tau(a^n).$$

Since power series of a function is uniquely determined it follows $\lambda_n = \tau(a^n)$ for all $n \geq 0$.

For each $\lambda \in B(0, \frac{1}{r(a)})$ then $|\lambda^n \tau(a^n)| \to 0$ for $n \to \infty$. Hence, if we fix $\lambda \in B(0, \frac{1}{r(a)})$ then the sequence $(\lambda^n \tau(a^n))$ is bounded. Since this is true for each $\tau \in \mathscr{A}^*$ it follows from the principle of uniform boundedness [6, Corollary 1.8.11] that if $(\lambda^n \tau(a^n)) = (\tau(\lambda^n a^n))$ is bounded then $(\lambda^n a^n)$ is bounded. Hence, there exists a positive number M such that $\|\lambda^n a^n\| \leq M$ for all $n \geq 0$.

So for $\lambda \neq 0$, $\|\lambda^n a^n\| = |\lambda^n| \, \|a^n\| \leq M$ which implies $\|a^n\|^{1/n} \leq \frac{M^{\frac{1}{n}}}{|\lambda|}$. Since $M^{\frac{1}{n}} \to 1$ for $n \to \infty$, we get

$$\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq |\lambda|^{-1}$$

Thus, we have shown if $|\lambda| < \frac{1}{r(a)}$, or in other words $r(a) < \frac{1}{|\lambda|}$, then

$$\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq |\lambda|^{-1}$$

and therefore $\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq r(a)$. By above result it follows that

$$\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq r(a) \leq \lim_{n\to\infty} \inf \|a^n\|^{\frac{1}{n}},$$

Thus, $r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$. □

Lemma 1.19. *Let $\mathscr{A}$ be a Banach algebra and assume that $x, y \in \mathscr{A}$ commute. Then $r(x+y) \leq r(x) + r(y)$.*

Proof. Let $p, q > 0$ with $p > r(x)$ and $q > r(y)$. Choose $m > 0$ such that for $n \geq m$,

$$\|x^n\|^{\frac{1}{n}} < p \qquad \text{and} \qquad \|y^n\|^{\frac{1}{n}} < q, \tag{1}$$

where we have used Theorem 1.18. Now choose $M \geq 1$ such that

$$\left(\frac{\|x\|}{p}\right)^k \leq M \quad \text{and} \quad \left(\frac{\|y\|}{p}\right)^k \leq M,$$

for $k = 0, 1, \dots, m-1, m$. Rewriting we get

$$\|x\|^k \leq Mp^k \quad \text{and} \quad \|y\|^k \leq Mq^k, \tag{2}$$

for each $k = 0, 1, \dots, m-1, m$.

Fix some $n > 2m$. Since x and y commutes, we get

$$\|(x+y)^n\|^{\frac{1}{n}} = \Big\| \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \Big\|^{\frac{1}{n}} \leq \Big(\sum_{k=0}^{n} \binom{n}{k} \|x^k\| \|y^{n-k}\| \Big)^{\frac{1}{n}},$$

where we have used the triangle inequality and the submultiplicativity of the norm. Now we estimate each term in the sum depending on the value of $k \in \{0, 1, \dots, m\}$.

For $k \in \{0, 1, \dots, m-1\}$ we have $n - k \geq n - m + 1 > 2m - m + 1 = m + 1$, so using (1) and (2), we get

$$\sum_{k=0}^{m-1} \binom{n}{k} \|x^k\| \|y^{n-k}\| \leq \sum_{k=0}^{m-1} \binom{n}{k} \|x\|^k q^{n-k} \leq \sum_{k=0}^{m-1} \binom{n}{k} Mp^k q^{n-k}.$$

Likewise, for $k \in \{n-m+1, n-m+2, \dots, n\}$ we can estimate such that $k \geq n - m + 1 > 2m - m + 1 = m + 1$, so again using (1) and (2), we get

$$\sum_{k=n-m+1}^{n} \binom{n}{k} \|x^k\| \|y^{n-k}\| \leq \sum_{k=n-m+1}^{n} \binom{n}{k} p^k \|y\|^{n-k} \leq \sum_{k=n-m+1}^{n} \binom{n}{k} Mp^k q^{n-k}.$$

At last, for $k \in \{m, \dots, n-m\}$ we get just using (1) and the fact that $M \geq 1$ that

$$\sum_{k=m}^{n-m} \binom{n}{k} \|x^k\| \|y^{n-k}\| \leq \sum_{k=m}^{n-m} \binom{n}{k} Mp^k q^{n-k}.$$

When we combine these inequalities we obtain

$$\begin{aligned}
&\Big(\sum_{k=0}^{n}\binom{n}{k}\|x^k\|\|y^{n-k}\|\Big)^{\frac{1}{n}} \\
&\le \Big(\sum_{k=0}^{m-1}\binom{n}{k}Mp^kq^{n-k} + \sum_{k=m}^{n-m}\binom{n}{k}Mp^kq^{n-k} + \sum_{k=n-m+1}^{n}\binom{n}{k}Mp^kq^{n-k}\Big)^{\frac{1}{n}} \\
&= M^{\frac{1}{n}}\Big(\sum_{k=0}^{n}\binom{n}{k}p^kq^{n-k}\Big)^{\frac{1}{n}} = M^{\frac{1}{n}}(p+q).
\end{aligned}$$

Therefore $\|(x+y)^n\|^{\frac{1}{n}} \le \Big(\sum_{k=0}^{n}\binom{n}{k}\|x^k\|\|y^{n-k}\|\Big)^{\frac{1}{n}} \le M^{\frac{1}{n}}(p+q)$. Since $M^{\frac{1}{n}}(p+q) \to p+q$ for $n \to \infty$, we conclude that $r(x+y) \le p+q$. Since $p > 0$ and $q > 0$ was arbitrarily with $r(x) < p$ and $r(y) < q$, we conclude $r(x+y) \le r(x) + r(y)$. □

1.4 Another spectral radius formula

Now we turn to the spectral radius of an operator. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space and $B(\mathcal{X})$ the space of all bounded linear operators which map $\mathcal{X}$ into itself. Note that the norm on $\mathcal{X}$ and another norm $|\cdot|$ is equivalent on $\mathcal{X}$ if $k\|\cdot\| \le |\cdot| \le c\|\cdot\|$, for some constants c, k. This mean the norms induces the same topology on $\mathcal{X}$.

Lemma 1.20. *Let $\mathcal{X}$ be a Banach space. If $|\cdot|$ is equivalent to $\|\cdot\|$ on $\mathcal{X}$. Then the correspondence operator norms $|\cdot|_\infty$ and $\|\cdot\|_\infty$ are equivalent on $B(\mathcal{X})$.*

Proof. Since $|\cdot|$ and $\|\cdot\|$ are equivalent then

$$k\|x\| \le |x| \le c\|x\|.$$

Likewise, we have the inequalities

$$k'\|Tx\| \le |Tx| \le c'\|Tx\|.$$

Thus,

$$|T|_\infty = \sup_{x\neq 0}\frac{|Tx|}{|x|} \le \sup_{x\neq 0}\frac{c'\|Tx\|}{k\|x\|} = \frac{c'}{k}\|T\|_\infty,$$

and

$$|T|_\infty = \sup_{x\neq 0}\frac{|Tx|}{|x|} \ge \sup_{x\neq 0}\frac{k'\|Tx\|}{c\|x\|} = \frac{k'}{c}\|T\|_\infty.$$

Hence, $\frac{k'}{c}\|T\|_\infty \le |T|_\infty \le \frac{c'}{k}\|T\|_\infty$, i.e., $|\cdot|_\infty$ and $\|\cdot\|_\infty$ are equivalent norms. □

The following result is found in [5].

Theorem 1.21. *Let $T \in B(\mathcal{X})$. Then $r(T) = \inf |T|_\infty$, where the infimum is taken over all norms $|\cdot|$ on $\mathcal{X}$ equivalent to $\|\cdot\|$.*

Proof. By the spectral radius formula we have

$$r(T) = \lim_{n\to\infty}\left\|T^n\right\|_\infty^{1/n} \le \|T\|_\infty.$$

Since $\|\cdot\|$ is equivalent to $|\cdot|$ on $\mathcal{X}$ the corresponding operator norms are equivalent by Lemma 1.20, therefore we can replace the norm such that $r(T) \leq \inf |T|_\infty$. To obtain $\inf |T|_\infty \leq r(T)$ we show that if $r(T) < s$ then there exists a norm $|\cdot|$ on $\mathcal{X}$, equivalent to $\|\cdot\|$ for which $|T|_\infty < s$. Let $r(T) < s$ and set $U = s^{-1}T$. Then

$$r(U) = r(s^{-1}T) = s^{-1}r(T) < s^{-1}s = 1.$$

So it suffices to produce an equivalent norm $|\cdot|$ for which $|U|_\infty < 1$. Observe, the series $\sum_{n=0}^\infty \|U^n\|$ converges by the root test, since

$$\lim_{n\to\infty} \|U^n\|^{\frac{1}{n}} = r(U) < 1.$$

Then for any $x \in \mathcal{X}$ we define

$$|x| = \sum_{n=0}^\infty \|U^n x\|.$$

We note that $|x|$ satisfies the properties of the norm, i.e., $|x| = 0$ if and only if $x = 0$. Further,

$$|\lambda x| = \sum_{n=0}^\infty \|\lambda U^n x\| = |\lambda| \sum_{n=0}^\infty \|U^n x\| = |\lambda||x|$$

and

$$\begin{aligned} |x+y| = \sum_{n=0}^\infty \|U^n(x+y)\| &\leq \sum_{n=0}^\infty (\|U^n x\| + \|U^n y\|) \\ &= \sum_{n=0}^\infty \|U^n x\| + \sum_{n=0}^\infty \|U^n y\| \\ &= |x| + |y|. \end{aligned}$$

Now we show that $|\cdot|$ is equivalent to $\|\cdot\|$. Since

$$\|x\| = \|U^0 x\| \leq \sum_{n=0}^\infty \|U^n x\| \leq \big(\sum_{n=0}^\infty \|U^n\|\big)\|x\|,$$

then $\|x\| \leq |x| \leq \big(\sum_{n=0}^\infty \|U^n\|\big)\|x\|$. Thus, $|\cdot|$ is equivalent to $\|\cdot\|$. Further, we notice

$$|Ux| = \sum_{n=0}^\infty \|U^{n+1}x\| = \sum_{n=1}^\infty \|U^n x\| = |x| - \|x\|.$$

So if $x \neq 0$ then

$$\frac{|Ux|}{|x|} = 1 - \frac{\|x\|}{|x|} \leq 1 - \inf_{x\neq 0} \frac{\|x\|}{|x|} \leq 1 - \big(\sum_{n=0}^\infty \|U^n\|\big)^{-1}.$$

Thus, $|U|_\infty = \sup_{x\neq 0} \frac{|Ux|}{|x|} < 1$, since $\sum_{n=0}^\infty \|U^n\| > 0$ and therefore

$$1 - \big(\sum_{n=0}^\infty \|U^n\|\big)^{-1} < 1.$$

Now we have produced a norm such that if $r(T) < s$ then $|T|_\infty < s$. Hence, $\inf |T|_\infty \leq r(T)$, so we conclude $r(T) = \inf |T|$. □

Now we have shown that if the norms are equivalent on the Banach space, $\mathcal{X}$, then the spectral radius is given by $r(T) = \inf |T|$ for $T \in B(\mathcal{X})$. The next proposition shows that if the norms are equivalent on the Banach algebra, $\mathscr{A}$, then the spectral radius is given by $r(a) = \inf |a|$, for $a \in \mathscr{A}$.

Proposition 1.22. *If $(\mathscr{A}, \|\cdot\|)$ is a Banach algebra then $r(a) = \inf |a|$ for $a \in \mathscr{A}$, where the infimum is taken over all algebra-norms on $\mathscr{A}$ equivalent to $\|\cdot\|$.*

Proof. We define the map $\pi\colon \mathscr{A} \to B(\mathscr{A})$ given by $\pi(a)b = ab$. The map is well-defined since

$$\|\pi(a)b\| = \|ab\| \leq \|a\|\|b\|,$$

so $\pi(a)$ is bounded. For an algebra norm $|\cdot|$ equivalent to $\|\cdot\|$ we have

$$|\pi(a)|_\infty = \sup_{b\neq 0} \frac{|\pi(a)b|}{|b|} = \sup_{b\neq 0} \frac{|ab|}{|b|} \leq \sup_{b\neq 0} \frac{|a||b|}{|b|} = |a|,$$

and

$$|\pi(a)|_\infty = \sup_{b\neq 0} \frac{|\pi(a)b|}{|b|} \geq \frac{|\pi(a)\mathbb{1}|}{|\mathbb{1}|} = \frac{|a\cdot \mathbb{1}|}{|\mathbb{1}|} = |a|.$$

Hence, $|\pi(a)|_\infty = |a|$. Further, π is a homomorphism since

$$(\pi(a)\pi(b))(x) = \pi(a)(bx) = abx = \pi(ab)x.$$

Now we want to show $\sigma_{\mathscr{A}}(a) = \sigma_{B(\mathscr{A})}(\pi(a))$. First we show that $\pi(a) \in B(\mathscr{A})$ is invertible if and only if $a \in \mathscr{A}$ is invertible. Assume a is invertible, then there exists $a^{-1} \in \mathscr{A}$ such that $aa^{-1} = a^{-1}a = \mathbb{1}$, and

$$\pi(a^{-1})\pi(a)x = \pi(a^{-1})ax = a^{-1}ax = x.$$

Likewise, we have $\pi(a)\pi(a^{-1})x = x$. Thus, $\pi(a^{-1})\pi(a) = \pi(a)\pi(a^{-1}) = \mathbb{1}$ and therefore there exists an inverse to $\pi(a) \in B(\mathscr{A})$, namely, $\pi(a^{-1}) = \pi(a)^{-1} \in B(\mathscr{A})$. Hence, $\pi(a)$ is invertible in $B(\mathscr{A})$. Now assume $\pi(a)$ is invertible. Then there exists $T \in B(\mathscr{A})$ such that

$$T\pi(a) = \pi(a)T = \mathbb{1}_{B(\mathscr{A})}.$$

Set $b = T(\mathbb{1}_{\mathscr{A}})$. Then a is invertible, since

$$ab = aT(\mathbb{1}_{\mathscr{A}}) = \pi(a)(T(\mathbb{1}_{\mathscr{A}})) = (\pi(a)T)(\mathbb{1}_{\mathscr{A}}) = \mathbb{1}_{\mathscr{A}},$$

and if we use the fact that $ab = \mathbb{1}_{\mathscr{A}}$, we get

$$\begin{aligned} ba = (T\pi(a))ba = T(\pi(a)ba) = T(aba) = T(\mathbb{1}_{\mathscr{A}}a) &= T(\pi(a)\mathbb{1}_{\mathscr{A}}) \\ &= T\pi(a)(\mathbb{1}_{\mathscr{A}}) = \mathbb{1}_{\mathscr{A}}. \end{aligned}$$

Hence, $a \in \mathscr{A}$ is invertible if and only if $\pi(a) \in B(\mathscr{A})$ is invertible. This implies that $\lambda - a \in \mathrm{Inv}(\mathscr{A})$ if and only if $\pi(\lambda - a) = \lambda - \pi(a) \in \mathrm{Inv}(B(\mathscr{A}))$. Thus, we have $\sigma_{B(\mathscr{A})}(\pi(a)) = \sigma_{\mathscr{A}}(a)$. In particular $r(a) = r(\pi(a))$ and by Theorem 1.21 we get

$$r(a) = r(\pi(a)) = \inf |\pi(a)|_\infty.$$

We conclude $r(a) = \inf |\pi(a)|_\infty = \inf |a|$ for an algebra norm $|\cdot|$ equivalent to $\|\cdot\|$ on $\mathscr{A}$. □

1.5 The Banach fixed point theorem

The result in this section is needed to prove the existence of square roots in Banach algebra. The definitions and the theorem is found in [4].

Definition 1.23. Let $\mathfrak{X}$ be a metric space and let $T\colon \mathfrak{X} \to \mathfrak{X}$. A point $x \in \mathfrak{X}$ is called a fixed point of T in $\mathfrak{X}$ if $Tx = x$.

Definition 1.24. Let $\mathfrak{X}$ be a metric space. A mapping $T\colon \mathfrak{X} \to \mathfrak{X}$ is a contraction mapping if there exists a positive real $\lambda < 1$ such that

$$d(Tx, Ty) \leq \lambda d(x, y), \qquad \text{for all } x, y \in \mathfrak{X}.$$

It is easy to see that contraction mappings are continuous.

Theorem 1.25. *Let $\mathfrak{X}$ be a non-empty, complete metric space and let $T\colon \mathfrak{X} \to \mathfrak{X}$ be a contraction mapping. Then T has exactly one fixed point in $\mathfrak{X}$, it is given by*

$$x = \lim_{n\to\infty} T^n y,$$

for some arbitrary $y \in \mathfrak{X}$.

Proof. First we want to show that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for each $n \geq N$ and $k \geq 0$

$$d(T^n y, T^{n+k} y) < \epsilon,$$

because this makes $(T^n y)_{n\in\mathbb{N}}$ a Cauchy sequence. Since T is a contraction mapping,

$$d(T^m y, T^{m+1} y) \leq \lambda d(T^{m-1} y, T^m y) \leq \lambda^2 d(T^{m-2} y, T^{m-1} y) \leq ... \leq \lambda^m d(y, Ty).$$

By the triangle inequality,

$$d(T^n y, T^{n+k} y) \leq d(T^n y, T^{n+1} y) + ... + d(T^{n+k-1} y, T^{n+k} y).$$

Combining the two inequalities and because $|\lambda| < 1$ we get

$$\begin{aligned} d(T^n y, T^{n+k} y) &\leq (\lambda^n + \lambda^{n+1} + ... + \lambda^{n+k-1}) d(y, Ty) \\ &\leq \lambda^n \Big(\sum_{i=0}^{\infty} \lambda^i\Big) d(y, Ty) = \frac{\lambda^n}{1-\lambda} d(y, Ty). \end{aligned}$$

Since $\lim_{n\to\infty} \frac{\lambda^n}{1-\lambda} = 0$ we can choose $N \in \mathbb{N}$ such that for $n \geq \mathbb{N}$, $d(T^n y, T^{n+k} y) < \epsilon$. Thus, $(T^n y)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $\mathfrak{X}$ is a complete metric space $(T^n y)_{n\in\mathbb{N}}$ converges in $\mathfrak{X}$. Let z denote the limit, i.e., $z = \lim_{n\to\infty} T^n y$. By continuity of the contraction mapping,

$$Tz = T \lim_{n\to\infty} T^n y = \lim_{n\to\infty} T^{n+1} y = \lim_{n\to\infty} T^n y = z.$$

Hence, z is a fixed point for T. To prove the uniqueness assume there exists another fixed point z'. We consider the distance between z and z'. For each $n \in \mathbb{N}$, we have $d(z, z') = d(T^n z, T^n z')$, so

$$d(z, z') = \lim_{n\to\infty} d(T^n z, T^n z') \leq \lim_{n\to\infty} \lambda^n d(z, z') = 0,$$

since $|\lambda| < 1$. Hence, $z = z'$ which proves that the fixed point is unique. □

1.6 The square root in a Banach algebra

In this section we prove the existence of the square roots in Banach algebras. The following result is found in [1].

Proposition 1.26. *Let $\mathscr{A}$ be a Banach algebra with norm $\|\cdot\|$ and let $a \in \mathscr{A}$ with $r(a) < 1$. Then there exists a unique element $x \in \mathscr{A}$ with $r(x) < 1$ so that $a = 2x - x^2$.*

Proof. By Proposition 1.22 $r(a) = \inf |a|$, where the infimum is taken over all algebra-norms $|\cdot|$ equivalent to the given norm $\|\cdot\|$. We choose such a norm $|\cdot|_1$ and a real number η with $|a|_1 < \eta < 1$. Let $\mathscr{C}$ denote the least closed subalgebra containing a, and let $\mathscr{C}_\eta = \{x \in \mathscr{C} \mid \ |x|_1 \leq \eta\}$.

We consider the map $T\colon \mathscr{C}_\eta \to \mathscr{C}_\eta$, given by $Tx = \frac{1}{2}(a + x^2)$. Let $y \in \mathscr{C}_\eta$. Since $\mathscr{C}$ is commutative,

$$\begin{aligned}
|Tx - Ty|_1 &= |\tfrac{1}{2}(a + x^2) - \tfrac{1}{2}(a + y^2)|_1 \\
&= \tfrac{1}{2}|x^2 - y^2|_1 \\
&= \tfrac{1}{2}|x^2 - xy + xy - y^2|_1 \\
&= \tfrac{1}{2}|x^2 - yx + xy - y^2|_1 \\
&= \tfrac{1}{2}|(x - y)(x + y)|_1 \\
&\leq \tfrac{1}{2}|x - y|_1|x + y|_1 \\
&\leq \tfrac{1}{2}|x - y|_1 2\eta = \eta|x - y|_1.
\end{aligned}$$

So T is a contraction mapping. By Theorem 1.25 there exists $x \in \mathscr{C}_\eta$ such that $x = Tx$. Thus, $x = \frac{1}{2}(a + x^2)$ which implies that $a = 2x - x^2$. Further, we have

$$r(x) = \inf |x| \leq |\tfrac{1}{2}(a + x^2)|_1 \leq \tfrac{1}{2}(|a|_1 + |x|_1^2) \leq \tfrac{1}{2}(\eta + \eta^2) < 1.$$

For the uniqueness we suppose there exists $y \in \mathscr{A}$ with $a = 2y - y^2$ and $r(y) < 1$. Since x contains in the closed subalgebra generated by a, i.e., $\mathscr{C}$, there exists a sequence $p_n(z) \in \mathbb{C}[z]$ such that for $p_n(a) \in \mathscr{A}$,

$$p_n(a) \to x \quad \text{for} \quad n \to \infty.$$

But y commutes with a, so y must commutes with $q(a)$ for all $q(z) \in \mathbb{C}[z]$, therefore

$$yx = y\big(\lim_{n\to\infty} p_n(a)\big) = \lim_{n\to\infty} yp_n(a) = \lim_{n\to\infty} p_n(a)y = \big(\lim_{n\to\infty} p_n(a)\big)y = xy.$$

Thus, x commutes with y. Then by Lemma 1.19 we know that $r(x+y) \leq r(x)+r(y) < 2$. We may again choose an equivalent norm $|\cdot|_2$ with $|x + y|_2 < 2$, so that since x and y commute,

$$|x - y|_2 = |\tfrac{1}{2}(a + x^2) - \tfrac{1}{2}(a + y^2)|_2 \leq \tfrac{1}{2}|x - y|_2|x + y|_2.$$

If $|x - y| \neq 0$ then $2 \leq |x + y|$ which contradicts $|x + y| < 2$. Hence, $|x - y| = 0$. This proves the uniqueness of x. □

Proposition 1.27. *Let $a \in \mathrm{Sym}(\mathscr{A})$ and $r(a) < 1$. Then there exists a unique $x \in \mathrm{Sym}(\mathscr{A})$ with $2x - x^2 = a$ and $r(x) < 1$.*

Proof. By Proposition 1.26 there exists a unique $x \in \mathscr{A}$ with $2x - x^2 = a$ and $r(x) < 1$. But $a = a^* = (2x - x^2)^* = 2x^* - (x^*)^2$ and by Proposition 1.10 $\sigma(x^*) = \overline{\sigma(x)}$ so $r(x^*) = r(x) < 1$. Therefore by uniqueness of x, $x = x^*$. □

Corollary 1.28. *If $a \in \mathrm{Sym}(\mathscr{A})$ with $r(a) < 1$. Then there exists a unique $h \in \mathrm{Sym}(\mathscr{A})$ with $h^2 = \mathbb{1} - a$ and $r(\mathbb{1} - h) < 1$. Moreover, h is invertible.*

Proof. Let $a \in \mathrm{Sym}(\mathscr{A})$. From Proposition 1.27 we may choose x with $a = 2x - x^2$, and so with $h = \mathbb{1} - x$,

$$h^2 = (\mathbb{1} - x)^2 = \mathbb{1} - 2x + x^2 = \mathbb{1} - a.$$

We have $\mathbb{1} - h = x$ so $r(\mathbb{1} - h) = r(x) < 1$. Since $\sigma(x) \subseteq (-1, 1)$ we get that $\sigma(h) \subseteq (0, 2)$ and so h is invertible. To prove the uniqueness assume there exists $\tilde{h} \in \mathrm{Sym}(\mathscr{A})$ with $\tilde{h}^2 = \mathbb{1} - a$ and $r(\mathbb{1} - \tilde{h}) < 1$, then

$$\mathbb{1} - a = \tilde{h}^2 = (\mathbb{1} - (\mathbb{1} - \tilde{h}))^2 = \mathbb{1} - 2(\mathbb{1} - \tilde{h}) + (\mathbb{1} - \tilde{h})^2,$$

so $a = 2(\mathbb{1} - \tilde{h}) - (\mathbb{1} - \tilde{h})^2$. This implies $x = \mathbb{1} - \tilde{h}$ and $r(\mathbb{1} - \tilde{h}) < 1$. However, we defined $h = \mathbb{1} - x$, so $\tilde{h} = h$. □

2 Shirali-Ford's Theorem

Now we are ready to prove Shirali-Ford's Theorem. This section is based on [2].

Definition 2.1. A Banach *-algebra $\mathscr{A}$ is called *Hermitian*, if every self-adjoint element of $\mathscr{A}$ has real spectrum. It is called *symmetric*, if the spectrum of a^*a is non-negative, for all $a \in \mathscr{A}$.

In the following $\mathscr{A}$ denotes a Banach *-algebra.

Theorem 2.2. *$\mathscr{A}$ is symmetric if and only if $\mathbb{1} + a^*a$ is invertible, for all $a \in \mathscr{A}$.*

Proof. Assume $\mathscr{A}$ is symmetric. Then for $a \in \mathscr{A}$ and $\lambda \in \sigma(a^*a)$ we have $\lambda \in [0, \infty)$, so in particular for $\lambda \in (0, \infty)$, $-\lambda \notin \sigma(a^*a)$ and it follows that $\lambda + a^*a$ is invertible. In particular, $\mathbb{1} + a^*a$ is invertible.

Now assume $a^*a + \mathbb{1}$ is invertible for all $a \in \mathscr{A}$. We observe for $\lambda > 0$ and $a \in \mathscr{A}$ that

$$a^*a + \lambda\mathbb{1} = \lambda(\lambda^{-1}a^*a + \mathbb{1}) = \lambda((a^*\lambda^{-\frac{1}{2}})(\lambda^{-\frac{1}{2}})a + \mathbb{1}) = \lambda((\lambda^{-\frac{1}{2}}a)^*(\lambda^{-\frac{1}{2}}a) + \mathbb{1}).$$

This identity proves that $a^*a + \lambda\mathbb{1}$ is invertible, since $b^*b + \mathbb{1}$ is invertible, for all $b \in \mathscr{A}$ by assumption. Therefore $\sigma(a^*a) \subseteq \mathbb{C} \setminus (-\infty, 0)$.

Let $b \in \mathrm{Sym}(\mathscr{A})$ and set $\mu = \alpha + i\beta$ for $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$. We consider $(b - \mu)^*(b - \mu)$ and $(b - \mu)(b - \mu)^*$,

$$\begin{aligned}(b - \mu)^*(b - \mu) = (b^* - \bar{\mu})(b - \mu) &= (b - \bar{\mu})(b - \mu) \\ &= (b - (\alpha - i\beta))(b - (\alpha + i\beta)) \\ &= b^2 - b(\alpha + i\beta) - b(\alpha - i\beta) + \alpha^2 + \beta^2 \\ &= b^2 - 2b\alpha + \alpha^2 + \beta^2 \\ &= (b - \alpha)^2 + \beta^2.\end{aligned}$$

Likewise, $(b-\mu)(b-\mu)^* = (b-\alpha)^2 + \beta^2$. Since

$$(b-\alpha)^*(b-\alpha) = (b^*-\alpha)(b-\alpha) = (b-\alpha)^2,$$

we know from above $\sigma((b-\alpha)^2) \subseteq \mathbb{C} \setminus (-\infty, 0)$. In particular, $(b-\alpha)^2+\beta^2$ is invertible and so $b-\mu$ is invertible with inverse given by

$$(b-\mu)^{-1} = ((b-\alpha)^2+\beta^2)^{-1}(b-\bar{\mu}) = (b-\bar{\mu})((b-\alpha)^2+\beta^2)^{-1}.$$

This shows for all self-adjoint, $b \in \mathscr{A}$, that $\sigma(b) \subseteq \mathbb{R}$, since $\mu \in \mathbb{C} \setminus \mathbb{R}$ was arbitrary. In particular, $\sigma(a^*a) \subseteq \mathbb{R}$, for all $a \in \mathscr{A}$, and since we already know that $\sigma(a^*a) \in \mathbb{C} \setminus (-\infty, 0)$ we conclude $\sigma(a^*a) \subseteq [0, \infty)$. □

Corollary 2.3. *If $\mathscr{A}$ is symmetric then $\mathscr{A}$ is hermitian.*

Proof. This was actually proved in the proof of Lemma 2.2. □

Definition 2.4. Let $\mathscr{A}$ be a Hermitian algebra, then we define a real-valued function $s\colon \mathscr{A} \to \mathbb{R}$ by

$$s(a) = (r(a^*a))^{\frac{1}{2}}, \qquad a \in \mathscr{A}.$$

Lemma 2.5. *Let $a \in \mathscr{A}$. If $(\mathbb{1}-a)(\mathbb{1}+a^*)$ and $(\mathbb{1}+a^*)(\mathbb{1}-a)$ are invertible, then $\mathbb{1}-a$ is invertible.*

Proof. Let $a \in \mathscr{A}$ with $(\mathbb{1}-a)(\mathbb{1}+a^*)$ and $(\mathbb{1}+a^*)(\mathbb{1}-a)$ be invertible. Then there exists $a_1, a_2 \in \mathscr{A}$ such that

$$a_1(\mathbb{1}+a^*)(\mathbb{1}-a) = (\mathbb{1}-a)(\mathbb{1}+a^*)a_2 = \mathbb{1}.$$

Set $b_1 = a_1(\mathbb{1}+a^*)$ and $b_2 = (\mathbb{1}+a^*)a_2$. Then we see that

$$b_1(\mathbb{1}-a) = (\mathbb{1}-a)b_2 = \mathbb{1}.$$

Therefore $b_2 = \mathbb{1}b_2 = b_1(\mathbb{1}-a)b_2 = b_1\mathbb{1} = b_1$, so it follows that $\mathbb{1}-a$ is invertible. □

Lemma 2.6. *Let $\mathscr{A}$ be a Hermitian algebra and let $a \in \mathscr{A}$. If $s(a) < 1$ then $1 \notin \sigma(a)$.*

Proof. We need to show that $\mathbb{1}-a$ is invertible. By Lemma 2.5 it suffices to show $(\mathbb{1}+a^*)(\mathbb{1}-a)$ and $(\mathbb{1}-a)(\mathbb{1}+a^*)$ are invertible. We only show $(\mathbb{1}+a^*)(\mathbb{1}-a)$ is invertible since the proof is analogous for $(\mathbb{1}-a)(\mathbb{1}+a^*)$.

We consider $(\mathbb{1}+a^*)(\mathbb{1}-a) = \mathbb{1}-a^*a-a+a^*$. By Corollary 1.28 there exist an invertible element $h \in \mathscr{A}$ with $h = h^*$ and $h^2 = \mathbb{1}-a^*a$.

Therefore

$$(\mathbb{1}+a^*)(\mathbb{1}-a) = h^2+a^*-a = h(\mathbb{1}+h^{-1}(a^*-a)h^{-1})h.$$

So we need to show that $\mathbb{1}+h^{-1}(a^*-a)h^{-1}$ is invertible or equivalently that $0 \notin \sigma(\mathbb{1}+h^{-1}(a^*-a)h^{-1})$. We observe that

$$(h^{-1}(a^*-a)h^{-1})^* = ((h^{-1})^*(a-a^*)(h^{-1})^* = -(h^{-1}(a^*-a)h^{-1}).$$

This implies that $(ih^{-1}(a^*-a)h^{-1})^* = ih^{-1}(a^*-a)h^{-1}$. Since $\mathscr{A}$ is Hermitian, all self-adjoint elements have real spectrum, therefore

$$\sigma(ih^{-1}(a^*-a)h^{-1}) \subseteq \mathbb{R} \qquad \text{so} \qquad \sigma(h^{-1}(a^*-a)h^{-1}) \subseteq i\mathbb{R}.$$

It follows that $\sigma(\mathbb{1}+h^{-1}(a^*-a)h^{-1}) \subseteq 1+i\mathbb{R}$ again by Theorem 1.8. Thus, $0 \notin \sigma(\mathbb{1}+h^{-1}(a^*-a)h^{-1})$ so we conclude that $\mathbb{1}-a$ is invertible. □

Lemma 2.7. *Let $\mathscr{A}$ be Hermitian. Then $r(a) \leq s(a) = s(a^*)$.*

Proof. By Lemma 1.16 we know $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$ for all $a, b \in \mathscr{A}$, therefore $r(a^*a) = r(aa^*)$, and so

$$s(a) = (r(a^*a))^{\frac{1}{2}} = (r(aa^*))^{\frac{1}{2}} = s(a^*).$$

First let us prove that $s(a) < 1$ implies $r(a) < 1$. Suppose $t \in \mathbb{C}$ with $|t| \leq 1$. If $s(a) < 1$ then

$$s(ta) = (r((ta)^*ta))^{\frac{1}{2}} = (r(|t|^2 a^*a))^{\frac{1}{2}} = (|t|^2 r(a^*a))^{\frac{1}{2}} = |t|s(a) < 1.$$

So by Lemma 2.6 $1 \notin \sigma(ta)$, therefore $t^{-1} \notin \sigma(a)$ since $\sigma(ta) = t\sigma(a)$ by Theorem 1.8. This means for $|\lambda| \geq 1$ then $\lambda \notin \sigma(a)$, hence $\sigma(a) \subseteq \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$ and it follows that $r(a) < 1$.

Consider $a \in \mathscr{A}$. For $t > 0$ with $t > s(a)$ we have $s(t^{-1}a) = t^{-1}s(a) < 1$. By the first part of the proof we conclude that $r(t^{-1}a) < 1$ and therefore $r(a) < t$, since $r(t^{-1}a) = t^{-1}r(a)$. So if $t > 0$ with $t > s(a)$ then $r(a) < t$. Thus, $r(a) \leq s(a)$. □

Lemma 2.8. *Let $\mathscr{A}$ be Hermitian and let $h, k \in \mathrm{Sym}(\mathscr{A})$. Then $r(hk) \leq r(h)r(k)$.*

Proof. Let $h, k \in \mathrm{Sym}(\mathscr{A})$. From Lemma 2.7 we have

$$r(hk) \leq s(hk) = (r((hk)^*hk))^{\frac{1}{2}} = (r(k^*h^*hk))^{\frac{1}{2}} = (r(khhk))^{\frac{1}{2}}.$$

Since $\sigma(k(h^2k)) \setminus \{0\} = \sigma((h^2k)k) \setminus \{0\}$, we get

$$r(hk) \leq (r(khhk))^{\frac{1}{2}} = (r(h^2k^2))^{\frac{1}{2}}.$$

Now we prove by induction that, for all $n \in \mathbb{N}$, $r(hk) \leq (r(h^{2^n}k^{2^n}))^{2^{-n}}$. We have already proved that the statement is true for $n = 1$. Now assume the statement is true for $n - 1$ and let us prove that it is true for n. By the same argument as before, we have

$$\begin{aligned} r(h^{2^{n-1}}k^{2^{n-1}}) \leq s(h^{2^{n-1}}k^{2^{n-1}}) &= (r(k^{2^{n-1}}h^{2^{n-1}}h^{2^{n-1}}k^{2^{n-1}}))^{\frac{1}{2}} \\ &= (r((hh)^{2^{n-1}}(kk)^{2^{n-1}})^{\frac{1}{2}} \\ &= (r(h^{2 \cdot 2^{n-1}}k^{2 \cdot 2^{n-1}})^{\frac{1}{2}} \\ &= (r(h^{2^n}k^{2^n}))^{\frac{1}{2}}. \end{aligned}$$

By induction hypotheses we have $r(hk) \leq (r(h^{2^{n-1}}k^{2^{n-1}}))^{2^{-(n-1)}}$, therefore

$$\begin{aligned} r(hk) \leq (r(h^{2^{n-1}}k^{2^{n-1}}))^{2^{-(n-1)}} &\leq \left((r(h^{2^n}k^{2^n}))^{2^{-1}}\right)^{2^{-(n-1)}} \\ &= (r(h^{2^n}k^{2^n}))^{2^{-n}}. \end{aligned}$$

Thus, the statement is true for all $n \in \mathbb{N}$. Further, we know from Lemma 1.11 that $\sigma(h^{2^n}k^{2^n})$ is contained in the closed disc with centre 0 and radius $\|h^{2^n}k^{2^n}\|$, therefore

$$r(h^{2^n}k^{2^n}) \leq \|h^{2^n}k^{2^n}\|.$$

Finally, we see that

$$r(hk) \leq (r(h^{2^n}k^{2^n}))^{2^{-n}} \leq \|h^{2^n}k^{2^n}\|^{2^{-n}} \leq \|h^{2^n}\|^{2^{-n}}\|k^{2^n}\|^{2^{-n}}.$$

So if we let $n \to \infty$, we get by Theorem 1.18 that $r(hk) \leq r(h)r(k)$. □

Definition 2.9. If $\mathscr{A}$ is a Banach *-algebra and $a \in \mathrm{Sym}(\mathscr{A})$ has non-negative spectrum, then a is said to be *positive*. We denote the set of positive elements in $\mathscr{A}$ by $\mathrm{Pos}(\mathscr{A})$.

Lemma 2.10. *Let $\mathscr{A}$ be Hermitian. If $h, k \in \mathrm{Pos}(\mathscr{A})$ then $h + k \in \mathrm{Pos}(\mathscr{A})$.*

Proof. Assume that $h, k \in \mathrm{Pos}(\mathscr{A})$. It is clear that $h + k \in \mathrm{Sym}(\mathscr{A})$. Let us prove that $-1 \notin \sigma(h+k)$. Since h and k have non-negative spectrum $\mathbb{1}+h$ and $\mathbb{1}+k$ are invertible. We define two self-adjoint elements $u, v \in \mathscr{A}$ by

$$u = h(\mathbb{1} + h)^{-1} \quad \text{and} \quad v = k(\mathbb{1} + k)^{-1}.$$

First we show $\mathbb{1} - uv$ is invertible. Observe that

$$\begin{aligned} u = h(\mathbb{1} + h)^{-1} = (\mathbb{1} + h - \mathbb{1})(\mathbb{1} + h)^{-1} &= (\mathbb{1} + h)(\mathbb{1} + h)^{-1} - (\mathbb{1} + h)^{-1} \\ &= \mathbb{1} - (\mathbb{1} + h)^{-1}. \end{aligned}$$

By assumption $\sigma(h) \subseteq [0, \infty)$ and by Theorem 1.8 this implies that $\sigma(\mathbb{1} + h) \subseteq [1, \infty)$. By Proposition 1.9 we have

$$\sigma((\mathbb{1} + h)^{-1}) = (\sigma(\mathbb{1} + h))^{-1},$$

and therefore $\sigma((\mathbb{1} + h)^{-1}) \subseteq (0, 1]$. In this way we get

$$\sigma(u) = \sigma(\mathbb{1} - (\mathbb{1} + h)^{-1}) \subseteq [0, 1),$$

i.e, $r(u) < 1$. Similarly, $r(v) < 1$ and by Lemma 2.8 we get that $r(uv) < 1$. Therefore $1 \notin \sigma(uv)$, so that $\mathbb{1} - uv$ is invertible. Now we know that $\mathbb{1} + h$, $\mathbb{1} + k$ and $\mathbb{1} + uv$ are invertible, so consider the product of the three invertible elements

$$\begin{aligned} (\mathbb{1} + h)(\mathbb{1} - uv)(\mathbb{1} + v) &= (\mathbb{1} + h)(\mathbb{1} - h(\mathbb{1} + h)^{-1}k(I + k)^{-1})(\mathbb{1} + k) \\ &= (\mathbb{1} + h)(\mathbb{1} + k) - hk \\ &= \mathbb{1} + h + k. \end{aligned}$$

Hence, $\mathbb{1} + h + k$ is invertible, and thereby $-1 \notin \sigma(h + k)$.

We notice that for $t > 0$ both $t^{-1}h$ and $t^{-1}k$ are positive, so from above we know that

$$-1 \notin \sigma(t^{-1}h + t^{-1}k) = t^{-1}\sigma(h + k),$$

which is equivalent to the fact that $-t \notin \sigma(h + k)$. Since $t > 0$ was arbitrarily $\sigma(h + k) \in [0, \infty)$.

□

Lemma 2.11. *Let $\mathscr{A}$ be Hermitian. Let $h, k \in \mathrm{Sym}(\mathscr{A})$. Then*

$$r(h + k) \leq r(h) + r(k).$$

Proof. Since $\mathscr{A}$ is Hermitian, we have $\sigma(h) \subseteq \mathbb{R}$ and $\sigma(k) \subseteq \mathbb{R}$. In fact

$$\sigma(h) \subseteq [-r(h), r(h)] \qquad \text{and} \qquad \sigma(k) \subseteq [-r(k), r(k)].$$

By adding the spectral radius of each of the elements, we translate their spectrum, i.e.,

$$\sigma(r(h) + h) \subseteq [0, 2r(h)] \qquad \text{and} \qquad \sigma(r(k) + k) \subseteq [0, 2r(k)].$$

Thus, $r(h)+h$ and $r(k)+k$ have non-negative spectrum and by Lemma 2.10 $r(h)+r(k)+(h+k)$ has non-negative spectrum, i.e.,

$$\sigma(r(h)+r(k)+(h+k)) \subseteq [0,\infty).$$

This means

$$\sigma(h+k) \subseteq [-r(h)-r(k),\infty).$$

The same argument applied to $-h$ and $-k$ yields

$$\sigma(h+k) \subseteq (-\infty, r(h)+r(k)].$$

If we combine the two inclusion, we get

$$\sigma(h+k) \subseteq [-r(h)-r(k), r(h)+r(k)].$$

Thus, $r(h+k) \leq r(h)+r(k)$. □

Lemma 2.12. *Let $\mathscr{A}$ be Hermitian. If $a \in \mathscr{A}$ with $s(a) < 1$ then*

$$\sigma(b^*b) = 1 - \sigma((\mathbb{1}+a^*a)^{-1}),$$

where $b = (\mathbb{1}+aa^*)^{-\frac{1}{2}}a$.

Proof. Let $a \in \mathscr{A}$ with $s(a) < 1$. By Lemma 2.7 we have $s(a) = s(a^*)$, therefore

$$r(aa^*) = s(a^*)^2 = s(a)^2 < 1.$$

In particular $-1 \notin \sigma(aa^*)$ so $-\mathbb{1} - aa^*$ is invertible and hence also $\mathbb{1}+aa^*$.

By Corollary 1.28 there exists an invertible element $h \in \mathscr{A}$ such that $h^2 = (\mathbb{1}+a^*a)$ and $h = h^*$.
Observe that since $\mathbb{1}+aa^*$ and $h^2 = \mathbb{1}+a^*a$ are invertible, we have

$$\begin{aligned}(\mathbb{1}+aa^*)^{-1}a(\mathbb{1}+a^*a)(\mathbb{1}+a^*a)^{-1} &= (\mathbb{1}+aa^*)^{-1}(a\cdot\mathbb{1}+aa^*a)(\mathbb{1}+a^*a)^{-1} \\ &= (\mathbb{1}+aa^*)^{-1}(\mathbb{1}+aa^*)a(\mathbb{1}+a^*a)^{-1}.\end{aligned}$$

This implies $(\mathbb{1}+aa^*)^{-1}a = a(\mathbb{1}+a^*a)^{-1}$. Now consider b^*b and using what we now know, we get

$$\begin{aligned}b^*b &= ((\mathbb{1}+aa^*)^{-1/2}a)^*(\mathbb{1}+aa^*)^{-1/2}a \\ &= a^*(\mathbb{1}+aa^*)^{-1/2}(\mathbb{1}+aa^*)^{-1/2}a \\ &= a^*(\mathbb{1}+aa^*)^{-1}a \\ &= a^*a(\mathbb{1}+a^*a)^{-1} \\ &= (\mathbb{1}+a^*a-\mathbb{1})(\mathbb{1}+a^*a)^{-1} \\ &= \mathbb{1}-(\mathbb{1}+a^*a)^{-1}.\end{aligned}$$

Hereby, we obtain $\sigma(b^*b) = \sigma(\mathbb{1}-(\mathbb{1}+a^*a)^{-1}) = 1-\sigma((\mathbb{1}+a^*a)^{-1})$. □

Theorem 2.13 (Shirali-Ford's Theorem)**.** *If $\mathscr{A}$ is Hermitian then $\mathscr{A}$ is symmetric.*

Proof. Let $\delta = \inf\{\epsilon \geq 0 \mid \sigma(a^*a) \subseteq [-\epsilon, 1), \forall a \in \mathscr{A} \text{ with } s(a) < 1\}$. Note $\delta \in [0,1]$, since $s(a) < 1$ implies that $\sigma(a^*a) \subseteq (-1,1)$. Let $a \in \mathscr{A}$ with $s(a) < 1$ and let $b = (1 + aa^*)^{-1/2}a$. First we want to show

$$\sigma(b^*b) \subseteq (-\tfrac{1}{2}, \tfrac{1}{2}).$$

Since $s(a) < 1$ then $\sigma(a^*a) \subseteq (-1,1)$ so $\sigma(\mathbb{1} + a^*a) \subseteq (0,2)$. By Proposition 1.9 we have $\sigma((\mathbb{1} + a^*a)^{-1}) = \sigma(\mathbb{1} + a^*a)^{-1}$, therefore

$$\sigma((\mathbb{1} + a^*a)^{-1}) \subseteq (\tfrac{1}{2}, \infty).$$

By Lemma 2.12 $\sigma(b^*b) = 1 - \sigma((\mathbb{1} + a^*a)^{-1})$. Hence,

$$\sigma(b^*b) = 1 - \sigma((\mathbb{1} + a^*a)^{-1}) \subseteq (-\infty, \tfrac{1}{2}).$$

We let $\beta = \max \sigma(b^*b) < \frac{1}{2}$. Then we have $\sigma(b^*b) \subseteq (-\infty, \beta]$ which implies that $\sigma(\beta - b^*b) \subseteq [0, \infty)$. Now let $b = h + ik$ where $h, k \in \mathrm{Sym}(\mathscr{A})$. We consider $bb^* + b^*b$,

$$\begin{aligned} bb^* + b^*b &= (h+ik)(h+ik)^* + (h+ik)^*(h+ik) \\ &= (h+ik)(h-ik) + (h-ik)(h+ik) = 2h^2 + 2k^2. \end{aligned}$$

This implies that $b^*b + \beta = 2h^2 + 2k^2 + (\beta - bb^*)$. Since $\mathscr{A}$ is Hermitian the spectrum of all self-adjoint elements is real and so $\sigma(h), \sigma(k) \subseteq \mathbb{R}$. Hence, $\sigma(h^2) = \sigma(h)^2 \subseteq [0, \infty)$, and therefore $2h^2 \in \mathrm{Pos}(\mathscr{A})$. Similarly, $2k^2 \in \mathrm{Pos}(\mathscr{A})$.

Since $(\beta - bb^*)^* = (\beta - bb^*)$ and $\sigma(\beta - bb^*) \subseteq [0, \infty)$ then $(\beta - bb^*) \in \mathrm{Pos}(\mathscr{A})$. By Lemma 2.10 $(b^*b + \beta) \in \mathrm{Pos}(\mathscr{A})$, so $\sigma(b^*b) \subseteq [-\beta, \infty)$, and because $\beta < \frac{1}{2}$, we have

$$\sigma(b^*b) \subseteq (-\tfrac{1}{2}, \infty).$$

So far we have proved that $\sigma(b^*b) \subseteq (-\frac{1}{2}, \infty)$ and $\sigma(b^*b) \subseteq (-\infty, \frac{1}{2})$. Thus, we have $\sigma(b^*b) \subseteq (-\frac{1}{2}, \frac{1}{2})$ and so $s(b^*b) < \frac{1}{2}$. If we multiply by $2^{1/2}$, we get

$$s((2^{1/2}b^*)(2^{1/2}b)) < 1.$$

So by definition of δ we have $\sigma((2^{1/2}b^*)(2^{1/2}b)) \subseteq [-\delta, 1)$. Hence, $\sigma(b^*b) \subseteq [-\frac{1}{2}\delta, \frac{1}{2})$. Since $b^*b = \mathbb{1} - (\mathbb{1} + a^*a)^{-1}$ by Lemma 2.12 then $(\mathbb{1} - b^*b)^{-1} = \mathbb{1} + a^*a$, which implies $a^*a = (\mathbb{1} - b^*b)^{-1} - \mathbb{1}$. This identity yields

$$\sigma(a^*a) = \sigma((\mathbb{1} - b^*b)^{-1} - \mathbb{1}) = \sigma((\mathbb{1} - b^*b)^{-1}) - 1.$$

We know that $\sigma(\mathbb{1} - b^*b) = 1 - \sigma(b^*b) \subseteq (\frac{1}{2}, 1 + \frac{1}{2}\delta]$, therefore by Proposition 1.9

$$\sigma((\mathbb{1} - b^*b)^{-1}) = \sigma((\mathbb{1} - b^*b))^{-1} \subseteq [(1 + \tfrac{1}{2}\delta)^{-1}, 2).$$

Hence, $\sigma(a^*a) = \sigma((\mathbb{1} - b^*b)^{-1}) - 1 \subseteq [-\frac{1}{2}\delta(1 + \frac{1}{2}\delta)^{-1}, 1) \subseteq [-\frac{1}{2}\delta, 1)$.

Since a was arbitrary and $\delta = \inf\{\epsilon \geq 0 \mid \sigma(a^*a) \subseteq [-\epsilon, 1), \forall a \in \mathscr{A} \text{ with } s(a) < 1\}$ we conclude that $\delta \leq \frac{1}{2}\delta$. Because $\delta \geq 0$ we must have $\delta = 0$. So $\sigma(a^*a) \subseteq [0,1)$, for all $a \in \mathscr{A}$, with $s(a) < 1$.

To prove the general case let $a \in \mathscr{A}$ and choose $\lambda \in (0, \infty)$ such that $s(a) < \lambda$. Then $s(\lambda^{-1}a) = \lambda^{-1}s(a) < 1$. By the first part of the proof $\sigma((\lambda^{-1}a)^*(\lambda^{-1}a)) \subseteq [0,1)$. However,

$$\sigma((\lambda^{-1}a)^*(\lambda^{-1}a)) = \sigma(\lambda^{-2}a^*a) = \lambda^{-2}\sigma(a^*a),$$

and therefore $\sigma(a^*a) \subseteq [0, \lambda^2)$, for all $a \in \mathscr{A}$. Thus, $\mathscr{A}$ is symmetric. □

3 A non-symmetric Banach *-algebra

In this section we show that there exists a non-Hermitian Banach *-algebra and therefore by Shirali-Ford's Theorem the Banach *-algebra is non-symmetric. The strategy is to consider a *-subalgebra of the Banach *-algebra and thereby conclude that the Banach *-algebra is non-symmetric.

3.1 The algebra $\ell^1(G)$ of a group G

Let G be a group. We denote by $\ell^1(G)$ the set of all functions $f\colon G \to \mathbb{C}$ satisfying

$$\sum_{g\in G} |f(g)| < \infty.$$

It is well known that $\ell^1(G)$ is a Banach space with componentwise addition and scalar multiplication and the norm given by

$$\|f\|_1 = \sum_{g\in G} |f(g)| < \infty.$$

For $f, g \in \ell^1(G)$ and $x \in G$ we define the *convolution* of f and g denoted $f \star g$, by

$$(f \star g)(x) = \sum_{y\in G} f(xy^{-1})g(y).$$

The convolution is well-defined, since

$$\begin{aligned}\sum_{x\in G} |(f\star g)(x)| = \sum_{x\in G} \Big|\sum_{y\in G} f(xy^{-1})g(y)\Big| &\le \sum_{x\in G}\sum_{y\in G} |f(xy^{-1})||g(y)| \\ &= \sum_{y\in G} |g(y)| \sum_{x\in G} |f(xy^{-1})| \\ &= \sum_{y\in G} |g(y)| \sum_{z\in G} |f(z)| \\ &= \|g\|_1 \|f\|_1 < \infty,\end{aligned}$$

where we have used the fact that $z \mapsto zy^{-1}$ is a bijection. The inequality implies $f \star g \in \ell^1(G)$ and $\|f \star g\|_1 \le \|f\|_1\|g\|_1$, so $\ell^1(G)$ becomes a Banach algebra with the multiplication $\star$, since one can easily show that $\star$ is associative and distributive with respect to $+$. Let $e \in G$ denote the identity and δ_e the indicator function on e, then

$$(f \star \delta_e)(x) = \sum_{y\in G} f(xy^{-1})\delta_e(y) = f(xe^{-1})\delta_e(e) = f(x),$$

since $\delta_e(y) = 0$ whenever $y \ne e$, so $f \star \delta_e = f$. Similarly, $\delta_e \star f = f$, therefore δ_e is the identity on $\ell^1(G)$. So all in all $\ell^1(G)$ is a Banach algebra. But in fact $\ell^1(G)$ is a Banach *-algebra. We may define an involution on $\ell^1(G)$ by the following; for $f \in \ell^1(G)$ we let $f^*\colon G \to \mathbb{C}$ be given by

$$f^*(g) = \overline{f(g^{-1})}.$$

It is easy to see that $f^* \in \ell^1(G)$ with $\|f^*\|_1 = \|f\|_1$, since

$$\sum_{g\in G} |f^*(g)| = \sum_{g\in G} |\overline{f(g^{-1})}| = \sum_{g\in G} |f(g^{-1})| = \sum_{h\in G} |f(h)| < \infty.$$

To show $\ell^1(G)$ is a Banach *-algebra we need to check that $\ell^1(G)$ satisfies the condition in Definition 1.2. Let $f, g \in \ell^1(G)$. Clearly, $f^{**} = f$. Let $x \in G$ then

$$(\lambda f + \mu g)^*(x) = \overline{(\lambda f + \mu g)(x^{-1})} = \overline{(\lambda f)(x^{-1})} + \overline{(\mu g)(x^{-1})} = \overline{\lambda} f^*(x) + \overline{\mu} g^*(x).$$

Further,

$$\begin{aligned}(f \star g)^*(x) = \overline{(f \star g)(x^{-1})} &= \sum_{y\in G} \overline{f(x^{-1}y^{-1})g(y)} = \sum_{y\in G} \overline{f(x^{-1}y^{-1})}\ \overline{g(y)} \\ &= \sum_{y\in G} \overline{f((yx)^{-1})}\ \overline{g(y)} = \sum_{z\in G} \overline{f(z^{-1})}\ \overline{g(zx^{-1})} \\ &= \sum_{z\in G} f^*(z) g^*(xz^{-1}) = (g^* \star f^*)(x),\end{aligned}$$

where we have substituted $z = yx$. Thus, $\ell^1(G)$ is a Banach *-algebra.

3.2 The free group

Let the generators of $\mathbb{F}_n$ be denoted by $x_1, x_2, \ldots, x_n$. We define a *word* in $\mathbb{F}_n$ to be a finite string of the generators and the inverse of the generators. The composition in $\mathbb{F}_n$ is defined by juxtaposition, e.g.,

$$x_7x_6x_1,\ x_1x_3 \rightsquigarrow x_7x_6x_1x_1x_3 = x_7x_6x_1^2x_3.$$

The empty word, e, denotes the identity. If a word is on the form

$$\ldots xx^{-1} \ldots,$$

then we can cancel out x and x^{-1}. A word is reduced if no cancellation can be made. For each element $g \in \mathbb{F}_n$ there is only one reduced form of a given word, i.e., there are unique $m \in \mathbb{N}$ and $n_i \in \mathbb{Z} \setminus \{0\}$ such that

$$g = x_{i_1}^{n_1} \cdot x_{i_2}^{n_2} \cdots x_{i_m}^{n_m}, \quad \text{where} \quad i_k \neq i_{k+1} \ \text{ for all } \ k = 1, \ldots, m-1.$$

The length of the word g is defined by $|g| = |n_1| + |n_2| + \cdots + |n_m|$ and $|e| = 0$.

3.3 A subalgebra of $\ell^1(\mathbb{F}_n)$

In this subsection we will consider a subalgebra of $\ell^1(\mathbb{F}_n)$. We set
$\mathscr{E}_r = \{g \in \mathbb{F}_n \mid |g| = r\}$. This implies

$$\begin{aligned}&\mathscr{E}_0 = \{e\},\\ &\mathscr{E}_1 = \{x_1, x_2, \ldots, x_n, x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\},\\ &\mathscr{E}_2 = \{x_{i_j} \cdot x_{i_l} \mid x_{i_j}, x_{i_l} \in \mathscr{E}_1,\ x_{i_j} \neq x_{i_l}^{-1}\}\\ &\quad\vdots\\ &\mathscr{E}_r = \{x_{i_1}x_{i_2} \cdots x_{i_r} \mid x_{i_j} \in \mathscr{E}_1, x_{i_j} \neq x_{i_{j+1}}^{-1},\ \text{for all } j = 1, 2, \ldots, r-1\}.\end{aligned}$$

Form this it is seen that the cardinality of $\mathscr{E}_r$ is $|\mathscr{E}_0| = 1$, $|\mathscr{E}_1| = 2n$ and $|\mathscr{E}_r| = 2n(2n-1)^{r-1}$, for $r \geq 2$. Now let $\mathscr{X}_r \in \ell^1(\mathbb{F}_n)$ be the indicator function on $\mathscr{E}_r$ that is,

$$\mathscr{X}_r(g) = \begin{cases} 1, & g \in \mathscr{E}_r \\ 0, & \text{otherwise,} \end{cases}$$

or in other words $\mathscr{X}_r = \sum_{g \in \mathscr{E}_r} \delta_g$, where δ is the indicator function on g. Let us investigate the relations between the $\mathscr{X}_r$'s.

Lemma 3.1. *If $g \in \mathbb{F}_n \setminus \left(\mathscr{E}_{k-1} \bigcup \mathscr{E}_{k+1}\right)$ and $z \in \mathscr{E}_k$ for some $k \geq 1$. Then $gz^{-1} \notin \mathscr{E}_1$.*

Proof. Let $z \in \mathscr{E}_k$. First we consider $g \in \mathbb{F}_n \setminus \left(\mathscr{E}_{k-1} \bigcup \mathscr{E}_k \bigcup \mathscr{E}_{k+1}\right)$. This means $|g| < k-1$ or $|g| > k+1$. First observe, if $x, y \in \mathbb{F}_n$ and we write $x = x_{i_1}^{n_1} \cdot x_{i_2}^{n_2} \cdots x_{i_r}^{n_r}$ and $y = x_{j_1}^{m_1} \cdot x_{j_2}^{m_2} \cdots x_{j_s}^{m_s}$, then

$$\begin{aligned} |x| + |y| &= |x_{i_1}^{n_1} \cdot x_{i_2}^{n_2} \cdots x_{i_r}^{n_r}| + |x_{j_1}^{m_1} \cdot x_{j_2}^{m_2} \cdots x_{j_s}^{m_s}| \\ &\geq |x_{i_1}^{n_1} \cdot x_{i_2}^{n_2} \cdots x_{i_r}^{n_r} \cdot x_{j_1}^{m_1} \cdot x_{j_2}^{m_2} \cdots x_{j_s}^{m_s}| \\ &= |xy|. \end{aligned}$$

This implies the following inequality $|xy| - |y| \leq |x|$. If we let $x = gz^{-1}$ and $y = z$, then

$$|gz^{-1}| \geq |g| - |z|. \tag{1}$$

On the other hand, if we let $x = zg^{-1}$ and $y = g$, then

$$|gz^{-1}| = |zg^{-1}| \geq |z| - |g|. \tag{2}$$

So if $|g| > k+1$ then by (1),

$$|gz^{-1}| \geq |g| - |z| > k + 1 - k = 1,$$

and if $|g| < k-1$ then by (2),

$$|gz^{-1}| \geq |z| - |g| > k - (k-1) = 1.$$

Thus, if $g \in \mathbb{F}_n \setminus \left(\mathscr{E}_{k-1} \bigcup \mathscr{E}_k \bigcup \mathscr{E}_{k+1}\right)$ then $gz^{-1} \notin \mathscr{E}_1$.

We prove the statement for $g \in \mathscr{E}_k$ by induction. Let $k = 0$ such that $g, z \in \mathscr{E}_0$ and therefore $g = z = e$. This implies $gz^{-1} = ee^{-1} = e \in \mathscr{E}_0$. Thus, $gz^{-1} \notin \mathscr{E}_1$. Assume now $k > 0$ and we have proved the statement for $n < k$. Let $g, z \in \mathscr{E}_k$ and we write the reduced form as $g = x_{i_1}^{n_1} \cdot x_{i_2}^{n_2} \cdots x_{i_r}^{n_r}$ and $z = x_{j_1}^{m_1} \cdot x_{j_2}^{m_2} \cdots x_{j_s}^{m_s}$, i.e., $n_1, n_2 \ldots, n_r, m_1, m_2 \ldots, m_s \in \mathbb{Z} \setminus \{0\}$. If $i_r \neq j_s$ then

$$\begin{aligned} |gz^{-1}| &= |x_{i_1}^{n_1} \cdot x_{i_2}^{n_2} \cdots x_{i_r}^{n_r} \cdot x_{j_s}^{-m_s} \cdot x_{j_{s-1}}^{-m_{s-1}} \cdots x_{j_1}^{-m_1}| \\ &= |n_1| + |n_2| + \cdots + |n_r| + |m_s| + \cdots + |m_2| + |m_1| \\ &= 2k \neq 1, \end{aligned}$$

since $k \in \mathbb{N}_0$. Thus, $gz^{-1} \notin \mathscr{E}_1$. If $i_r = j_s$ and n_r and m_s have different signs, then

$$\begin{aligned} |gz^{-1}| &= |n_1| + |n_2| + \cdots + |n_r - m_s| + |m_{s-1}| + \cdots + |m_1| \\ &= |n_1| + |n_2| + \cdots + |n_r| + |m_s| + |m_{s-1}| + \cdots + |m_1|, \end{aligned}$$

and again we get $|gz^{-1}| = 2k$, and therefore $gz^{-1} \notin \mathscr{E}_1$. Now we consider the case when $i_r = j_s$ and n_r and m_s have the same sign. Since $n_r \neq 0$ and $m_s \neq 0$ then

$$|n_r - sign(n_r)| = |n_r - \tfrac{n_r}{|n_r|}| = |\tfrac{n_r}{|n_r|}(|n_r| - 1)| = |\tfrac{n_r}{|n_r|}| \; ||n_r| - 1| = |n_r| - 1.$$

Likewise, $|m_s - sign(m_s)| = |m_s| - 1$. We set $g' = gx_{i_r}^{-sign(n_r)}$, then

$$\begin{aligned} |g'| &= |x_{i_1}^{n_1} \cdot x_{i_2}^{n_2} \cdots x_{i_r}^{n_r} x_{i_r}^{-sign(n_r)}| \\ &= |n_1| + |n_2| + \cdots + |n_{r-1}| + |n_r - sign(n_r)| \\ &= |n_1| + |n_2| + \cdots + |n_{r-1}| + |n_r| - 1 = k - 1. \end{aligned}$$

Now we set $z' = zx_{j_s}^{-sign(m_s)} = zx_{i_r}^{-sign(n_r)}$. Similarly, we get $|z'| = k - 1$. Further,

$$g'(z')^{-1} = gx_{i_r}^{-sign(n_r)}(zx_{i_r}^{-sign(n_r)})^{-1} = gx_{i_r}^{-sign(n_r)}x_{i_r}^{sign(n_r)}z^{-1} = gz^{-1}.$$

Since $g', z' \in \mathscr{E}_{k-1}$ then by the induction hypothesis we get $gz^{-1} \notin \mathscr{E}_1$. Thus, we have proved whenever $z \in \mathscr{E}_k$ and $g \in \mathbb{F} \setminus \left(\mathscr{E}_{k-1} \bigcup \mathscr{E}_{k+1}\right)$ then $gz^{-1} \notin \mathscr{E}_1$. □

The symbol $\mathscr{X}_k^n$ simply means $\underbrace{\mathscr{X}_k \star \mathscr{X}_k \star \mathscr{X}_k \star \cdots \star \mathscr{X}_k}_{n \text{ times}}$.

Lemma 3.2. *Let $\mathscr{X}_r \in \ell^1(\mathbb{F}_n)$. Then for $k \geq 2$ we have the following two identities $\mathscr{X}_1^2 = \mathscr{X}_2 + 2n\mathscr{X}_0$ and $\mathscr{X}_1 \star \mathscr{X}_k = \mathscr{X}_{k+1} + (2n-1)\mathscr{X}_{k-1}$.*

Proof. Let $g \in \mathbb{F}_n$ then $\mathscr{X}_1^2(g) = \sum_{z \in \mathbb{F}_n} \mathscr{X}_1(gz^{-1})\mathscr{X}_1(z)$. If $z \notin \mathscr{E}_1$ then $\mathscr{X}_1(z) = 0$, so we consider

$$\sum_{z \in \mathbb{F}_n} \mathscr{X}_1(gz^{-1})\mathscr{X}_1(z) = \sum_{z \in \mathscr{E}_1} \mathscr{X}_1(gz^{-1})\mathscr{X}_1(z) = \sum_{z \in \mathscr{E}_1} \mathscr{X}_1(gz^{-1}) \cdot 1.$$

If $g = e$ then $\sum_{z \in \mathscr{E}_1} \mathscr{X}_1(gz^{-1}) = \sum_{z \in \mathscr{E}_1} \mathscr{X}_1(z^{-1}) = |\mathscr{X}_1| = 2n$. On the other hand if $g \in \mathscr{E}_2$ then $z \in \mathscr{E}_1$ is uniquely determined such that $gz^{-1} \in \mathscr{E}_1$ and therefore $\sum_{z \in \mathscr{E}_1} \mathscr{X}_1(gz^{-1}) = 1$. By Lemma 3.1 we get that for $g \notin \mathscr{E}_0 \bigcup \mathscr{E}_1$ then $gz^{-1} \notin \mathscr{E}_1$. Hence, $\mathscr{X}_1^2 = |\mathscr{E}_1|\mathscr{X}_1 + \mathscr{X}_2 = \mathscr{X}_2 + 2n\mathscr{X}_0$.

Now we want to prove the other equality. If $g \in \mathbb{F}_n \setminus \left(\mathscr{E}_{k-1} \bigcup \mathscr{E}_{k+1}\right)$ then by Lemma 3.1

$$(\mathscr{X}_1 \star \mathscr{X}_k)(g) = \sum_{z \in \mathscr{E}_k} \mathscr{X}_1(gz^{-1})\mathscr{X}_k(z) = \sum_{z \in \mathscr{E}_k} \mathscr{X}_1(gz^{-1}) \cdot 1 = 0.$$

So let $g \in \mathbb{F}_n$ and $z \in \mathbb{F}_n$ such that $gz^{-1} \in \mathscr{E}_1$. This implies $gz^{-1} = x_j^{\epsilon}$ for $\epsilon \in \{\pm 1\}$ and $j = 1, 2, \ldots, n$. Hence, $z = x_j^{-\epsilon}g$. We define

$$y_g = \{x_j^{\epsilon}g \mid \epsilon \in \{\pm 1\}, \; j = 1, 2, \ldots, n\},$$

so $z \in y_g$. Hereby, we obtain

$$(\mathscr{X}_1 \star \mathscr{X}_k)(g) = \sum_{z \in \mathscr{E}_k} \mathscr{X}_1(gz^{-1})\mathscr{X}_k(z) = \sum_{z \in \mathscr{E}_k \cap y_g} \mathscr{X}_1(gz^{-1}) \cdot 1 = |\mathscr{E}_k \cap y_g|.$$

Now we write $g = x_{i_1}^{n_1} \cdot x_{i_2}^{n_2} \cdots x_{i_r}^{n_r}$ such that $n_1 \neq 0$, and find the elements of y_g which have length k.

If $j \neq i_1$ and $\epsilon \in \{\pm 1\}$, then $|x_j^{\epsilon} g| = |g| + 1$. If $j = i_1$ and $\epsilon = sign(n_1)$ then

$$\begin{aligned} |n_1 + \epsilon| &= |n_1 + \tfrac{n_1}{|n_1|}| = |\tfrac{n_1}{|n_1|}| \; ||n_1| + 1| = |n_1| + 1 \quad \text{and} \\ |n_1 - \epsilon| &= |n_1 - \tfrac{n_1}{|n_1|}| = |\tfrac{n_1}{|n_1|}| \; ||n_1| - 1| = |n_1| - 1, \end{aligned}$$

since $n_1 \neq 0$. This implies

$$\begin{aligned} |x_j^{\epsilon} g| &= |n_1 + \epsilon| + |n_2| + \cdots + |n_r| \\ &= |n_1| + 1 + |n_2| + \cdots + |n_r| = |g| + 1 \quad \text{and} \\ |x_j^{-\epsilon} g| &= |n_1 - \epsilon| + |n_2| + \cdots + |n_r| \\ &= |n_1| - 1 + |n_2| + \cdots + |n_r| = |g| - 1. \end{aligned}$$

So if we combine the three different cases, we get

$$|x_j^{\delta} g| = \begin{cases} |g| + 1, & \text{if } j \neq i_1 \text{ and } \delta \in \{\pm 1\}, \\ |g| + 1, & \text{if } j = i_1 \text{ and } \delta = \frac{n_1}{|n_1|}, \\ |g| - 1, & \text{if } j = i_1 \text{ and } \delta = -\frac{n_1}{|n_1|}. \end{cases}$$

Hence $|\mathscr{E}_{|g|+1} \cap y_g| = 2(n-1)+1 = 2n-1$ and $|\mathscr{E}_{|g|-1} \cap y_g| = 1$. Therefore when $|g| = k+1$ then $|\mathscr{E}_k \cap y_g| = |\mathscr{E}_{|g|-1} \cap y_g| = 1$ and when $|g| = k-1$ then $|\mathscr{E}_k \cap y_g| = |\mathscr{E}_{|g|+1} \cap y_g| = 2n-1$. Thus,

$$(\mathscr{X}_1 \star \mathscr{X}_k)(g) = \sum_{z \in \mathscr{E}_k \cap y_g} \mathscr{X}_1(gz^{-1}) = |\mathscr{E}_k \cap y_g| = \mathscr{X}_{k+1}(g) + (2n-1)\mathscr{X}_{k-1}(g).$$

This proves the statement. □

In the following we let $\mathscr{A}$ denote the algebra generated by $\mathscr{X}_0$ and $\mathscr{X}_1$ and $\mathscr{B}$ the closure of $\mathscr{A}$ with respect to $\|\cdot\|_1$.

Lemma 3.3. *Consider $\mathscr{X}_r \in \ell^1(\mathbb{F}_n)$. Then $\mathscr{A}$ is commutative and*

$$\mathscr{A} = \operatorname{span}\{\mathscr{X}_0, \mathscr{X}_1, \mathscr{X}_2 \ldots\}.$$

*Moreover, $\mathscr{B}$ is a commutative Banach *-subalgebra of $\ell^1(\mathbb{F}_n)$.*

Proof. By Lemma 3.2 we have the following equality

$$\mathscr{X}_1^2 = \mathscr{X}_2 + 2n\mathscr{X}_0 \quad \text{and} \quad \mathscr{X}_1 \star \mathscr{X}_k = \mathscr{X}_{k+1} + (2n-1)\mathscr{X}_{k-1}, \quad \text{for } k \geq 2.$$

This implies $\mathscr{X}_2 = \mathscr{X}_1^2 - 2n\mathscr{X}_0$ and $\mathscr{X}_{k+1} = \mathscr{X}_1 \star \mathscr{X}_k - (2n-1)\mathscr{X}_{k-1}$, therefore $\mathscr{X}_2 \in \mathscr{A}$ and if $\mathscr{X}_{k-1}, \mathscr{X}_k \in \mathscr{A}$ then $\mathscr{X}_{k+1} \in \mathscr{A}$, so it follows by induction that $\mathscr{X}_n \in \mathscr{A}$ for all $n \in \mathbb{N}$. Thus, $\operatorname{span}\{\mathscr{X}_0, \mathscr{X}_1, \mathscr{X}_2 \ldots\} \subseteq \mathscr{A}$. By same above equalities we see that $\mathscr{X}_1^2 \in \operatorname{span}\{\mathscr{X}_0, \mathscr{X}_1, \mathscr{X}_2 \ldots\}$ and

$$\mathscr{X}_1^3 = \mathscr{X}_1 \star \mathscr{X}_1^2 = \mathscr{X}_1 \star (\mathscr{X}_2 + 2n\mathscr{X}_0) = \mathscr{X}_1 \star \mathscr{X}_2 + 2n\mathscr{X}_1 = \mathscr{X}_3 + (2n-1)\mathscr{X}_1 + 2n\mathscr{X}_1.$$

Thus, $\mathscr{X}_1^3 \in \operatorname{span}\{\mathscr{X}_0, \mathscr{X}_1, \mathscr{X}_2 \ldots\}$. So if we are continuing in this way we will get that $\mathscr{A} \subseteq \operatorname{span}\{\mathscr{X}_0, \mathscr{X}_1, \mathscr{X}_2 \ldots\}$. Hence, $\mathscr{A} = \operatorname{span}\{\mathscr{X}_0, \mathscr{X}_1, \mathscr{X}_2 \ldots\}$. Now we prove the commutativity of $\mathscr{A}$. Let $\mathscr{Z}, \mathscr{Y} \in \mathscr{A}$ then for some $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $\mu_0, \mu_1, \ldots, \mu_m \in \mathbb{C}$ we have

$$\begin{aligned}\mathscr{Z} &= \lambda_0 \mathscr{X}_0 + \lambda_1 \mathscr{X}_1 + \lambda_2 \mathscr{X}_1^2 + \cdots + \lambda_n \mathscr{X}_1^n \quad \text{and} \\ \mathscr{Y} &= \mu_0 \mathscr{X}_0 + \mu_1 \mathscr{X}_1 + \mu_2 \mathscr{X}_1^2 + \cdots + \mu_m \mathscr{X}_1^m.\end{aligned}$$

Hereby, we obtain

$$\begin{aligned}\mathscr{Z} \star \mathscr{Y} &= \Big(\sum_{i=0}^{n} \lambda_i \mathscr{X}_1^i\Big) \star \Big(\sum_{s=0}^{m} \mu_s \mathscr{X}_1^s\Big) \\ &= \sum_{i=0}^{n} \sum_{s=0}^{m} \lambda_i \mu_s \big(\mathscr{X}_1^i \star \mathscr{X}_1^s\big) \\ &= \sum_{i=0}^{n} \sum_{s=0}^{m} \lambda_i \mu_s \big(\mathscr{X}_1^s \star \mathscr{X}_1^i\big) \\ &= \Big(\sum_{s=0}^{m} \mu_s \mathscr{X}_1^s\Big) \star \Big(\sum_{i=0}^{n} \lambda_i \mathscr{X}_1^i\Big) \\ &= \mathscr{Y} \star \mathscr{Z}.\end{aligned}$$

Hence, $\mathscr{A}$ is commutative. Since $\mathscr{B}$ is closed we get $\mathscr{B}$ is a Banach algebra. Further, if $a, b \in \mathscr{B}$ then we choose $a_n, b_n \in \mathscr{A}$ such that $a_n \to a$ and $b_n \to b$ for $n \to \infty$. Then

$$a_n b_n \to ab \in \mathscr{B}.$$

Thus, $\mathscr{B}$ is closed under multiplication. We observe that

$$\|a\|_1 = \sum_{g \in \mathbb{F}_n} |a(g)| = \sum_{g \in \mathbb{F}_n} |\overline{a(g^{-1})}| = \|a^*\|_1.$$

Hereby, we obtain

$$\|a_n^* - a^*\|_1 = \|(a_n - a)^*\|_1 = \|a_n - a\|_1 \to 0, \qquad \text{for } n \to \infty.$$

Thus, $a_n^* \to a^*$ and therefore $a^* \in \mathscr{B}$ since $a_n^* \in \mathscr{A}$. By above $\mathscr{A}$ is commutative and hereby is the subalgebra $\mathscr{B}$ also commutative. Hence, we have proved that $\mathscr{B}$ is a commutative *-subalgebra of $\ell^1(\mathbb{F}_n)$. □

3.4 A recurrence relation

Now we construct a bounded linear character whose values satisfy the recurrence relation below. Let $\omega_k \in \mathbb{C}$ and we define a recurrence relation by

$$\omega_{k+1} - \omega \omega_k + (2n-1)\omega_{k-1} = 0 \tag{1}$$

with initial conditions

$$\omega_1 = \omega \tag{2}$$

$$\omega_2 = \omega^2 - 2n. \tag{3}$$

It is well-known that the general solution to the recurrence relation is determined by the solution of the characteristic equation,

$$r^2 - \omega r + (2n-1). \tag{4}$$

There exists different solutions to the characteristic equation depending on the discriminant given by $d = \omega^2 - 4(2n-1)$. If $d \neq 0$ then the roots are given by

$$r_1 = \frac{\omega}{2} + \sqrt{\frac{\omega^2}{4} - (2n-1)} \qquad \text{and} \qquad r_2 = \frac{\omega}{2} - \sqrt{\frac{\omega^2}{4} - (2n-1)},$$

so the solution to the recurrence relation is given by $\omega_k = c_1 r_1^k + c_2 r_2^k$, where c_1 and c_2 are a fixed pair of constants determined by (1) and (2). If $d = 0$ then the root is given by $r = \frac{\omega}{2}$ and the solution to the recurrence relation is given by $\omega_k = c_1 r^k + c_2 k r^k$.

Remark 3.4. We obtain, that $r_1 + r_2 = \omega$ and $r_1 \cdot r_2 = 2n-1$ as well as $2r = \omega$ and $r^2 = 2n-1$. Conversely, if the roots satisfy these conditions then they are a solution to (4).

Definition 3.5. Let $\mathscr{D}$ be an algebra. A *character* is a non-zero linear map $\phi\colon \mathscr{D} \to \mathbb{C}$, which is multiplicative.

Note, that $\phi(\mathbb{1}) = 1$ since ϕ is non-zero and $\phi(\mathbb{1}) = \phi(\mathbb{1}^2) = \phi(\mathbb{1})^2$.

Lemma 3.6. *Let $\phi\colon \mathscr{A} \to \mathbb{C}$ be linear. Then ϕ is a character if and only if the complex numbers $(\omega_k)_{k\in\mathbb{N}}$ with $\omega_k = \phi(\mathscr{X}_k)$ satisfy the recurrence relation* (1) *with initial conditions* (2) *and* (3).

Proof. Assume ϕ is a character on $\mathscr{A}$ and $\omega_k = \phi(\mathscr{X}_k)$. This implies, that

$$\begin{aligned} \phi(\mathscr{X}_1) = \omega_1 = \omega \quad \text{and} \quad \omega_2 = \phi(\mathscr{X}_2) &= \phi(\mathscr{X}_1^2 - 2n\mathscr{X}_0) \\ &= \phi(\mathscr{X}_1)^2 - 2n\phi(\mathscr{X}_0) \\ &= \omega^2 - 2n, \end{aligned}$$

where we have used Lemma 3.2. Again by Lemma 3.2 we have for $k \geq 2$

$$\begin{aligned} \omega_{k+1} = \phi(\mathscr{X}_{k+1}) &= \phi(\mathscr{X}_1 \star \mathscr{X}_k - (2n-1)\mathscr{X}_{k-1}) \\ &= \phi(\mathscr{X}_1 \star \mathscr{X}_k) - \phi((2n-1)\mathscr{X}_{k-1}) \\ &= \phi(\mathscr{X}_1)\phi(\mathscr{X}_k) - (2n-1)\phi(\mathscr{X}_{k-1}) \\ &= \omega\omega_k - (2n-1)\omega_{k-1}. \end{aligned}$$

Thus, $\omega_{k+1} - \omega\omega_k + (2n-1)\omega_{k-1} = 0$. So $\phi(\mathscr{X}_k) = \omega_k$ satisfy the recurrence relation.

Now we show that if ϕ satisfies the recurrence relation, then ϕ is a character. First we prove that $\phi(\mathscr{X}_1 \star \mathscr{X}_k) = \phi(\mathscr{X}_1)\phi(\mathscr{X}_k)$ for all $k \in \mathbb{N}_0$. For $k = 0$ it is clear and for $k = 1$, we have

$$\phi(\mathscr{X}_1)^2 = \omega^2 = \omega_2 + 2n = \phi(\mathscr{X}_2 + 2n\mathscr{X}_0) = \phi(\mathscr{X}_1 \star \mathscr{X}_1).$$

Now we show that $\phi(\mathscr{X}_1 \star \mathscr{X}_k) = \phi(\mathscr{X}_1)\phi(\mathscr{X}_k)$ for all $k \geq 2$. We know that

$$\phi(\mathscr{X}_{k+1}) - \phi(\mathscr{X}_1)\phi(\mathscr{X}_k) + (2n-1)\phi(\mathscr{X}_{k-1}) = 0,$$

and by the identity $\mathscr{X}_{k+1} = \mathscr{X}_1 \star \mathscr{X}_k - (2n-1)\mathscr{X}_{k-1}$, we get

$$\begin{aligned}
0 &= \phi(\mathscr{X}_{k+1}) - \phi(\mathscr{X}_1)\phi(\mathscr{X}_k) + (2n-1)\phi(\mathscr{X}_{k-1}) \\
&= \phi(\mathscr{X}_1 \star \mathscr{X}_k - (2n-1)\mathscr{X}_{k-1}) - \phi(\mathscr{X}_1)\phi(\mathscr{X}_k) + (2n-1)\phi(\mathscr{X}_{k-1}) \\
&= \phi(\mathscr{X}_1 \star \mathscr{X}_k) - (2n-1)\phi(\mathscr{X}_{k-1}) - \phi(\mathscr{X}_1)\phi(\mathscr{X}_k) + (2n-1)\phi(\mathscr{X}_{k-1}) \\
&= \phi(\mathscr{X}_1 \star \mathscr{X}_k) - \phi(\mathscr{X}_1)\phi(\mathscr{X}_k).
\end{aligned}$$

Hence, $\phi(\mathscr{X}_1 \star \mathscr{X}_k) = \phi(\mathscr{X}_1)\phi(\mathscr{X}_k)$ for all $k \in \mathbb{N}_0$. Now we prove by induction that for $n \geq 1$, $\phi(\mathscr{X}_n \star \mathscr{X}_k) = \phi(\mathscr{X}_n)\phi(\mathscr{X}_k)$ for all $k \geq 1$. By Lemma 3.2 we have for $k \geq 2$ that

$$\begin{aligned}
\mathscr{X}_2 \star \mathscr{X}_k &= (\mathscr{X}_1^2 - 2n\mathscr{X}_0) \star \mathscr{X}_k \\
&= \mathscr{X}_1 \star (\mathscr{X}_1 \star \mathscr{X}_k) - 2n\mathscr{X}_0 \star \mathscr{X}_k \\
&= \mathscr{X}_1 \star (\mathscr{X}_{k+1} + (2n-1)\mathscr{X}_{k-1}) - 2n\mathscr{X}_k \\
&= \mathscr{X}_1 \star \mathscr{X}_{k+1} + (2n-1)\mathscr{X}_1 \star \mathscr{X}_{k-1} - 2n\mathscr{X}_k.
\end{aligned}$$

We apply ϕ to $\mathscr{X}_2 \star \mathscr{X}_k$ and get

$$\begin{aligned}
\phi(\mathscr{X}_2 \star \mathscr{X}_k) &= \phi(\mathscr{X}_1 \star \mathscr{X}_{k+1} + (2n-1)\mathscr{X}_1 \star \mathscr{X}_{k-1} - 2n\mathscr{X}_k) \\
&= \phi(\mathscr{X}_1)\phi(\mathscr{X}_{k+1}) + (2n-1)\phi(\mathscr{X}_1)\phi(\mathscr{X}_{k-1}) - 2n\phi(\mathscr{X}_k) \\
&= \phi(\mathscr{X}_1)(\phi(\mathscr{X}_{k+1}) + (2n-1)\phi(\mathscr{X}_{k-1})) - 2n\phi(\mathscr{X}_k) \\
&= \phi(\mathscr{X}_1)\phi(\mathscr{X}_1)\phi(\mathscr{X}_k) - 2n\phi(\mathscr{X}_k) \\
&= (\phi(\mathscr{X}_1)^2 - 2n)\phi(\mathscr{X}_k) \\
&= \phi(\mathscr{X}_2)\phi(\mathscr{X}_k).
\end{aligned}$$

Now let $n > 2$ and assume $\phi(\mathscr{X}_j \star \mathscr{X}_k) = \phi(\mathscr{X}_j)\phi(\mathscr{X}_k)$, for all $k \geq 2$ and $j < n$, then we prove the statement is true for $j = n$. Since $\mathscr{X}_1$ commutes with all $\mathscr{X}_n$ and $\mathscr{X}_k = \mathscr{X}_1 \star \mathscr{X}_{k-1} - (2n-1)\mathscr{X}_{k-2}$ for $k \geq 3$, then

$$\begin{aligned}
\mathscr{X}_n \star \mathscr{X}_k &= (\mathscr{X}_1 \star \mathscr{X}_{n-1} - (2n-1)\mathscr{X}_{n-2}) \star \mathscr{X}_k \\
&= \mathscr{X}_{n-1} \star (\mathscr{X}_1 \star \mathscr{X}_k) - (2n-1)\mathscr{X}_{n-2} \star \mathscr{X}_k \\
&= \mathscr{X}_{n-1} \star (\mathscr{X}_{k+1} + (2n-1)\mathscr{X}_{k-1}) - (2n-1)\mathscr{X}_{n-2} \star \mathscr{X}_k \\
&= \mathscr{X}_{n-1} \star \mathscr{X}_{k+1} + (2n-1)\mathscr{X}_{n-1} \star \mathscr{X}_{k-1} - (2n-1)\mathscr{X}_{n-2} \star \mathscr{X}_k.
\end{aligned}$$

We apply ϕ to $\mathscr{X}_n \star \mathscr{X}_k$ and use the induction hypothesis and we get

$$\begin{aligned}
\phi(\mathscr{X}_n \star \mathscr{X}_k) &= \phi(\mathscr{X}_{n-1} \star \mathscr{X}_{k+1} + (2n-1)\mathscr{X}_{n-1} \star \mathscr{X}_{k-1} - (2n-1)\mathscr{X}_{n-2} \star \mathscr{X}_k) \\
&= \phi(\mathscr{X}_{n-1})\phi(\mathscr{X}_{k+1}) + (2n-1)\phi(\mathscr{X}_{n-1})\phi(\mathscr{X}_{k-1}) - (2n-1)\phi(\mathscr{X}_{n-2})\phi(\mathscr{X}_k) \\
&= \phi(\mathscr{X}_{n-1})(\phi(\mathscr{X}_{k+1}) + (2n-1)\phi(\mathscr{X}_{k-1})) - (2n-1)\phi(\mathscr{X}_{n-2})\phi(\mathscr{X}_k).
\end{aligned}$$

Since $\omega_{k+1} + (2n-1)\omega_{k-1} = \omega\omega_k$, we get

$$\begin{aligned}
\phi(\mathscr{X}_n \star \mathscr{X}_k) &= \phi(\mathscr{X}_{n-1})(\phi(\mathscr{X}_{k+1}) + (2n-1)\phi(\mathscr{X}_{k-1})) - (2n-1)\phi(\mathscr{X}_{n-2})\phi(\mathscr{X}_k) \\
&= \phi(\mathscr{X}_{n-1})\phi(\mathscr{X}_1)\phi(\mathscr{X}_k) - (2n-1)\phi(\mathscr{X}_{n-2})\phi(\mathscr{X}_k) \\
&= (\phi(\mathscr{X}_{n-1})\phi(\mathscr{X}_1) - (2n-1)\phi(\mathscr{X}_{n-2}))\phi(\mathscr{X}_k) \\
&= \phi(\mathscr{X}_n)\phi(\mathscr{X}_k).
\end{aligned}$$

Thus, $\phi(\mathscr{X}_n \star \mathscr{X}_k) = \phi(\mathscr{X}_n)\phi(\mathscr{X}_k)$ for all $n, k \in \mathbb{N}_0$. At last, we need to prove that if $\mathscr{Y}_1, \mathscr{Y}_2 \in \operatorname{span}\{\mathscr{X}_0, \mathscr{X}_1, \mathscr{X}_2, \ldots\}$ then we have that $\phi(\mathscr{Y}_1 \star \mathscr{Y}_2) = \phi(\mathscr{Y}_1)\phi(\mathscr{Y}_2)$. We write $\mathscr{Y}_1 = \sum_{i=0}^{n} \lambda_i \mathscr{X}_i$ and $\mathscr{Y}_2 = \sum_{j=0}^{m} \mu_j \mathscr{X}_j$ for $n, m \in \mathbb{N}$ and $\lambda_i, \mu_j \in \mathbb{C}$ for $i = 0, 1, 2, \ldots, n$ and $j = 0, 1, 2, \ldots, m$. Then we have the following,

$$\begin{aligned}
\phi(\mathscr{Y}_1 \star \mathscr{Y}_2) &= \phi\big(\big(\sum_{i=0}^{n} \lambda_i \mathscr{X}_i\big) \star \big(\sum_{j=0}^{m} \mu_j \mathscr{X}_j\big)\big) = \phi\big(\sum_{i=0}^{n}\sum_{j=0}^{m} \lambda_i \mu_j (\mathscr{X}_i \star \mathscr{X}_j)\big) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m} \lambda_i \mu_j \phi(\mathscr{X}_i \star \mathscr{X}_j) = \sum_{i=0}^{n}\sum_{j=0}^{m} \lambda_i \mu_j \phi(\mathscr{X}_i) \star \phi(\mathscr{X}_j) \\
&= \sum_{i=0}^{n} \lambda_i \phi(\mathscr{X}_i) \sum_{j=0}^{m} \mu_j \phi(\mathscr{X}_j) = \phi\big(\sum_{i=0}^{n} \lambda_i \mathscr{X}_j\big)\phi\big(\sum_{j=0}^{m} \mu_j \mathscr{X}_j\big) \\
&= \phi(\mathscr{Y}_1)\phi(\mathscr{Y}_2).
\end{aligned}$$

Thus, ϕ is a character. □

Lemma 3.7. *Let ϕ be a character on $\mathscr{A}$. Then with $\omega_k = \phi(\mathscr{X}_k)$, ϕ is bounded if and only if there exists a $C \geq 1$ such that $|\omega_k| \leq C2n(2n-1)^{k-1}$, for all $k \in \mathbb{N}$.*

Proof. Let ϕ be the linear functional which satisfies the recurrence relation (1) with the initial conditions (2) and (3). Then for $N \in \mathbb{N}$ and $\sum_{i=0}^{N} c_i \mathscr{X}_i \in \mathscr{A}$, we have

$$\phi\Big(\sum_{i=0}^{N} c_i \mathscr{X}_i\Big) = \sum_{i=0}^{N} c_i \phi(\mathscr{X}_i) = c_0 \phi(\mathscr{X}_0) + \sum_{i=1}^{N} c_i \phi(\mathscr{X}_i) = c_0 + \sum_{i=1}^{N} c_i \omega_i.$$

If ϕ is bounded then there exists a constant $C \geq 1$ such that

$$\big|c_0 + \sum_{i=1}^{N} c_i \omega_i\big| = \big|\phi\Big(\sum_{i=0}^{N} c_i \mathscr{X}_i\Big)\big| \leq C \big\|\sum_{i=0}^{N} c_i \mathscr{X}_i\big\|_1.$$

However,

$$\big\|\sum_{i=0}^{N} c_i \mathscr{X}_i\big\|_1 = \sum_{g \in \mathbb{F}_n} \big|\sum_{i=0}^{N} c_i \mathscr{X}_i(g)\big| = \sum_{k=0}^{\infty}\big(\sum_{g \in \mathscr{E}_k}\big|\sum_{i=0}^{N} c_i \mathscr{X}_i(g)\big|\big),$$

and for $g \in \mathscr{E}_k$ is $\mathscr{X}_i(g) = 0$ whenever $i \neq k$, therefore

$$\sum_{i=0}^{N} c_i \mathscr{X}_i\ (g) = \begin{cases} c_k, & \text{if } k \in \{0, 1, 2, \ldots, N\} \\ 0, & \text{if } k > N. \end{cases}$$

This implies,

$$\sum_{k=0}^{\infty}\big(\sum_{g \in \mathscr{E}_k}\big|\sum_{i=0}^{N} c_i \mathscr{X}_i(g)\big|\big) = \sum_{k=0}^{N}\big(\sum_{g \in \mathscr{E}_k}\big|\sum_{i=0}^{N} c_i \mathscr{X}_i(g)\big|\big) = \sum_{k=0}^{N}\sum_{g \in \mathscr{E}_k} |c_k| = \sum_{k=0}^{N} |c_k||\mathscr{E}_k|.$$

Hereby, we obtain

$$\begin{aligned}\Big|c_0+\sum_{i=1}^{N}c_i\omega_i\Big| \le C\Big\|\sum_{i=0}^{N}c_i\mathscr{X}_i\Big\|_1 \le C\sum_{k=0}^{N}|c_k||\mathscr{E}_k| &= C(|c_0|+\sum_{k=1}^{N}|c_k||\mathscr{E}_k|)\\ &= C(|c_0|+\sum_{k=1}^{N}|c_k|2n(2n-1)^{k-1}).\end{aligned}$$

If we set

$$c_k=\begin{cases}1, & \text{if } i=k\\ 0, & \text{if } i\neq k,\end{cases}$$

then we get $|\omega_k|\le C2n(2n-1)^{k-1}$. Therefore we have that if ϕ is bounded on $\mathscr{A}$ then $|\omega_k|\le C2n(2n-1)^{k-1}$.

Assume now $|\phi(\mathscr{X}_k)|=|\omega_k|\le C2n(2n-1)^{k-1}$. Let $y=\sum_{k=0}^{N}c_n\mathscr{X}_n\in\mathscr{A}\setminus\{0\}$ and $C\ge 1$ then using above calculation, we get

$$\begin{aligned}\frac{|\phi(y)|}{\|y\|_1}=\frac{|c_0+\sum_{k=1}^{N}c_k\omega_k|}{\|\sum_{k=0}^{N}c_n\mathscr{X}_n\|_1} &= \frac{|c_0+\sum_{k=1}^{N}c_k\omega_k|}{|c_0|+\sum_{k=1}^{N}|c_k|2n(2n-1)^{k-1}}\\ &\le \frac{|c_0|+\sum_{k=1}^{N}|c_k||\omega_k|}{|c_0|+\sum_{k=1}^{N}|c_k|2n(2n-1)^{k-1}}\\ &\le \frac{|c_0|+\sum_{k=1}^{N}|c_k|C2n(2n-1)^{k-1}}{|c_0|+\sum_{k=1}^{N}|c_k|2n(2n-1)^{k-1}}\\ &\le \frac{C(|c_0|+\sum_{k=1}^{N}|c_k|2n(2n-1)^{k-1})}{|c_0|+\sum_{k=1}^{N}|c_k|2n(2n-1)^{k-1}}=C.\end{aligned}$$

Hence, ϕ is bounded. □

In the following we consider the roots of (4) such that the value of ω_k makes ϕ bounded.

Lemma 3.8. *If the roots of the characteristic equation* $r^2-\omega r+(2n-1)=0$ *satisfy*

$$|r|\le 2n-1,$$

then there exists a constant $C\ge 1$ *such that*

$$|\omega_k|\le C2n(2n-1)^{k-1},$$

where ω_k *is the numbers defined by the recurrence relation* (1) *with initial conditions* (2) *and* (3).

Proof. First we consider the recurrence relation when the solution have the form

$$\omega_k=c_1r_1^k+c_2r_2^k.$$

If $|r_i|\le 2n-1$ for $i=1,2$, then we get

$$\begin{aligned}|\omega_k|=|c_1r_1^k+c_2r_2^k| &\le |c_1|(2n-1)^k+|c_2|(2n-1)^k\\ &=(|c_1|+|c_2|)\frac{(2n-1)}{2n}2n(2n-1)^{k-1}.\end{aligned}$$

Thus, we set $C = \max\{1, (|c_1| + |c_2|)^{\frac{(2n-1)}{2n}}\}$ and the statement is proved.

Now we consider the solution to the recurrence relation when the characteristic equation only has one root, i.e., $r = \frac{\omega}{2}$ and the solution have the form

$$\omega_k = c_1 r^k + c_2 k r^k.$$

By Remark 3.4 we get

$$r^2 = \frac{\omega^2}{4} = 2n - 1,$$

which implies $|\omega| = 2\sqrt{2n-1}$ and therefore $|r| = \left|\frac{\omega}{2}\right| = \sqrt{2n-1}$. Hence,

$$|\omega_k| = |c_1 r^k + c_2 k r^k| = |c_1|(2n-1)^{\frac{k}{2}} + |c_2| k (2n-1)^{\frac{k}{2}}.$$

Rewritting $k(2n-1)^{\frac{k}{2}}$ we get

$$k(2n-1)^{\frac{k}{2}} = k(2n-1)^k (2n-1)^{-\frac{k}{2}},$$

and since $k(2n-1)^{-\frac{k}{2}} \to 0$ when $k \to \infty$ then there exists $K > 0$ such that $k(2n-1)^{-\frac{k}{2}} \leq K$ for all $k \in \mathbb{N}$. Therefore

$$\begin{aligned} |\omega_k| &= |c_1(2n-1)^{\frac{k}{2}} + c_2 k (2n-1)^{\frac{k}{2}}| \\ &= |c_1(2n-1)^{\frac{k}{2}} + c_2 k (2n-1)^{-\frac{k}{2}} (2n-1)^k| \\ &\leq (|c_1| + |c_2| K)(2n-1)^k, \end{aligned}$$

where we have used the fact that $(2n-1)^{\frac{k}{2}} \leq (2n-1)^k$. Thus, we set $C = \max\{1, (|c_1| + K|c_2|)^{\frac{(2n-1)}{2n}}\}$ and the statement is proved. □

Theorem 3.9. *Let $\omega \in \mathbb{C}$, $n \in \mathbb{N}$ and set $\Omega = \left\{x + iy \mid (\frac{x}{2n})^2 + (\frac{y}{2n-2})^2 \leq 1\right\}$. Then the following statements are equivalent*

(i) $\omega \in \Omega$,

(ii) Every root, r, of the characteristic equation (4) *satisfy* $|r| \leq 2n - 1$.

Proof. We prove (ii) implies (i) by contraposition. Let $\omega = x + iy$ be fixed and assume $\omega \notin \Omega$, i.e.,

$$\left(\frac{x}{2n}\right)^2 + \left(\frac{y}{2n-2}\right)^2 > 1.$$

Let $u(t)$ and $v(t)$ be the coordinate functions of the given by

$$u(t) = t + \tfrac{2n-1}{t} \qquad \text{and} \qquad v(t) = t - \tfrac{2n-1}{t}.$$

For $t = 2n - 1$ we have $u(t) = 2n$ and $v(t) = 2n - 2$. Notice that for $t \to \infty$ then $u(t) \to \infty$ and $v(t) \to \infty$. Consider now

$$g(t) = \left(\frac{x}{u(t)}\right)^2 + \left(\frac{y}{v(t)}\right)^2.$$

This function is continuous for $t \in [2n-1, \infty)$ since $u(t)$ and $v(t)$ are continuous. We know $g(2n-1) > 1$ and $g(t) \to 0$ for $t \to \infty$, therefore there exists $t_0 \in [2n-1, \infty)$ such that

$$1 = g(t_0) = \Big(\frac{x}{u(t_0)}\Big)^2 + \Big(\frac{y}{v(t_0)}\Big)^2.$$

Thus, there exists $\theta \in [0, 2\pi)$ such that $\frac{x}{u(t_0)} = \cos(\theta)$ and $\frac{y}{v(t_0)} = \sin(\theta)$. So

$$\begin{aligned} x + iy = u(t_0)\frac{x}{u(t_0)} + iv(t_0)\frac{y}{v(t_0)} &= u(t_0)\cos(\theta) + iv(t_0)\sin(\theta) \\ &= (t_0 + \tfrac{2n-1}{t_0})\cos(\theta) + i(t_0 - \tfrac{2n-1}{t_0})\sin(\theta). \end{aligned}$$

Hence, $\omega = (t_0 + \frac{2n-1}{t_0})\cos(\theta) + i(t_0 - \frac{2n-1}{t_0})\sin(\theta)$. Since $e^{iz} = \cos(z) + i\sin(z)$ for $z \in \mathbb{C}$, we obtain

$$\omega = t_0 e^{i\theta} + \tfrac{2n-1}{t_0} e^{-i\theta}.$$

If we set $r_1 = t_0 e^{i\theta}$ and $r_2 = \frac{2n-1}{t_0} e^{-i\theta}$, then

$$r_1 + r_2 = \omega \quad \text{and} \quad r_1 \cdot r_2 = (t_0 e^{i\theta}) \cdot (\tfrac{2n-1}{t_0} e^{-i\theta}) = (2n-1)e^{i\theta - i\theta} = 2n-1.$$

Thus, by Remark 3.4 r_1 and r_2 are solutions to the characteristic equation (4). Additionally, $|r_1| = |t_0 e^{i\theta}| = |t_0| > 2n-1$. So (ii) is not satisfied. Hence, we have proved (ii) implies (i). Now we prove (i) implies (ii). First we assume that ω is in the interior of the ellipse Ω, and set

$$f_1(t) = t + \tfrac{2n-1}{t} \quad \text{and} \quad f_2(t) = t - \tfrac{2n-1}{t}, \qquad \text{for } t \in [\sqrt{2n-1}, 2n-1].$$

Both functions are monotonic increasing in the interval, and we see that f_1 and f_2 are mapping, respectively

$$f_1(t) \in [2\sqrt{2n-1}, 2n] \quad \text{and} \quad f_2(t) \in [0, 2n-2], \qquad \text{for } t \in [\sqrt{2n-1}, 2n-1].$$

So by same argument as above there exists $t \in [\sqrt{2n-1}, 2n-1]$ and $\theta \in [0, 2\pi)$ such that

$$\omega = (t + \tfrac{2n-1}{t})\cos(\theta) + i(t - \tfrac{2n-1}{t})\sin(\theta),$$

which implies $\omega = te^{i\theta} + \frac{2n-1}{t}e^{-i\theta}$. If we set $r_1 = te^{i\theta}$ and $r_2 = \frac{2n-1}{t}e^{-i\theta}$ then again we get that r_1 and r_2 are solutions to the characteristic equation. Hereby, we obtain

$$|r_1| = |te^{i\theta}| = |t| \le 2n-1 \quad \text{and} \quad |r_2| = |\tfrac{2n-1}{t_0} e^{-i\theta}| = |\tfrac{2n-1}{t_0}| \le \sqrt{2n-1} \le 2n-1.$$

Finally, we need to consider when ω is on the boundary of the ellipse Ω. This means for $\theta \in [0, 2\pi)$,

$$\omega = 2n\cos(\theta) + i(2n-2)\sin(\theta) = 2n(\cos(\theta) + i\sin(\theta)) - 2i\sin(\theta)$$

and therefore $\omega = 2ne^{i\theta} - e^{i\theta} + e^{-i\theta} = (2n-1)e^{i\theta} + e^{-i\theta}$. So the solutions of the characteristic equation are

$$r_1 = (2n-1)e^{i\theta} \quad \text{and} \qquad r_2 = e^{-i\theta},$$

which satisfy $r_1 + r_2 = \omega$ and $r_1 \cdot r_2 = 2n-1$. Hence, we get

$$|r_1| = |(2n-1)e^{i\theta}| \le 2n-1 \qquad \text{and} \qquad |r_2| = |e^{-i\theta}| = 1 \le 2n-1.$$

Thus, (i) implies (ii) both when ω is on the boundary and in the interior. □

Theorem 3.10. *Let ϕ be a character. Then the following are equivalent*

(i) ϕ is bounded

(ii) Every root, r, of the characteristic equation (4) satisfy $|r| \leq 2n-1$.

Proof. If $|r| \leq 2n-1$ then by Lemma 3.8 $|\omega_k| \leq C2n(2n-1)^{k-1}$. So ϕ is bounded by Lemma 3.7. Thus, (ii) implies (i).

Now we prove (i) implies (ii). Assume ϕ is bounded. First we prove the statement when the constants c_1 and c_2 of the recurrence relation are non-zero. Without loss of generality we assume $|r_1| \geq |r_2|$. If $|r_1| = |r_2|$ then we know by Remark 3.4 that $r_1 \cdot r_2 = 2n-1$. This implies $|r_1| = |r_2| = \sqrt{2n-1}$. Thus, $|r_1| = |r_2| = \sqrt{2n-1} \leq 2n-1$.

Now consider $|r_1| > |r_2|$. We have

$$\begin{aligned}|\omega_k| = |c_1 r_1^k + c_2 r_2^k| \geq |c_1||r_1|^k - |c_2||r_2|^k &= |c_1||r_1|^k\Big(1 - \frac{|c_2||r_2|^k}{|c_1||r_1|^k}\Big)\\ &= |c_1||r_1|^k\Big(1 - \frac{|c_2|}{|c_1|}\Big(\frac{|r_2|}{|r_1|}\Big)^k\Big).\end{aligned}$$

Since $|r_1| > |r_2|$ then $\frac{|c_2|}{|c_1|}\left(\frac{|r_2|}{|r_1|}\right)^k \to 0$ for $k \to \infty$. So for $\epsilon \in (0,1)$ there exists $k_0 \in \mathbb{N}$ such that

$$\frac{|c_2|}{|c_1|}\left(\frac{|r_2|}{|r_1|}\right)^k \leq \epsilon, \qquad \text{for} \qquad k \geq k_0.$$

Hence $|\omega_k| \geq |c_1||r_1|^k(1-\epsilon)$. This implies $|r_1|^k \leq |\omega_k|\frac{1}{|c_1|(1-\epsilon)}$. However, ϕ is bounded so by Lemma 3.7 $|\omega_k| \leq C2n(2n-1)^{k-1}$. Thus,

$$|r_1|^k \leq (2n-1)^k \tfrac{C'}{|c_1|(1-\epsilon)}, \qquad \text{for} \qquad C' = C\tfrac{2n}{2n-1},$$

which implies $|r_1| \leq (2n-1)\Big(\frac{C'}{|c_1|(1-\epsilon)}\Big)^{\frac{1}{k}}$. Since $\Big(\frac{C'}{|c_1|(1-\epsilon)}\Big)^{\frac{1}{k}} \to 1$ for $k \to \infty$, then

$$|r_2| \leq |r_1| \leq 2n-1.$$

Hence, the statement is proved when the constants c_1 and c_2 are non-zero.

Now we prove the statement when c_1 or c_2 is zero. Without loss of generality we assume $c_2 = 0$. Then we have $\omega_k = c_1 r_1^k$ and therefore we get $\omega = \omega_1 = c_1 r_1$ and $c_1 r_1^2 = \omega_2 = \omega^2 - 2n = (c_1 r_1)^2 - 2n$. Hence, $2n = (c_1^2 - c_1)r_1^2$. We know for $k \geq 2$ that $\omega_{k+1} + (2n-1)\omega_{k-1} - \omega_k\omega = 0$, so for $k = 2$ we get

$$c_1 r_1^3 + (2n-1)c_1 r_1 - (c_1 r_1^2)c_1 r_1 = 0.$$

Since $r_1 \neq 0$ because $r_1 \cdot r_2 = 2n-1$ we can divide the equation by r_1 such that

$$c_1 r_1^2 + (2n-1)c_1 - c_1^2 r_1^2 = 0,$$

which implies $(2n-1)c_1 = (c_1^2 - c_1)r_1^2 = 2n$. Thus, $c_1 = \frac{2n}{2n-1}$. So we obtain

$$\omega = \tfrac{2n}{2n-1} r_1,$$

and therefore $\omega^2 = \left(\frac{2n}{2n-1}\right)^2 r_1^2$. This implies $r_1^2 = \omega^2\left(\frac{2n-1}{2n}\right)^2$. At the same time we have

$$\omega^2 - 2n = \omega_2 = \tfrac{2n}{2n-1} r_1^2.$$

If we use the expression of r_1^2, we get

$$\omega^2 - 2n = \tfrac{2n}{2n-1}\omega^2\left(\tfrac{2n-1}{2n}\right)^2 = \omega^2 \tfrac{2n-1}{2n},$$

which implies $\omega^2\left(1 - \frac{2n-1}{2n}\right) = 2n$. Thus, $\omega^2 = 4n^2$, and therefore $|\omega| = 2n$. Hence,

$$|r_1| = |\omega||\tfrac{2n-1}{2n}| = |2n||\tfrac{2n-1}{2n}| = 2n - 1.$$

Since $r_1 \cdot r_2 = 2n - 1$, then $|r_2| = 1$. Thus, $|r_i| \leq 2n - 1$ for $i = 1, 2$. □

3.5 A non-symmetric Banach *-algebra

Now we are ready to prove that in $\mathscr{B}$ the spectrum of $\mathscr{X}_1$ is non-real.

Proposition 3.11. *Let $\phi\colon \mathscr{A} \to \mathbb{C}$ be linear and bounded character. Then there exists a unique bounded linear extension $\widetilde{\phi}\colon \mathscr{B} \to \mathbb{C}$ such that $\widetilde{\phi}|_{\mathscr{A}} = \phi$. Moreover, $\widetilde{\phi}$ is a character.*

Proof. Since $\mathscr{A}$ is dense in $\mathscr{B}$ then there exists a Cauchy sequence $(x_n) \in \mathscr{A}$ and $x \in \mathscr{B}$ such that $(x_n) \to x$, for $n \to \infty$. By assumption ϕ is bounded so $\phi(x_n)$ is a Cauchy sequence by the following inequality,

$$\|\phi(x_n) - \phi(x_m)\| \leq \|\phi\|_\infty \|x_n - x_m\|.$$

Since $\mathbb{C}$ is complete then $(\phi(x_n))$ has a limit. So we define $\widetilde{\phi} : \mathscr{B} \to \mathbb{C}$ given by $\widetilde{\phi}(x) = \lim_{n\to\infty} \phi(x_n)$ for $x_n \in \mathscr{A}$ with $\lim_{n\to\infty} x_n = x$. This is well-defined by the following; let (y_n) and (x_n) converge to x for $n \to \infty$ then

$$\begin{aligned}
\|\lim_{n\to\infty} \phi(x_n) - \lim_{n\to\infty} \phi(y_n)\| &= \|\lim_{n\to\infty} \phi(x_n - y_n)\| \\
&\leq \|\phi\|_\infty \lim_{n\to\infty} \|x_n - y_n\|_1 \\
&= \|\phi\|_\infty \lim_{n\to\infty} \|x_n - x + x - y_n\|_1 \\
&\leq \|\phi\|_\infty \lim_{n\to\infty} (\|x_n - x\|_1 + \|x - y_n\|_1) = 0.
\end{aligned}$$

Thus, $\widetilde{\phi}$ is independent of choice of sequence and therefore well-defined. Now we show $\widetilde{\phi}$ is bounded and linear. Let $y_n \to y$ and $x_n \to x$, for $n \to \infty$ and let $\mu, \lambda \in \mathbb{C}$, we use the linearity of ϕ to obtain

$$\widetilde{\phi}(\mu y + \lambda x) = \lim_{n\to\infty} \phi(\mu y_n + \lambda x_n) = \lim_{n\to\infty} (\mu\phi(y_n) + \lambda\phi(x_n)) = \mu\widetilde{\phi}(y) + \lambda\widetilde{\phi}(x).$$

Hence, $\widetilde{\phi}$ is linear. The boundedness of $\widetilde{\phi}$ can be seen from the inequality,

$$\|\widetilde{\phi}(x)\| = \|\lim_{n\to\infty} \phi(x_n)\| \leq \|\phi\|_\infty \lim_{n\to\infty} \|x_n\|_1 = \|\phi\|_\infty \|x\|_1.$$

If $x_n \to x \in \mathscr{A}$, for $n \to \infty$, we obtain

$$\widetilde{\phi}(x) = \lim_{n\to\infty} \phi(x_n) = \phi(x),$$

since a bounded linear functional is continuous. Thus, $\widetilde{\phi}|_{\mathscr{A}} = \phi$. To show the multiplicativity of $\widetilde{\phi}$ we use the continuity of the multiplication. Let $(x_n) \to x$ and $(y_n) \to y$ then $(x_n y_n) \to xy$, therefore

$$\widetilde{\phi}(xy) = \lim_{n\to\infty} \phi(x_n y_n) = \lim_{n\to\infty} \phi(x_n) \lim_{n\to\infty} \phi(y_n) = \widetilde{\phi}(x)\widetilde{\phi}(y).$$

To prove the uniqueness of $\widetilde{\phi}$ assume there exists $\widehat{\phi}\colon \mathscr{B} \to \mathbb{C}$ with $\widehat{\phi}(x) = \phi(x)$ for all $x \in \mathscr{A}$. Let $x \in \mathscr{B}$ and let $x_n \to x$ for $x_n \in \mathscr{A}$. Since $\widehat{\phi}|_{\mathscr{A}} = \phi$, we get

$$\|\widehat{\phi}(x) - \phi(x_n)\| = \|\widehat{\phi}(x) - \widehat{\phi}(x_n)\| \le \|\widehat{\phi}\|_\infty \|x - x_n\| \to 0, \quad \text{for } n \to \infty.$$

Thus, $\phi(x_n) \to \widehat{\phi}(x)$. However, $\phi(x_n) \to \widetilde{\phi}(x)$ for $n \to \infty$, so by uniqueness of the limit $\widehat{\phi}(x) = \widetilde{\phi}(x)$ for $x \in \mathscr{B}$. □

A multiplicative linear functional on $\mathscr{B}$ is automatically bounded [3, Proposition 2.22]. This is not necessarily the case for $\mathscr{A}$, but the above proposition shows that a bounded multiplicative linear functional on $\mathscr{A}$ extends to a bounded multiplicative linear functional on $\mathscr{B}$.

Proposition 3.12. *Let $\mathscr{D}$ be a Banach algebra and let ϕ be a character on $\mathscr{D}$. Then $\phi(x) \in \sigma(x)$ for all $x \in \mathscr{D}$.*

Proof. If $y \in \mathrm{Inv}(\mathscr{D})$ then $\phi(y)\phi(y^{-1}) = \phi(yy^{-1}) = \phi(\mathbb{1}) = 1$, which implies $\phi(y) \neq 0$ or in other words if $\phi(y) = 0$ then $y \notin \mathrm{Inv}(\mathscr{D})$. Since

$$\phi(x - \phi(x)\mathbb{1}) = \phi(x) - \phi(\phi(x)\mathbb{1}) = \phi(x) - \phi(x)\phi(\mathbb{1}) = \phi(x) - \phi(x) = 0,$$

then $x - \phi(x)\mathbb{1} \notin \mathrm{Inv}(\mathscr{D})$. This implies $\phi(x) \in \sigma(x)$. □

Theorem 3.13. *The Banach *-algebra, $\mathscr{B}$, is not symmetric. More precisely $\mathscr{X}_1 \in \mathscr{B}$ is self-adjoint with $\left\{x + iy \mid (\frac{x}{2n})^2 + (\frac{y}{2n-2})^2 \le 1\right\} = \Omega \subseteq \sigma_{\mathscr{B}}(\mathscr{X}_1)$.*

Proof. Let $\omega \in \Omega$ and set $\omega_1 = \omega$, $\omega_2 = \omega_1^2 + 2n$ and $\omega_{k+1} = \omega_1\omega_k - (2n-1)\omega_{k+1}$, for $k \ge 2$. Define $\phi\colon \mathscr{A} \to \mathbb{C}$ linearly by $\phi(\mathscr{X}_k) = \omega_k$. This is well-defined since $\mathscr{A} = \mathrm{span}\{\mathscr{X}_k,\ k \in \mathbb{N}_0\}$ and $\mathscr{X}_0, \mathscr{X}_1, \mathscr{X}_2, \ldots$ are linearly independent. By Lemma 3.6 ϕ is a character which is bounded by Theorem 3.9 and Theorem 3.10. According to Proposition 3.11 ϕ extends to a bounded character $\widetilde{\phi}\colon \mathscr{B} \to \mathbb{C}$, so by Lemma 3.12 $\omega = \phi(\mathscr{X}_1) = \widetilde{\phi}(\mathscr{X}_1) \in \sigma_{\mathscr{B}}(\mathscr{X}_1)$. This shows that $\Omega \subseteq \sigma_{\mathscr{B}}(\mathscr{X}_1)$. Thus, $\mathscr{B}$ is not Hermitian since $\Omega \setminus \mathbb{R} \neq \emptyset$ and by Shirali-Ford's Theorem $\mathscr{B}$ is not symmetric. □

In fact one can show that $\Omega = \sigma(\mathscr{X}_1)$. If we denote the set of all bounded multiplicative linear functionals on $\mathscr{A}$ and $\mathscr{B}$ by $\mathrm{M}_{\mathscr{A}}$ and $\mathrm{M}_{\mathscr{B}}$, respectively, then by Theorem 3.9 and Theorem 3.10 we have

$$\Omega = \{\phi(\mathscr{X}_1) \mid \phi|_{\mathscr{A}} \in \mathrm{M}_{\mathscr{A}}\}.$$

One can show that in general

$$\sigma_{\mathscr{B}}(\mathscr{X}_1) = \{\phi(\mathscr{X}_1) \mid \phi \in \mathrm{M}_{\mathscr{B}}\},$$

holds for all Banach algebras [8, Theorem 18.17]. From Theorem 3.13 we have $\Omega \subseteq \sigma_{\mathscr{B}}(\mathscr{X}_1)$. If $\phi \in \mathrm{M}_{\mathscr{B}}$ then $\phi|_{\mathscr{A}} \in \mathrm{M}_{\mathscr{A}}$, so by Theorem 3.9 and Theorem 3.10 $\phi(\mathscr{X}_1) = \phi|_{\mathscr{A}}(\mathscr{X}_1) \subseteq \Omega$. Thus, $\sigma_{\mathscr{B}}(\mathscr{X}_1) = \Omega$.

Now we have shown that $\mathscr{B} \in \ell^1(\mathbb{F}_n)$ is a non-symmetric Banach *-algebra. However, it is possible to prove that a Banach *-algebra which contains a non-symmetric Banach *-subalgebra is itself a non-symmetric algebra, and by that statement $\ell^1(\mathbb{F}_n)$ is a non-symmetric algebra.

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