Proof. Let A' be defined to be A if A has an identity e and $A' = A_e$, the unitisation of A, otherwise. If $x \in A$ is such that $\|x - u\| < 1$, then $e - (u - x)$ is invertible in A' by Lemma 1.2.6. Write $(e - (u - x))^{-1} = y + \lambda e$, where $y \in A$ and $\lambda \in \mathbb{C}$. Then

$$e = \lambda e + y - \lambda u - yu + \lambda x + yx.$$

Towards a contradiction, assume that $x \in I$. If $e \in A$, then

$$e = \lambda e - (\lambda e)u + y - yu + (\lambda e + y)x \in I,$$

which is impossible. If $e \notin A$, then

$$(1 - \lambda)e = y - \lambda u - yu + \lambda x + yx \in A,$$

which forces $\lambda = 1$ and $u = y - yu + x + yx \in I$, a contradiction. Thus x cannot be contained in I. □

As an application of Urysohn's lemma, we now determine all the closed ideals of $C_0(X)$.

Theorem 1.4.6. *Let X be a locally compact Hausdorff space, and for each subset E of X let*

$$I(E) = \{f \in C_0(X) : f(x) = 0 \text{ for all } x \in E\}.$$

Then the map $E \to I(E)$ is a bijection between the collection of nonempty closed subsets of X and the proper closed ideals of $C_0(X)$. Moreover, $I(E)$ is a modular ideal if and only if E is compact, and $I(E) \in \operatorname{Max}(C_0(X))$ if and only if E is a singleton.

Proof. It is clear that $I(E)$ is a closed ideal of $C_0(X)$. Since, given any point $x \in X$, there exists $f \in C_0(X)$ such that $f(x) \neq 0$, it follows that $I(E)$ is proper whenever $E \neq \emptyset$. Moreover, if E is a closed subset of X and $x \in X \setminus E$, then by Urysohn's lemma there exists $f \in C_0(X)$ such that $f|_E = 0$ and $f(x) \neq 0$. This in particular implies that the assignment $E \to I(E)$ is injective.

Now let I be a proper closed ideal and set

$$E = \{x \in X : f(x) = 0 \text{ for all } f \in I\}.$$

Then E is a closed subset of X and $I \subseteq I(E)$. To prove that actually $I = I(E)$, we show first that every $g \in C_c(G)$ with $E \cap \operatorname{supp} g = \emptyset$ belongs to I. To that end, let C be any compact subset of X with $C \cap E = \emptyset$. For every $x \in C$ there exists $h_x \in I$ such that $h_x(x) \neq 0$. Then $|h_x|^2 \in I, |h_x|^2 \geq 0$ and $|h_x|^2(x) > 0$. Because C is compact, there exists a finite subset F of C such that the function h defined by

$$h(y) = \left(\sum_{x \in F} h_x \cdot \overline{h_x}\right)(y) = \sum_{x \in F} |h_x|^2(y), \ y \in X,$$

is strictly positive on C. Note that $h \in I$.

Now, let J be the set of all $g \in C_c(X)$ such that $E \cap \operatorname{supp} g = \emptyset$. By what we have just seen, for any $g \in J$ there exists $h \in I$ such $h(y) > 0$ for all $y \in \operatorname{supp} g$. Define a function f on X by $f(x) = 0$ for $x \in X \setminus \operatorname{supp} g$ and $f(x) = g(x)/h(x)$ for $x \in \operatorname{supp} g$. It is easily verified that f is continuous. Thus $f \in C_0(X)$ and $g = fh \in I$. This shows that $J \subseteq I$, as announced above.

On the other hand, J is dense in $I(E)$. To see this, let $f \in I(E)$ and $\epsilon > 0$ be given and let $C = \{x \in X : |f(x)| \geq \epsilon\}$. Then C is compact and $C \cap E = \emptyset$. Again, by Urysohn's lemma, there exists $h \in C_c(X)$ such that $h(X) \subseteq [0,1]$, $h|_C = 1$ and $\operatorname{supp} h \subseteq X \setminus E$. Then $g = fh \in J$ and $\|f - g\|_\infty \leq \epsilon$.

Since I is closed, combining what we have shown yields that

$$I(E) \subseteq \overline{J} \subseteq \overline{I} = I \subseteq I(E),$$

so that $I(E) = I$. Clearly, $E \neq \emptyset$ since otherwise $C_c(X) \subseteq I$, whence $I = C_0(X)$.

Finally, if E is compact then there exists $u \in C_0(X)$ with $u(x) = 1$ for all $x \in E$, and this shows that $C_0(X)(1-u) \subseteq I(E)$. Conversely, if $I(E)$ is modular, there exists $u \in C_0(X)$ such that $C_0(X)(1-u) \subseteq I(E)$. This implies that $u = 1$ on E and hence E is compact since $u \in C_0(X)$. The remaining assertion concerning maximal modular ideals is now obvious. □

Let G be a locally compact group. The closed ideals of $L^1(G)$ turn out to be nothing but the closed translation invariant subspaces of $L^1(G)$.

Proposition 1.4.7. *A closed linear subspace I of $L^1(G)$ is an ideal in $L^1(G)$ if and only if I is two-sided translation invariant.*

Proof. Suppose that I is two-sided translation invariant. We have to show that $g * f \in I$ and $f * g \in I$ for each $f \in I$ and $g \in L^1(G)$. Let $\varphi \in L^\infty(G)$ be such that $\int_G f(x)\varphi(x)dx = 0$ for all $f \in I$. Then, for $f \in I$ and any $g \in L^1(G)$,

$$\begin{aligned}
\int_G (g * f)(x)\varphi(x)dx &= \int_G \varphi(x)\left(\int_G g(xy)f(y^{-1})dy\right)dx \\
&= \int_G \varphi(x)\left(\int_G g(y)f(y^{-1}x)dy\right)dx \\
&= \int_G g(y)\left(\int_G L_y f(x)\varphi(x)dx\right) \\
&= 0.
\end{aligned}$$

Since $L^1(G)^* = L^\infty(G)$, the Hahn–Banach theorem implies that $g * f \in I$ for all $f \in I$ and $g \in L^1(G)$. Thus I is a left ideal, and using the right translation invariance of I, it is shown in the same way that I is a right ideal.

Conversely, let I be a closed ideal of $L^1(G)$ and $x \in G$. Let V be a symmetric compact neighbourhood of e in G and let $|V|$ denote the Haar measure of V. Then