1. Chapter 2.1-2.3 Harmonic functions: Poisson kernel, Hardy-Littlewood Maximal function

1.1. **Summation methods and Poisson kernel.** We are studying the convergence of the Fourier series of functions $f \in L^1(L^p)$ and in chapter one we tried Cesáro summation method that gives nice convergence in L^1. Now we look at Abel summation. For $f \in L^1$ define

$$u_f(re(\theta)) = \sum_n \hat{f}(n) r^{|n|} e(n\theta).$$

Using complex notation $z = re(\theta)$ we get

$$u_f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n + \sum_{n=1}^{\infty} \hat{f}(-n) \bar{z}^n.$$

The series converges when $|z| < 1$ and the result is a harmonic function. On the circle of radius r we get

$$u(r\cdot) = f * P_r, \quad P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos 2\pi\theta + r^2}.$$

P_r is called the Poisson kernel, it is a positive even function and the family $\{P_r\}_r$ forms an approximate identity when $r \to 1$.

1.2. **Little Hardy spaces.** We define

$$h^p(\mathbb{D}) = \{u : \mathbb{D} \to \mathbb{C}; \Delta u = 0, \sup_r \|u(r\cdot)\|_{L^p(\mathbb{T})} < \infty\}.$$

Then the following holds:
(i) If $f \in L^p$ then $u_f \in h^p$ for $1 \le p \le \infty$.
(ii) $u \in h^1$ if and only if there exists a Borel measure μ on $\mathbb{T}$ such that $u(r\cdot) = P_r * \mu$.
(iii) $u \in h^p$, $1 < p < \infty$ if and only if $u = u_f$ for some $f \in L^p$.
(iv) $u \in h^\infty \cap C(\bar{\mathbb{D}})$ if and only if $u = u_f$ for some $f \in C(\mathbb{T})$.

To prove (ii) and (iii) we apply the sequential version of the Banach-Alaoglu theorem. It says that closed unit ball of the dual space of a separable normed vector space is sequentially compact in the weak* topology. In our case $L^p(\mathbb{T}) = (L^q(\mathbb{T}))^*$ where $1/p + 1/q = 1$ when $p > 1$ and $L^1(\mathbb{T}) \subset M(\mathbb{T}) = (C(\mathbb{T}))^*$. The proof consists of taking a dense sequence in the predual space and, using the standard diagonal construction, finding a weakly convergent subsequence from a bounded sequence. Further, we show that in $f \in L^p(\mathbb{T})$ or $\mu \in M(\mathbb{T})$ is the limiting function or measure then

$$u(re(\theta)) - P_r * f(\theta) = P_{r/s} * u_s - P_r * f = (P_{r/s} - P_r) * u_s + P_r * (u_s - f),\ 0 < r < s.$$

We let $s \to 1$ along the weakly convergent subsequence and get

$$|u(re(\theta) - P_r * f(\theta)| \le \|P_{r/s} - P_r\|_q \|u_s\|_p + |P_r * u_s - P_r * f|.$$

The last term goes to zero since $u_s \to f$ weakly and $P_r \in C(\mathbb{T}) \subset L^q(\mathbb{T})$ and the first term goes to zero since $\|u_s\|_p$ is uniformly bounded and $\|P_{r/s} - P_r\|_\infty \to 0$ as $s \to 1$.

1.3. **Hardy-Littlewood maximal function.** Suppose that ν is a measure on $\mathbb{R}^d$. For any $f \in L^1(d\nu)$ we define the maximal function of f with respect to ν by

$$M_\nu f(x) = \sup_{B \ni x} \frac{1}{\nu(B)} \int_B |f(y)| d\nu(y).$$

It is clear that if $f \in L^\infty(d\nu)$ then $Mf(x) \leq \|f\|_\infty$.

Lemma 1. *Assume that ν satisfies the doubling property, there exists a constant A such that*

$$\nu(2B) \leq A\nu(B)$$

for any ball $B \in \mathbb{R}^d$. Then for any $f \in L^1(\nu)$ and $t > 0$ the following inequality holds

$$\nu\{x : M_\nu f > t\} \leq A^2 t^{-1} \|f\|_1.$$

Proof. Let $K \subset \{x : M_\nu f > t\}$ be a compact set. By the definition of the maximal function $K \subset \cup_{j=1}^N B_j$, where $\int_{B_j} |f(y)| d\nu(y) > t\nu(B_j)$. We choose disjoint sub-collection of balls $B_{j'}$ such that $\cup_{j'} 3B_{j'} \supset K$. then

$$\nu(K) \leq \sum_{j'} \nu(3B_{j'}) \leq A^2 \sum_{j'} \nu(B_{j'}) \leq A^2 t^{-1} \int_{B_{j'}} |f(y)| d\nu(y) \leq A^2 t^{-1} \|f\|_1.$$

□

1.4. L^p **and weak-**L^p **spaces.** . Let (X, A, μ) be a measure space. For f be a measurable function on X, we define its distribution function by

$$\mu_f(t) = \mu\{x \in X : |f(x)| > t\}.$$

Then

$$\|f\|_p^p = \int_0^\infty p t^{p-1} \mu_f(t) dt.$$

The weak L^p space is the space of functions f such that

$$\mu_f(t) \leq C^p t^{-p}, \quad t > 0.$$

The smallest constant C for which the inequality above holds is called the weak-L^p norm of f. Clearly an L^p function belongs to weak-L^p and $\|f\|_{weak-L^p} \leq \|f\|_p$ since

$$\mu\{x \in X : |f(x)| > t\} \leq t^{-p} \int_X |f|^p d\mu = t^{-p} \|f\|_p^p.$$

2. Chapter 1.6, 2.4: An interpolation theorem and convergence almost everywhere

2.1. **Marcinkiewicz interpolation theorem.** Let D be a linear subset of measurable functions on (X, A, μ) such that D contains all finite linear combinations of characteristic functions of sets of finite measure and if $f \in D$ and $C > 0$ then $\min\{f, C\}$ is also in D. We say that an operator T from D to measurable functions on (Y, B, ν) is sublinear if

$$(i)\ |T(af)(y)| = a|Tf(y)|, \quad (ii) |T(f_1 + f_2)(y)| \leq |Tf_1(y)| + |Tf_2(y)|.$$

Theorem 1. *Suppose that T is a sublinear operator such that*

$$\|Tf\|_{weak-q_j} \leq C_j \|f\|_{p_j}$$

for $f \in L^{p_j}(X) \cap D$ and $j = 0, 1$, where $q_0 \neq q_1$ and $p_j \leq q_j$. Then $\|Tf\|_{q_t} \leq C_t \|f\|_{p_t}$, where $0 < t < 1$ and

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

We will prove it for the case $p_0 = q_0$ and $p_1 = q_1$.

Proof. Let $f \in D \cap L^{p_t}$, we want to estimate the distribution function $\mu_{Tf}(t)$. Assume that $p_0 < p_1$.

We fix $t > 0$ and decompose f into sum of two functions $f = f_0 + f_1$, where

$$f_0 = \begin{cases} 0, \ |f| \leq At \\ f, \ |f| > At \end{cases}, \quad f_1 = \begin{cases} f, \ |f| \leq At \\ 0, \ |f| > At \end{cases}.$$

By our assumption $f_1, f_2 \in D$ and $|Tf| \leq |Tf_1| + |Tf_2|$. Then

$$\nu_{Tf}(t) \leq \nu_{Tf_0}(t/2) + \nu_{Tf_1}(t/2).$$

We note that $f_1 \in L^{p_0} \cap D$ and $f_2 \in L^{p_1} \cap D$ since $p_0 \leq p_t \leq p_1$. Further,

$$\mu_{f_0}(s) = \begin{cases} \mu_f(At), \ s < At \\ \mu_f(s), \quad s > At \end{cases}, \quad \mu_{f_1}(s) = \begin{cases} \mu_f(s) - \mu_f(At), \ s < At \\ 0, \quad s > At \end{cases}.$$

Applying the weak estimate for T in L^{p_0} we get

$$\nu_{Tf_0}(t/2) \leq C_0^{p_0} 2^{p_0} t^{-p_0} \|f_0\|_{p_0}^{p_0} = (2C_0)^{p_0} t^{-p_0} \int_0^\infty p_0 s^{p_0-1} \mu_{f_0}(s) ds.$$

Using the formula for μ_{f_0} we get

$$\nu_{Tf_0}(t/2) \leq (2C_0)^{p_0} t^{-p_0} \left((At)^{p_0} \mu_f(At) + \int_{At}^\infty p_0 s^{p_0-1} \mu_f(s) ds \right).$$

On the other hand for $f_1 \in L^{p_1}$ we get

$$\nu_{Tf_1}(t/2) \le (2C_1)^{p_1} t^{-p_1} \int_0^{At} p_1 s^{p_1-1} \mu_f(s) ds.$$

Thus for any $t > 0$ we obtain

$$\nu_{Tf}(t) \le (2C_0)^{p_0} t^{-p_0} \left((At)^{p_0} \mu_f(At) + \int_{At}^\infty p_0 s^{p_0-1} \mu_f(s) ds \right) + (2C_1)^{p_1} t^{-p_1} \int_0^{At} p_1 s^{p_1-1} \mu_f(s) ds.$$

We forget about our decomposition $f = f_0 + f_1$ after we obtained this inequality and start to vary t.

Now we integrate the inequality above

$$\int_0^\infty p t^{p-1} \nu_{Tf}(t) dt \le (2C_0 A)^{p_0} A^{-p} \int_0^\infty p s^{p-1} \mu_f(s) ds + \frac{(2C_0)^{p_0} A^{p_0-p}}{p-p_0} \int_0^\infty p s^{p-1} \mu_f(s) ds + \frac{(2C_1)^{p_1} A^{p_1-p}}{p_1-p} \int_0^\infty p s^{p-1} \mu_f(s) ds.$$

This implies $\|Tf\|_{L^p(\nu)} \le C\|f\|_{L^p(\mu)}$. To minimize the constant we should choose A in an appropriate way. We see that C blows up when p approaches p_0 or p_1, this is natural as we assumed only weak inequalities at the end points. □

2.2. **Almost everywhere convergence.** Using estimates for the maximal function we can now prove that if $f \in L^1(\mathbb{T})$ then $\lim_{r\to 1-} u_f(re(\theta)) = f(\theta)$ a.e.

The idea is to approximate f by a continuous function g in L^1-norm. We know that $g * P_r$ converges to g uniformly (since P_r is an approximate identity). Further we know that $(f-g) * P_r(\theta) \le M(f-g)(\theta)$ for each r, thus

$$\begin{aligned} &|\{\theta : \limsup_{r\to 1-} f * P_r(\theta) - \liminf_{r\to 1-} f * P_r > \epsilon\}| \\ &\quad = |\{\theta : \limsup_{r\to 1-} (f-g) * P_r(\theta) - \liminf_{r\to 1-} (f-g) * P_r > \epsilon\}| \\ &\quad\quad \le |\theta : M(f-g)(\theta) > \epsilon/2| \le 6\|f-g\|_1 \epsilon^{-1}. \end{aligned}$$

Similarly, using the box kernel instead of the Poisson kernel we see that if $f \in L^1$ that for almost every θ we have $f(\theta) = \lim_{t\to 0} \frac{1}{2t} \int_{\theta-t}^{\theta+t} f(\tau) d\tau$. The same result holds in $L^1(\mathbb{R}^d)$. It is called the Lebesgue differentiation theorem.

3. Chapter 2.5: Weighted estimates for the maximal function

3.1. **Calderón-Zygmund decomposition.** Let $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$. Then there is a sequence of dyadic cubes $\{Q_j\}$ and a corresponding decomposition of f into sum of two functions $f = g + b$ with $g = \sum_j f\chi_{Q_j}$ such that

(i) $\lambda|Q_j| \le \int_{Q_j} |f| \le 2^d\lambda|Q_j|$

(ii) $|b| \le \lambda$ a.e.

The construction starts with large dyadic cubes and uses simple stopping time argument. Property (ii) follows from the Lebesgue differentiation theorem.

Now we clearly have that $\{x : Mf(x) \ge \lambda\} \supset \cup_j Q_j$. We will show that in some sense the opposite inclusion holds. More precisely,

$$\{x : Mf > 4^d\lambda\} \subset \cup_j 3Q_j.$$

Assume that $Mf(x) > 4^d\lambda$ then there is a cube Q such that $x \in Q$ and $\int_Q |f| > 4^d\lambda|Q|$. This cube Q can be covered by 2^d equal dyadic cubes $\{Q_{l_k}\}$ such that $|Q| < |Q_{l_k}| \le 2^d|Q|$. Then there is at least one Q_{l_k} such that

$$\int_{Q_{l_k}} |f| > 2^d\lambda|Q| \ge \lambda|Q_{l_k}|.$$

It means that Q_{l_k} is contained in a dyadic cube from the family constructed in the Calderón-Zygmund decomposition, $Q_{l_k} \subseteq Q_j$. Then $x \in Q \subset 3Q_{l_k} \subseteq 3Q_j$.

3.2. **Muckenhaupt weights.** We know want to discuss for which positive functions w in $\mathbb{R}^d$, $w \in L^1_{loc}(\mathbb{R}^d)$, the L^p-inequality for maximal functions holds. More precisely, we want to know when for any $f \in L^p(\mathbb{R}^d, w)$

$$\int_{\{Mf>t\}} w(x)dx \le K_p t^{-p} \int_{\mathbb{R}^d} |f(x)|^p w(x)dx. \tag{1}$$

Lemma 2. *Suppose that $w > 0$ in $\mathbb{R}^d$ and (1) holds. Then for any $f \in L^p(\mathbb{R}^d, w)$ and any cube Q*

$$\int_Q w(x)dx \left(\frac{1}{|Q|}\int_Q f(x)dx\right)^p \le K_p \int_Q |f(x)|^p w(x)dx. \tag{2}$$

In particular, for a measurable set $E \subset Q$ with $|E| > 0$ we have

$$\int_Q w(x)dx \le K_p \left(\frac{|Q|}{|E|}\right)^p \int_E w(x)dx. \tag{3}$$

Proof. It is clear that $Q \subset \{x : Mf(x) > |Q|^{-1}\int_Q |f|\}$. Then (1) with $t = |Q|^{-1}\int_Q |f|$ implies (2). Then if we take $f = \chi_E$ we get (3). □

We remark that if w satisfies (3) then $w(A) = \int_A w(x)dx$ is a doubling measure.

Definition. We say that a positive function $w \in L^1_{loc}$ satisfies Muckenhoupt A_1 condition if

$$Mw(x) \leq C_1 w(x) \quad \text{a.e.} \tag{4}$$

and that it satisfies Muckenhoupt A_p condition with $1 < p < \infty$ if for any cube Q

$$\frac{1}{|Q|}\int_Q w \left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}\right)^{p-1} < C_p. \tag{5}$$

Proposition 1. *Suppose that $w > 0$ in $\mathbb{R}^d$ and (1) holds with $1 \leq p < \infty$. Then w satisfies A_p*

It follows from the inequality (3) in lemma above and the Lebesgue differentiation theorem when $p = 1$ and from the inequality (2) for $p > 1$ when we take $f = w^{-1/(p-1)}\chi_B$.

Lemma 3. *If w satisfies A_p with $1 \leq p < \infty$ then (2) holds.*

It follows from the Hölder inequality when $p > 1$. See also solutions to problems for this chapter.

Theorem 2. *If w satisfies A_p with $1 \leq p \leq \infty$ then (1) holds.*

Proof. We use calderón-Zygmund decomposition of the function f on the level $t/4^d$ such that $\{Mf > t\} \subset \cup_j 3Q_j$ from this decomposition. We already know that A_p implies (2) which implies doubling. Thus

$$\int_{Mf>t} w(x)dx \leq C\sum_j \int_{Q_j} w(x)dx \leq CK_p \sum_j \left(\frac{1}{|Q_j|}\int_{Q_j}|f|\right)^{-p}\int_{Q_j}|f|^p w.$$

Since Q_j are from the Calderón-Zygmund decomposition, we can estimate the first factor by $4^{pd}t^{-p}$. Then

$$\int_{Mf>t} w(x)dx \leq CK_p 4^{pd} t^{-p}\int |f(x)|^p w(x)dx.$$

□