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MORITA EQUIVALENCE FOR OPERATOR ALGEBRAS

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Let A be a C*-algebra, and let B be a hereditary subalgebra of A, that is, a C*-subalgebra such that if $a \in A$ and if $0 \leq a \leq b$ for some $b \in B$, then $a \in B$. Recall [20] that there is a natural bijection between hereditary subalgebras of A and (closed) left ideals of A given by

$$L \to L \cap L^*, \quad B \to (\text{closed span of } AB).$$

We are interested in the relation between B and the two-sided ideal which B generates. Then for simplicity of notation we may as well replace A by this ideal, that is, only consider full hereditary subalgebras, where by a <u>full</u> hereditary subalgebra we mean one which is contained in no proper two-sided ideal of the containing algebra.

It is more-or-less well-known [20] that the representation theories of A and B are equivalent (when B is full), as follows: to any Hermitian A-module V (the Hilbert space for a non-degenerate *-representation of A) one associates the Hermitian B-module $\overline{BV}$ consisting of the part of V on which B acts non-degenerately, and one notices that A-intertwining operators (that is, module homomorphisms) restrict to B-intertwining operators, so that one obtains in this way a functor from the category of Hermitian A-modules to the category of Hermitian B-modules. This functor is an equivalence of categories. At the level of irreducible representations this is usually seen by discussing the extension of pure states of B to pure states of A [20]. But this approach is not functorial, so let us describe here a slightly different approach which is functorial, and which is important for the rest of our

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discussion. We want to start with a Hermitian B-module W and construct from it in a canonical way a Hermitian A-module. Let $L = AB$ (closed span), the left ideal of A corresponding to B, and view L as a left-A-right-B-bimodule. Form the purely algebraic tensor product $L \otimes_B W$, which is a left A-module, and define a pre-inner-product on it by

$$\langle x \otimes w, y \otimes w_o \rangle = \langle (y^* x) w, w_o \rangle_W ,$$

which makes sense since $y^* x \in B$ for $x, y \in L$. Notice the relation with positive linear functionals. Factoring by the space of vectors of length zero and completing, we obtain a Hilbert space ${}^A W$. It is not difficult to verify that the action of A on $L \otimes_B W$ extends to give a non-degenerate $*$-representation of A on ${}^A W$. If W_1 is another Hermitian B-module and $T \in \mathrm{Hom}_B(W, W_1)$, then $\mathrm{id}_L \otimes T$ gives in an evident way an element of $\mathrm{Hom}_A({}^A W, {}^A W_1)$, and one obtains this way a functor from the category of Hermitian B-modules to the category of Hermitian A-modules. It is not difficult to verify that this functor is the other half of the equivalence between these categories.

Now let C be another full hereditary subalgebra of A, so that the representation theory of C is equivalent to that of A, and so to that of B. The equivalence between the representation theories of C and B can be seen directly as follows. Let $X = CAB$ (the closed linear span), which is a C-B-bimodule. If W is a Hermitian B-module, let ${}^C W$ be the completion of $X \otimes_B W$ for the inner product defined as above. In this way we obtain a functor from Hermitian B-modules to Hermitian C-modules. To go in the other direction, use $X^* = BAC$ in the analogous way to construct ${}^B V = X^* \otimes_C V$ for any Hermitian C-module V. To see that these two functors give an equivalence, one notices, for example, that

$${}^C({}^B V) = X \otimes_B (X^* \otimes_C V),$$

which is naturally isomorphic to V as C-modules by means of the unitary map defined on elementary tensors by

$$x \otimes (y^* \otimes v) \mapsto x y^* v.$$

The concept of strong Morita equivalence involves the recognition that the relation between B and C given by X as above may well occur naturally even when the algebra A is not at first in evidence. To see how this works one must abstract the relations between B,C and X. Of crucial importance is the fact that X is not just a C-B-bimodule, but that it also has inner-products with values in B and C, given by

$$\langle x, y\rangle_C = xy^*, \ \langle x, y\rangle_B = x^*y.$$

These were vital in defining the ordinary inner products on the modules $X\otimes_B W$ and $X^*\otimes_C V$. By a B-valued inner-product on a right B-module X we mean a B-valued sesquilinear form $\langle\ ,\ \rangle_B$, conjugate linear in the first variable, such that

1. $\langle x, x\rangle_B$ is a positive element of B for each $x \in X$,
2. $\langle x, y\rangle_B^* = \langle y, x\rangle_B$ for $x, y \in X$,
3. $\langle x, yb\rangle_B = \langle x, y\rangle_B b$ for $x, y \in X$, $b \in B$.

A C-valued inner-product on a left C-module X is defined analogously except that it is conjugate linear in the second variable and 3 is replaced by

3'. $\langle cx, y\rangle_C = c\langle x, y\rangle_C$ for $x, y \in X$, $c \in C$.

The relation between B, C and X can then be abstracted as follows:

DEFINITION. Let C and B be C*-algebras. By a C-B-<u>equivalence bimodule</u> is meant a C-B-bimodule X on which are defined a C-valued and a B-valued inner-product such that

1. $\langle x, y\rangle_C z = x\langle y, z\rangle_B$ for $x, y, z \in X$,
2. The representation of C on X is a continuous *-representation by operators which are bounded for $\langle\ ,\ \rangle_B$, that is $\langle cx, cx\rangle_B \le \|c\|^2\langle x, x\rangle$ as positive elements of B, etc., and similarly for the right representation of B,
3. The linear span of $\langle X, X\rangle_B$, which is an ideal in B, is dense in B, and similarly for $\langle X, X\rangle_C$.

(For historical reasons [11, 24] such X were at first called "imprimitivity bimodules".)

We say that two C*-algebras B and C are <u>strongly Morita equivalent</u> if there exists a C-B-equivalence bimodule.

It is not difficult to verify [21] that

$$\|x\| = \|\langle x, x\rangle_B\|^{\frac{1}{2}} \ (= \|\langle x, x\rangle_C\|^{\frac{1}{2}})$$

defines a norm on X, and that all the structure extends to the completion of X. From now on we will assume that X is complete for this norm.

Let us now see how to recapture A from a C-B-equivalence bimodule. This is done by means of the "linking" algebra introduced in [5]. Specifically, let A be the algebra of matrices of form

$$\begin{pmatrix} c & x \\ \tilde{y} & b \end{pmatrix} \qquad x,y \in X,\ c \in C,\ b \in B$$

where $\tilde{y}$ means y viewed as an element of $\tilde{X}$, which is X with conjugate linear operations (the analogue of X^* used earlier, with $b\cdot\tilde{y} = yb^*$ etc.). The product, involution and norm on A are defined in the natural way in terms of the operations on B and C, their actions on X and the B-valued and C-valued inner products. Then it is not difficult to verify that A is a C*-algebra, and that B and C, which are hereditary subalgebras in the obvious way, are full.

Before giving some examples, it is useful to remark that, in fact, C is not needed for the above construction, since it is already determined by B and X. Specifically, we will say, following [24], that a right B-module X is a right <u>B-rigged space</u> if it has a B-valued inner-product such that the span of $\langle X, X\rangle_B$ is dense in B. These objects, under various names, have also appeared in the work of Paschke [19], and very recently in the work of Miscenko and Fomenko [17] concerned with the index of elliptic operators over C*-algebras, and the work of Kasparov [14, 15] concerned with K-theory and extensions of C*-algebras. (See also [34, 35].) The condition that the closed span of $\langle X, X\rangle_B$, which is an ideal of B, must coincide with B, is sometimes dropped (but may be recovered by replacing B by this ideal), and X is sometimes required to complete with respect to the norm $\|\langle x, x\rangle_B\|^{\frac{1}{2}}$ used above. One can then think of X as being the analogue of a Hilbert space, and it is then natural to define the algebra, $L(X)$, of "bounded operators" on X as being the algebra of continuous B-module homomorphisms of X into itself which have adjoints with respect to $\langle\ ,\ \rangle_B$ in the evident sense. It is then not difficult to verify that $L(X)$ is a pre-C*-algebra, which will be complete if X is [24]. Carrying further the analogy with Hilbert spaces, one defines the corresponding ideal, $K(X)$, of "compact operators" to be the closed span of the "rank one" operators $\langle x, y\rangle_C$ defined for $x, y \in X$ by

$$\langle x, y\rangle_C z = x\langle y, z\rangle_B.$$

Then $K(X)$ is the appropriate algebra to take for C, since it then follows readily that X is a C-B-equivalence bimodule [24].

If B and C are C*-algebras and if X is a C-B-equivalence bimodule, then the equivalence between the representation theories of B and C which X determines will preserve many properties. To begin with, it can easily be shown [27] that X determines an isomorphism between the lattices of two-sided ideals of B and C, and in particular, a homeomorphism between the primitive ideal spaces of B and C. Furthermore, the equivalence of representation theories will preserve weak containment, CCR-ness, GCR-ness, and also direct integrals to the extent that they make sense.

EXAMPLE 1. Let G be a locally compact group and H a closed subgroup with both unimodular for simplicity of notation. Let G act on G/H, and let $C = C^*(G, G/H)$ be the corresponding transformation group C*-algebra [20]. Let $B = C^*(H)$ be the group C*-algebra of H. We show how these are naturally strongly Morita equivalent [24]. For this purpose we work with the dense subalgebras $C_c(G, G/H)$ and $C_c(H)$ of continuous functions of compact support. We let $X = C_c(G)$, with $C_c(H)$ acting on the right by convolution (viewing elements of $C_c(H)$ as measures on G supported on H), and $C_c(G, G/H)$ acting on the left by

$$(Ff)(x) = \int_G F(y, \dot{x})f(y^{-1}x)dy$$

for $F \in C_c(G, G/H)$ and $f \in C_c(G)$.
For $f, g \in C_c(G)$ let

$$\langle f, g\rangle_B(t) = (f^{*}*g)(t) \quad t \in H$$

$$\langle f, g\rangle_C(x, \dot{y}) = \int_H (f(yt)g^*(t^{-1}y^{-1}x)dt$$

for $x,y \in G$. All this can be completed to give a C-B-equivalence bimodule. The corresponding equivalence of the representation theories is essentially Mackey's imprimitivity theorem, as is discussed in [24]. Notice that the linking algebra for this situation is not at first in evidence.

EXAMPLE 2. Let M be a connected compact space, and let E be a finite-dimensional vector bundle over M on which there is a Hermitian form. Let $B = C(M)$, the C*-algebra of continuous finctions on M, and let X be the space of continuous sections of E. Then X is a B-module in the obvious way, and the Hermitian form on E defines a B-valued inner product on X so that X becomes a B-rigged space. Notice that C is not around at first, but it can be defined as above, and is just the homogeneous C*-algebra [9] corresponding to the field of Hilbert spaces E. Notice also that the linking algebra is not at first in evidence.

This example suggests that more generally one should consider B-rigged spaces over arbitrary C*-algebras to be vector bundles over "non-commutative locally compact spaces", at least if in the commutative case we allow the fibers to vary in dimension (dropping local triviality) and even to be infinite dimensional (so we are really talking about continuous fields of Hilbert spaces [9]), though if one also allows zero-dimensional fibers, then one should drop the requirement that $\langle X, X\rangle_B$ have dense span in B. The appropriate definition of the analogue of the usual locally trivial bundles with finite dimensional fibers is suggested by:

Swan's Theorem [31]. Let M be a compact space and let $B = C(M)$. Let E be a vector bundle over M (locally trivial with finite-dimensional fibers), and let X be the space of continuous cross-sections of E, which is a B-module in the obvious way. Then X is finitely generated and projective as a B-module. Conversely, every finitely generated projective B-module arises in this way from a vector bundle over M. Furthermore, bundle maps correspond to module homomorphisms, so that the category of vector bundles over M is equivalent to the category of finitely generated projective B-modules.

We include here for one direction of this theorem a sketch of a proof which makes somewhat closer contact with C*-algebras than the proofs which appear in the literature at present. So suppose we are given a vector bundle E over M, and let X be the space of continuous sections of E. By a standard argument using a partition of the identity, E can be equipped with a Hermitian form, so that X becomes a B-rigged space (with the span of $\langle X, X\rangle_B$ not dense if there are zero-dimensional fibers). It is easily seen that X is complete for the norm defined above, so that $L(X)$ and its ideal $K(X)$ are C*-algebras. We wish to show that X is a projective finitely-generated B-module. To do this we will apply the following criterion (closely related to theorem II 3.4(3) of [2]), which is equally valid in the non-commutative case:

PROPOSITION. Let B be a C*-algebra with identity element, and let X be a B-rigged space (with the span of $\langle X, X\rangle_B$ perhaps not dense). If $1_X \in K(X)$, so that $K(X) = L(X)$, then X is finitely generated and projective as a B-module.

Proof. For notational simplicity let $C = K(X)$ as above. Then by hypothesis there are two finite sequences, (y_i) and (z_i) of elements of X such that

$$1_X = \Sigma\langle y_i, z_i\rangle_C .$$

Then for any $x \in X$ we have

$$x = 1_X x = \Sigma\langle y_i, z_i\rangle_C x = \Sigma y_i\langle z_i, x\rangle_B,$$

so that X is finitely generated by the y_i. Let n be the length of the sequences (y_i) and (z_i). We now embed X as a direct summand of B^n, showing that X is projective. The embedding map, ϕ, is defined by

$$\phi(x) = (\langle z_i, x\rangle_B)_i \quad x \in X,$$

while the projection, ψ, of B^n onto X is defined by

$$\psi((b_i)) = \Sigma y_i b_i \quad (b_i) \in B^n.$$

It is easily seen that $1_X = \psi\circ\phi$ as required. Q.E.D.

We remark that the condition that $1_X \in K(X)$ also implies that X is self-dual, in the sense that the mapping of X into $\mathrm{Hom}_B(X, B)$ which to $x \in X$ assigns the B-valued functional $y \to \langle x, y\rangle_B$, is surjective [19, 25]. Conversely, one can show that if X is finitely generated and self-dual, then $1_X \in K(X)$. The self-duality condition has been used by Alain Connes in recent lectures concerning these matters.

We return to our proof of one direction of Swan's theorem.

With $B = C(M)$ and X the space of cross-sections of E, we must show that $1_X \in K(X)$. Let (p_i) be a partition of unity each element of which is supported in an open set over which E is trivial. From this triviality it is easily seen that for each i one can find a finite sequence (x_k^i) of elements of X such that

$$p_i \le \Sigma_k \langle x_k^i, x_k^i \rangle_C,$$

where we use here the fact that $B \subseteq L(X)$ since B is commutative, and where the inequality is to be understood as one between positive operators in $L(X) \supseteq K(X) = C$.

It follows that

$$1_X = \Sigma p_i \le \Sigma_{i,k} \langle x_k^i, x_k^i \rangle_C.$$

But the right-hand-side must then be invertible. Thus the ideal $K(X)$ contains an invertible element of $L(X)$ and so must be all of $L(X)$. This concludes the proof.

The point of view that finitely generated projective modules over unital C*-algebras should be thought of as the analogues over "non-commutative compact spaces" of ordinary vector bundles has been used very fruitfully by Alain Connes in his recent announcement [8] concerning differential geometry on "non-commutative compact spaces".

Another indication that viewing B-rigged spaces as non-commutative vector bundles is not an unreasonable point of view, comes as follows. If the B-rigged space is thought of as some kind of vector bundle, then C and B might be pictured (incorrectly) as made up of fiber algebras of different sizes over some space. This suggests that if one tensors both algebras with the algebra, K, of compact operators on a separable Hilbert space one should obtain isomorphic algebras. This works remarkably often. Specifically, let us say that C*-algebras B and C are <u>stably</u> isomorphic if $B \otimes K$ and $C \otimes K$ are isomorphic. Then a theorem of L. Brown [4] gives, by means of the linking algebra as discussed in [5]:

THEOREM. Any two C*-algebras which are stably isomorphic are strongly Morita equivalent. Two C*-algebras which are strongly Morita equivalent, and which both have countable approximate identities, are stably isomorphic.

But it should be noted that in many examples, such as those above and below, there will be no natural isomorphism between $B \otimes K$ and $C \otimes K$.

EXAMPLE 3. The algebras in this example will be at the opposite extreme of those in Example 2, namely, they will be simple (with identity element). Let w be an irrational number, let Z denote the integers as a subgroup of the real numbers, R, and let Zw act on R/Z in the evident way, that is as powers of the irrational rotation of the circle by angle $2\pi w$. Let $B = C^*(Zw, R/Z)$, the corresponding transformation group C^*-algebra, with dense subalgebra $C_c(Zw, R/Z)$. It is known that B is simple. (Discussion of this example, with references, can be found in [29].) Define the structure of a right B-rigged space on $X = C_c(R)$ (or at least its eventual completion) as follows: If $f, g \in C_c(R)$ and $F \in C_c(Zw, R/Z)$ let

$$(fF)(r) = \Sigma_{n\in Z} f(r-nw)F(nw, r-nw)$$

$$\langle f, g\rangle_B(mw, r) = \Sigma_{n\in Z}\overline{f}(r-n)g(r-n+mw),$$

for $r \in R$ and $m \in Z$. If one then calculates what the algebra C must be which is determined by X and B, one finds [26, 29] that it is just $C = C^*(Z, R/Zw)$ with operations given, for $f, g \in X$, $G \in C$, by

$$(Gf)(r) = \Sigma_{n\in Z} G(n, r)f(r-n)$$

$$\langle f,g\rangle_C(m, r) = \Sigma_{n\in Z} f(r-nw)\overline{g}(r-nw-m).$$

(Notice again that the linking algebra is not initially in evidence.) Since C contains the identity operator on X, it follows from the proposition given above that X is a finitely generated projective B-module, and so should be considered a "vector bundle" of the nicest kind over the simple C^*-algebra B. This is an important example in Alain Connes' announcement concerning differential geometry on "non-commutative compact spaces" [7].

It is easily seen that C above is isomorphic to $C^*(Zw^{-1}, R/Z)$. From this it can be shown, more generally, that if v and w are irrational numbers which are in the same orbit for the action of $GL(2, Z)$ by linear fractional transformations, then the corresponding algebras are strongly

Morita equivalent [29]. That the converse is true follows from recent work of Pimsner and Voiculescu [21] and myself [29] in which the images of the K_o groups of these algebras under their unique trace are calculated. (The K_o groups themselves were subsequently calculated by Pimsner and Voiculescu in [22].) Specifically, it is shown that for the algebra B above, the image is just $(Z+Zw)\cap[0, 1]$. This requires showing the existence of many projections in these algebras, and, in fact, it was the finding of the strong Morita equivalences described above which first led me to finding these projections. The connection is the following standard argument from the purely algebraic theory of Morita equivalence and projective modules [1, 2]. Let (y_i) and (z_i) be as in the above proposition so that

$$1_X = \Sigma\langle y_i, z_i\rangle_C.$$

If we let $D = M_n(B)$, the algebra of $n\times n$ matrices over B, and if we view X^n as a C-D-equivalence bimodule in the evident way, and let $y = (y_i)$ and $z = (z_i)$ as elements of X^n, then $\langle y, z\rangle_C = 1_X$. But then a simple calculation shows from this that $\langle y, x\rangle_D$ must be idempotent in D, and so defines an element of $K_o(B)$. So, in the specific example of the irrational rotation algebras, what one must do is to find specific x and y such that $\langle x, y\rangle_B = 1$, and then calculate the corresponding idempotent in a sufficiently explicit way so that its trace can be calculated [29].

As suggested by the first and third examples above, the situations in which the concept of strong Morita equivalence has been most useful up to now involve actions of groups. For the case of induced representations of groups this can be seen in [24, 27]. Strong Morita equivalence plays an important role in Phil Green's work [12, 13] which illuminates, simplifies and extends Pukanszky's deep work on characters of connected Lie groups [23]. (See Green's survey in the section of these proceedings concerned with dynamical systems.) Applications of strong Morita equivalence to the subject of transformation group C*-algebras are also described in the section of these proceedings concerned with dynamical systems (see also [28]) and further results are contained in [32, 33]. An application of strong Morita equivalence to crossed product algebras occurs in

[30]. Most recently, strong Morita equivalence plays an important role in the thesis of Alex Kumjian [16], in which semigroups of local homeomorphisms, and associated C*-algebras, are used to unify and study a number of recent constructions of simple C*-algebras. (See Jean Renault's article in these proceedings.)

Let me also mention some unpublished results of Phil Green. For one thing, he has shown that any C*-algebra which is strongly Morita equivalent to a continuous trace algebra must be a continuous trace algebra, and that any continuous trace algebra is locally strongly Morita equivalent to a commutative C*-algebra. Moreover, if T is a paracompact locally compact space, then the Morita equivalence classes of continuous trace algebras whose primitive ideal space is homeomorphic to T form a kind of Brauer group isomorphic to the cohomology group $H^3(T, Z)$ which is discussed in [9]. Secondly, he has shown that if A and B are AF algebras, then A and B are strongly Morita equivalent if and only if their Elliott groups are isomorphic as ordered groups, but disregarding the hereditary generating subset.

Why have we used the word "strong" in the above discussion of Morita equivalence? In pure algebra, two rings are said to be Morita equivalent if their categories of left modules are equivalent. The fundamental theorem of Morita [18, 1, 2] then says that this is the same as the existence of an equivalence bimodule for the rings (analogous to the equivalence bimodules defined above). For a C*-algebra the appropriate category to use is the category of Hermitian modules. It is then certainly true, for the reasons sketched earlier, that any equivalence bimodule gives an equivalence of the categories of Hermitian modules. But the converse is false for C*-algebras, since the equivalence given by an equivalence bimodule will preserve weak containment of representations and direct integrals of representations, while an arbitrary equivalence need not [25]. Thus we must keep the two concepts distinct. In view of Morita's work, we will say that two C*-algebras are (ordinary) Morita equivalent if their categories of Hermitian modules are equivalent (where we require that the equivalence preserve the *-operation on the morphisms of these categories). But we should hasten to remark that this terminology clashes somewhat with the purely algebraic terminology, for if two C*-algebras both have identity elements, which is the only situation in which purely algebraic Morita equivalence has been defined so far, then these two C*-algebras will be purely algebraically Morita equivalent if and only if they are strongly Morita equivalent as defined above [3].

Let us now explore briefly the consequences of (ordinary) Morita equivalence of C*-algebras. Now the representations of a C*-algebra correspond to the normal representations of its von Neumann enveloping algebra. That is, the category of Hermitian modules over a C*-algebra is canonically isomorphic to the category of normal modules over its von Neumann enveloping algebra, where by a normal module we mean the Hilbert space of a normal non-degenerate representation. It is then natural to say that two von Neumann algebras are Morita equivalent if their categories of normal representations are equivalent, and we see then that the question of (ordinary) Morita equivalence for C*-algebras just becomes a question about Morita equivalence of von Neumann algebras. But for von Neumann algebras the analogue of Morita's fundamental theorem <u>is</u> true, that is, if two von Neumann algebras are Morita equivalent then there exists an equivalence bimodule between them [25]. For von Neumann algebras M and N the definition of an M-N-equivalence bimodule is the same as that given earlier for C*-algebras except that condition 2 there must be strengthened by requiring that for any $x, y \in X$ the functional $M \to \langle mx, y\rangle_M$ is normal, while condition 3 must be weakened by requiring only that the linear span of $\langle X, X\rangle_M$ be w*-dense in M (and similarly for N). Thus we can form the corresponding linking algebra, and this will be a von Neumann algebra as long as X is complete in an appropriate w* sense, which can always be arranged [25]. Thus we see that two von Neumann algebras are Morita equivalent if they occur as reductions of some bigger von Neumann algebra by projections of central carrier 1. This can provide a pleasant point of view. For example, one can <u>define</u> type I von Neumann algebras as being those which are Morita equivalent to (i.e. have the same representation theory as) commutative von Neumann algebras. Recently Walter Beer has shown [3] that similar statements can be made for C*-algebras. In particular, a C*-algebra is of type I if and only if it is Morita equivalent to a commutative C*-algebra, while a separable C*-algebra is nuclear [6] if and only if it is Morita equivalent to an AF C*algebra. (This fact depends on work of Elliott [10], as well as Connes' results describing injective factors [7].

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