1. Chapter 2.1-2.3 Harmonic functions: Poisson kernel, Hardy-Littlewood Maximal function

1.1. Summation methods and Poisson kernel. We are studying the convergence of the Fourier series of functions $f \in L^1(L^p)$ and in chapter one we tried Cesàro summation method that gives nice convergence in $L^1$. Now we look at Abel summation. For $f \in L^1$ define

$$u_f(r\cdot(e^i\theta)) = \sum_{n} \hat{f}(n)r^{|n|}e(n\theta).$$

Using complex notation $z = re^{i\theta}$ we get

$$u_f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n + \sum_{n=1}^{\infty} \hat{f}(-n)\overline{z}^n.$$

The series converges when $|z| < 1$ and the result is a harmonic function. On the circle of radius $r$ we get

$$u(r\cdot) = f \ast P_r, \quad P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos 2\pi\theta + r^2}.$$

$P_r$ is called the Poisson kernel, it is a positive even function and the family $\{P_r\}$, forms an approximate identity when $r \to 1$.

1.2. Little Hardy spaces. We define

$$h^p(D) = \{u : D \to \mathbb{C}; \Delta u = 0, \sup_{r} \|u(r\cdot)\|_{L^p(T)} < \infty\}.$$

Then the following holds:

(i) If $f \in L^p$ then $u_f \in h^p$ for $1 \leq p \leq \infty$.

(ii) $u \in h^1$ if and only if there exists a Borel measure $\mu$ on $T$ such that $u(r\cdot) = P_r \ast \mu$.

(iii) $u \in h^p$, $1 < p < \infty$ if and only if $u = u_f$ for some $f \in L^p$.

(iv) $u \in h^\infty \cap C(D)$ if and only if $u = u_f$ for some $f \in C(T)$.

To prove (ii) and (iii) we apply the sequential version of the Banach-Alaoglu theorem. It says that closed unit ball of the dual space of a separable normed vector space is sequentially compact in the weak* topology. In our case $L^p(T) = (L^q(T))^*$ where $1/p + 1/q = 1$ when $p > 1$ and $L^1(T) \subset M(T) = (C(T))^*$. The proof consists of taking a dense sequence in the predual space and, using the standard diagonal construction, finding a weakly convergent subsequence from a bounded sequence. Further, we show that in $f \in L^p(T)$ or $\mu \in M(T)$ is the limiting function or measure then

$$u(re^i\theta) - P_r*f(\theta) = P_{r/s}*u_{s} - P_r*f = (P_{r/s} - P_r)*u_{s} + P_r*(u_{s} - f), \quad 0 < r < s.$$
The last term goes to zero since \( u_s \to f \) weakly and \( P_r \in C(\mathbb{T}) \subset L^q(\mathbb{T}) \) and the first term goes to zero since \( \|u_s\|_p \) is uniformly bounded and \( \|P_{r/s} - P_r\|_\infty \to 0 \) as \( s \to 1 \).

1.3. **Hardy-Littlewood maximal function.** Suppose that \( \nu \) is a measure on \( \mathbb{R}^d \). For any \( f \in L^1(d\nu) \) we define the maximal function of \( f \) with respect to \( \nu \) by

\[
M_\nu f(x) = \sup_{B \ni x} \frac{1}{\nu(B)} \int_B |f(y)|d\nu(y).
\]

It is clear that if \( f \in L^\infty(d\nu) \) then \( Mf(x) \leq \|f\|_\infty \).

**Lemma 1.** Assume that \( \nu \) satisfies the doubling property, there exists a constant \( A \) such that

\[
\nu(2B) \leq A \nu(B)
\]

for any ball \( B \in \mathbb{R}^d \). Then for any \( f \in L^1(\nu) \) and \( t > 0 \) the following inequality holds

\[
\nu\{x : M_\nu f > t\} \leq A^2 t^{-1} \|f\|_1.
\]

**Proof.** Let \( K \subset \{x : M_\nu f > t\} \) be a compact set. By the definition of the maximal function \( K \subset \bigcup_{j=1}^N B_j \), where \( \int_{B_j} |f(y)|d\nu(y) > t \nu(B_j) \). We choose disjoint sub-collection of balls \( B_{j'} \) such that \( \bigcup_{j'} 3B_{j'} \supset K \). then

\[
\nu(K) \leq \sum_{j'} \nu(3B_{j'}) \leq A^2 \sum_{j'} \nu(B_{j'}) \leq A^2 t^{-1} \int_{B_{j'}} |f(y)|d\nu(y) \leq A^2 t^{-1} \|f\|_1.
\]

\[ \square \]

1.4. **\( L^p \) and weak-\( L^p \) spaces.** Let \( (X, A, \mu) \) be a measure space. For \( f \) be a measurable function on \( X \), we define its distribution function by

\[
\mu_f(t) = \mu\{x \in X : |f(x)| > t\}.
\]

Then

\[
\|f\|_p^p = \int_0^\infty pt^{p-1}\mu_f(t)dt.
\]

The weak \( L^p \) space is the space of functions \( f \) such that

\[
\mu_f(t) \leq C t^{-p}, \quad t > 0.
\]

The smallest constant \( C \) for which the inequality above holds is called the weak-\( L^p \) norm of \( f \). Clearly an \( L^p \) function belongs to weak-\( L^p \) and \( \|f\|_{\text{weak-} L^p} \leq \|f\|_p \) since

\[
\mu\{x \in X : |f(x)| > t\} \leq t^{-p} \int_X |f|^p d\mu = t^{-p} \|f\|_p^p.
\]
2. Chapter 1.6, 2.4: An interpolation theorem and convergence almost everywhere

2.1. Marcinkiewicz interpolation theorem. Let $D$ be a linear subset of measurable functions on $(X, A, \mu)$ such that $D$ contains all finite linear combinations of characteristic functions of sets of finite measure and if $f \in D$ and $C > 0$ then $\min\{f, C\}$ is also in $D$. We say that an operator $T$ from $D$ to measurable functions on $(Y, B, \nu)$ is sublinear if

\[ (i) \ |T(af)(y)| = a|Tf(y)|, \quad (ii) |T(f_1 + f_2)(y)| \leq |Tf_1(y)| + |Tf_2(y)|. \]

**Theorem 1.** Suppose that $T$ is a sublinear operator such that

\[ \|Tf\|_{\text{weak-}q} \leq C_j \|f\|_{p_j} \]

for $f \in L^{p_j}(X) \cap D$ and $j = 0, 1$, where $q_0 \neq q_1$ and $p_j \leq q_j$. Then

\[ \|Tf\|_{q_t} \leq C_t \|f\|_{p_t}, \]

where $0 < t < 1$ and

\[ \frac{1}{p_t} = \frac{1 - t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1 - t}{q_0} + \frac{t}{q_1}. \]

We will prove it for the case $p_0 = q_0$ and $p_1 = q_1$.

**Proof.** Let $f \in D \cap L^{p_t}$, we want to estimate the distribution function $\mu_{Tf}(t)$.

Assume that $p_0 < p_1$. We fix $t > 0$ and decompose $f$ into sum of two functions $f = f_0 + f_1$, where

\[ f_0 = \begin{cases} 0, & |f| \leq At \\ f, & |f| > At \end{cases}, \quad f_1 = \begin{cases} f, & |f| \leq At \\ 0, & |f| > At \end{cases}. \]

By our assumption $f_1, f_2 \in D$ and $|Tf| \leq |Tf_1| + |Tf_2|$. Then

\[ \nu_{Tf}(t) \leq \nu_{Tf_0}(t/2) + \nu_{Tf_1}(t/2). \]

We note that $f_1 \in L^{p_0} \cap D$ and $f_2 \in L^{p_1} \cap D$ since $p_0 \leq p_t \leq p_1$. Further,

\[ \mu_{f_0}(s) = \begin{cases} \mu_f(At), & s < At \\ \mu_f(s), & s > At \end{cases}, \quad \mu_{f_1}(s) = \begin{cases} \mu_f(s) - \mu_f(At), & s < At \\ 0, & s > At \end{cases}. \]

Applying the weak estimate for $T$ in $L^{p_0}$ we get

\[ \nu_{Tf_0}(t/2) \leq C_0^{p_0} 2^{p_0} t^{-p_0} \|f_0\|_{p_0} = (2C_0)^{p_0} t^{-p_0} \int_0^\infty p_0 s^{p_0-1} \mu_{f_0}(s) ds. \]

Using the formula for $\mu_{f_0}$ we get

\[ \nu_{Tf_0}(t/2) \leq (2C_0)^{p_0} t^{-p_0} \left( (At)^{p_0} \mu_f(At) + \int_{At}^\infty p_0 s^{p_0-1} \mu_f(s) ds \right). \]
On the other hand for $f_1 \in L^{p_1}$ we get
\[ \nu_{Tf_1}(t/2) \leq (2C_1)^{p_1} t^{-p_1} \int_0^t p_1 s^{p_1-1} \mu_f(s) ds. \]

Thus for any $t > 0$ we obtain
\[ \nu_{Tf}(t) \leq (2C_0)^{p_0} t^{-p_0} \left( (At)^{p_0} \mu_f(At) + \int_A^t p_0 s^{p_0-1} \mu_f(s) ds \right) \]
\[ + (2C_1)^{p_1} t^{-p_1} \int_0^t p_1 s^{p_1-1} \mu_f(s) ds. \]

We forget about our decomposition $f = f_0 + f_1$ after we obtained this inequality and start to vary $t$.

Now we integrate the inequality above
\[ \int_0^r p t^{p-1} \nu_{Tf}(t) dt \leq (2C_0 A)^{p_0} A^{-p} \int_0^r p s^{p-1} \mu_f(s) ds \]
\[ + (2C_0)^{p_0} A^{p_0-p} \int_0^r p s^{p-1} \mu_f(s) ds + \frac{(2C_1)^{p_1} A^{p_1-p}}{p_1-p} \int_0^r p s^{p-1} \mu_f(s) ds. \]

This implies $\|Tf\|_{L^p(\nu)} \leq C \|f\|_{L^p(\mu)}$. To minimize the constant we should choose $A$ in an appropriate way. We see that $C$ blows up when $p$ approaches $p_0$ or $p_1$, this is natural as we assumed only weak inequalities at the end points. \(\square\)

### 2.2. Almost everywhere convergence.

Using estimates for the maximal function we can now prove that if $f \in L^1(\mathbb{T})$ then $\lim_{r \to 1-} u_f(re(\theta)) = f(\theta)$ a.e.

The idea is to approximate $f$ by a continuous function $g$ in $L^1$-norm. We know that $g * P_r$ converges to $g$ uniformly (since $P_r$ is an approximate identity). Further we know that $(f - g) * P_r(\theta) \leq M(f - g)(\theta)$ for each $r$, thus

\[ \left| \{\theta : \limsup_{r \to 1-} f \ast P_r(\theta) - \liminf_{r \to 1-} f \ast P_r > \epsilon \} \right| \]
\[ \leq |\theta : M(f - g)(\theta) > \epsilon/2| \leq 6 \|f - g\|_1 \epsilon^{-1}. \]

Similarly, using the box kernel instead of the Poisson kernel we see that if $f \in L^1$ that for almost every $\theta$ we have $f(\theta) = \lim_{t \to 0} \frac{1}{4\pi} \int_{[0, t]} f'(\tau) d\tau$. The same result holds in $L^1(\mathbb{R}^d)$. It is called the Lebesgue differentiation theorem.
3. Chapter 2.5: Weighted estimates for the maximal function

3.1. Calderón-Zygmund decomposition. Let \( f \in L^1(\mathbb{R}^d) \) and \( \lambda > 0 \). Then there is a sequence of dyadic cubes \( \{Q_j\} \) and a corresponding decomposition of \( f \) into sum of two functions \( f = g + b \) with \( g = \sum_j f1_{Q_j} \) such that

(i) \( \lambda |Q_j| \leq \int_{Q_j} |f| \leq 2^d \lambda |Q_j| \)

(ii) \( |b| \leq \lambda \) a.e.

The construction starts with large dyadic cubes and uses simple stopping time argument. Property (ii) follows from the Lebesgue differentiation theorem.

Now we clearly have that \( \{ x : Mf(x) \geq \lambda \} \supset \cup_j Q_j \). We will show that in some sense the opposite inclusion holds. More precisely,

\[ \{ x : Mf > 4^d \lambda \} \subset \cup_j 3Q_j. \]

Assume that \( Mf(x) > 4^d \lambda \) then there is a cube \( Q \) such that \( x \in Q \) and \( \int_Q |f| > 4^d \lambda |Q| \). This cube \( Q \) can be covered by \( 2^d \) equal dyadic cubes \( \{Q_{lk}\} \) such that \( |Q| < |Q_{lk}| \leq 2^d |Q| \). Then there is at least one \( Q_{lk} \) such that

\[ \int_{Q_{lk}} |f| > 2^d \lambda |Q| \geq \lambda |Q_{lk}|. \]

It means that \( Q_{lk} \) is contained in a dyadic cube from the family constructed in the Calderón-Zygmund decomposition, \( Q_{lk} \subseteq Q_j \). Then \( x \in Q \subset 3Q_{lk} \subseteq 3Q_j \).

3.2. Muckenhaup weights. We know want to discuss for which positive functions \( w \) in \( \mathbb{R}^d \), \( w \in L^1_{loc}(\mathbb{R}^d) \), the \( L^p \)-inequality for maximal functions holds. More precisely, we want to know when for any \( f \in L^p(\mathbb{R}^d, w) \)

\[
\int_{\{Mf > t\}} w(x)dx \leq K_p t^{-p} \int_{\mathbb{R}^d} |f(x)|^p w(x)dx.
\]

Lemma 2. Suppose that \( w > 0 \) in \( \mathbb{R}^d \) and (1) holds. Then for any \( f \in L^p(\mathbb{R}^d, w) \) and any cube \( Q \)

\[
\int_Q w(x)dx \left( \frac{1}{|Q|} \int_Q |f(x)|dx \right)^p \leq K_p \int_Q |f(x)|^p w(x)dx.
\]

In particular, for a measurable set \( E \subset Q \) with \( |E| > 0 \) we have

\[
\int_Q w(x)dx \leq K_p \left( \frac{|Q|}{|E|} \right)^p \int_E w(x)dx.
\]

Proof. It is clear that \( Q \subset \{ x : Mf(x) > |Q|^{-1} \int_Q |f| \} \). Then (1) with \( t = |Q|^{-1} \int_Q |f| \) implies (2). Then if we take \( f = \chi_E \) we get (3). \( \square \)
We remark that if $w$ satisfies (3) then $w(A) = \int_A w(x)dx$ is a doubling measure.

**Definition.** We say that a positive function $w \in L^1_{loc}$ satisfies Muckenhoupt $A_1$ condition if
\begin{equation}
Mw(x) \leq C_1 w(x) \quad \text{a.e.}
\end{equation}
and that it satisfies Muckenhoupt $A_p$ condition with $1 < p < \infty$ if for any cube $Q$
\begin{equation}
\frac{1}{|Q|} \int_Q w \left( \frac{1}{|Q|} \int_Q w^{-1/(p-1)} \right)^{p-1} < C_p.
\end{equation}

**Proposition 1.** Suppose that $w > 0$ in $\mathbb{R}^d$ and (1) holds with $1 \leq p < \infty$. Then $w$ satisfies $A_p$.

It follows from the inequality (3) in lemma above and the Lebesgue differentiation theorem when $p = 1$ and from the inequality (2) for $p > 1$ when we take $f = w^{-1/(p-1)} \chi_B$.

**Lemma 3.** If $w$ satisfies $A_p$ with $1 \leq p < \infty$ then (2) holds.

It follows from the Hölder inequality when $p > 1$. See also solutions to problems for this chapter.

**Theorem 2.** If $w$ satisfies $A_p$ with $1 \leq p \leq \infty$ then (1) holds.

**Proof.** We use Calderón-Zygmund decomposition of the function $f$ on the level $t/4^d$ such that $\{Mf > t\} \subset \cup_j 3Q_j$ from this decomposition. We already know that $A_p$ implies (2) which implies doubling. Thus
\[
\int_{Mf > t} w(x)dx \leq C \sum_j \int_{Q_j} w(x)dx \leq CK_p \sum_j \left( \frac{1}{|Q_j|} \int_{Q_j} |f| \right)^{-p} \int_{Q_j} |f|^p w.
\]
Since $Q_j$ are from the Calderón-Zygmund decomposition, we can estimate the first factor by $4^d t^{-p}$. Then
\[
\int_{Mf > t} w(x)dx \leq CK_p 4^d t^{-p} \int |f(x)|^p w(x)dx.
\]
\[\square\]