EXERCISES FOR MA8107 - FALL 2014

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(1) Let $C^n(\mathbb{T})$ be the space of n-times continuously differentiable functions on the circle $\mathbb{T}$. Show that $C^n(\mathbb{T})$ is a Banach algebra with respect to pointwise multiplication for the following norm:
$$\|f\|_{C^n} := \sum_{j=0}^{n} \frac{\|f^{(j)}\|_\infty}{j!}.$$
(2) Let v be a submultiplicative weight function on the integers $\mathbb{Z}$. Show that there exists a constant $a \geq 0$ such that $v(k) \leq e^{a|k|}$ for all $k \in \mathbb{Z}$.
(3) Show the following: If f is in $C^1(\mathbb{T})$, then f is in the Wiener algebra $\mathcal{W}$ and
$$\|f\|_{\mathcal{W}} \leq \|f\|_\infty + \frac{\pi^2}{3} \sum_{k \neq 0} k^2 |\widehat{f}(k)|^2.$$
Hint: Use that $\widehat{f'}(k) = 2\pi i k \widehat{f}(k)$ and Parseval's Theorem for Fourier series.
(4) Let $\mathcal{A}$ be a Banach algebra and we denote by $\sigma(a)$ the spectrum of an element a in $\mathcal{A}$. Then we have that
$$\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}.$$
(5) Let $a = (a_n)_{n \geq 0}$ be a sequence in $\ell^2(\mathbb{N})$. Then we define the shift operator $Sa = (a_{n+1})$ on $\ell^2(\mathbb{N})$. Determine S^* and compute the spectrum of SS^* and S^*S.
(6) Suppose $\mathcal{A}$ is a unital commutative Banach algebra. Show that the spectral radius $r_{\mathcal{A}}$ is subadditive and submultiplicative.
(7) Show that $\mathcal{A}$ is semisimple if and only if $r_{\mathcal{A}}$ is a norm on $\mathcal{A}$.
(8) Let $\mathcal{A} = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{C}\}$ be the subalgebra of the Banach algebra of 2×2 complex matrices. Show that $\mathcal{A}$ is **not** semisimple.
(9) Let $\mathcal{A}$ be the Banach algebra of absolutely convergent series on $\mathbb{Z}$, $\ell^1(\mathbb{Z})$, with $\|a\|_1 = \sum_{k \in \mathbb{Z}} |a_k|$ as its norm and with convolution as multiplication and with $a^* = (\overline{a_{-k}})$ as involution. Show that $\mathcal{A}$ is not a C^*-algebra.
(10) Show that a positive linear functional on a C^*-algebra is bounded.
(11) Show that a bounded linear functional φ on a C^*-algebra $\mathcal{A}$ is positive if and only if $\lim \varphi(u_\lambda) = \|\varphi\|$ for some bounded approximate unit $(u_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{A}$.

In particular, if $\mathcal{A}$ is unital, then a bounded linear functional φ on $\mathcal{A}$ is positive if and only if $\varphi(1) = \|\varphi\|$.