MA8107 OPERATOR ALGEBRA EXERCISES

EXERCISE 1: Let X be a locally compact non-compact Hausdorff space. Show that $\widetilde{C_0(X)}$ is isomorphic to $C(X \cup \{\infty\})$ where $X \cup \{\infty\}$ is the one-point compactification of X.

EXERCISE 2: Show that each element in a C^*-algebra A has a unique decomposition $x = a + ib$ where a and b are self-adjoint elements in A. (Hint: calculate $(a+ib)^*$.)

EXERCISE 3: Let X be a topological space. Say that $C_0(X)$ *separates points in* X if for each pair of distinct points x_1, x_2 in X there is an $f \in C_0(X)$ such that $f(x_1)$ and $f(x_2)$ are distinct and non-zero. Show that $C_0(X)$ separates points in X if and only if X is a locally compact Hausdorff space.

EXERCISE 4: Let A be a C^*-algebra and let X be a locally compact Hausdorff space. Consider the vector space $C_0(X,A)$ of all continuous functions $f : X \to A$ such that for every $\varepsilon > 0$ there is a compact subset K of X such that $||f(x)|| < \varepsilon$ for every $x \in X \setminus K$. For $f, g \in C_0(X,A)$ define fg and f^* in $C_0(X,A)$ by $(fg)(x) = f(x)g(x)$ and $f^*(x) = (f(x))^*$ for $x \in X$, and set $||f|| = \sup\{||f(x)|| \mid x \in X\}$. Show that $C_0(X,A)$ is a C^*-algebra.

EXERCISE 5: Let A be a unital Banach algebra.

1. Show that $\sigma_A(a+b) \subseteq \sigma_A(a) + \sigma_A(b)$ and $\sigma_A(ab) \subseteq \sigma_A(a)\sigma_A(b)$ for all $a, b \in A$. Show that this is not true for all Banach algebras.
2. Show that if $a \in \text{Inv}(A)$, then $\sigma_A(a) = \{\lambda^{-1} \mid \lambda \in \sigma_A(a)\}$.

EXERCISE 6: Let A be a Banach algebra. Show that the spectral radius function $r : A \to \mathbf{R}$ is upper semi-continuous.