1. Chapter 1: Solutions to selected problems

1.1. **Problem 1.1.** *Suppose that $f \in L^1(\mathbb{T})$ and that $\{S_n f\}_{n=1}^{\infty}$ converges to g in $L^p(\mathbb{T})$ with $1 \le p \le \infty$. Prove that $f = g$ and if $p = \infty$ conclude that f is continuous.*

Solution We will show that $\hat{f}(n) = \hat{g}(n)$ for all n then from the uniqueness theorem we know that $f = g$. Clearly for $N \ge n$ we have $\hat{f}(n) = \widehat{S_N f}(n)$ and by the Hölder inequality

$$|\hat{f}(n) - \hat{g}(n)| = \left| \int_0^1 (S_N f - g)(x) e(-nx) dx \right| \le \|S_N f - g\|_p \|e(-n\cdot)\|_{p'}.$$

The last expression goes to zero as $N \to \infty$ and it concludes the proof. When $p = \infty$ then f is the uniform limit of continuous functions and thus f is continuous.

1.2. **Problem 1.5.** *Suppose that $\sum_1^\infty n|a_n|^2 < +\infty$ and $\sum_1^\infty a_n$ is Cesáro summable. Show that $\sum_1^\infty a_n$ converges.*
Use this to prove that any $f \in C(\mathbb{T}) \cap H^{1/2}(\mathbb{T})$ satisfies $S_n f \to f$ uniformly.

Solution We compare the partial sums of the series to its Cesáro sums,

$$\sum_{k=1}^{N} a_k - \sum_{k=1}^{N} \frac{N-k}{N} a_k = \frac{1}{N} \sum_{1}^{N} k a_k$$

Then by the Cauchy-Schwarz inequality, we get

$$\left| \frac{1}{N} \sum_1^N k a_k \right| \le \frac{1}{N} \left(\sum_1^N k \right)^{1/2} \left(\sum_1^N k |a_k|^2 \right)^{1/2} \le C \left(\sum_1^N k |a_k|^2 \right)^{1/2}.$$

If $f \in C(\mathbb{T}) \cap H^{1/2}(\mathbb{T})$ then $\sum_n |n| |\hat{f}(n)|^2$ converges by the definition of the Sobolev space $H^{1/2}$ and $\sum_n \hat{f}(n) e(n\theta)$ is Cesáro summable to $f(\theta)$. Then $S_n f \to f$ by the previous result.

1.3. **Problem 1.6.** *Show that there exists an absolute constant C such that*

$$C^{-1} \sum_{n \ne 0} n |\hat{f}(n)|^2 \le \int\int \frac{|f(x) - f(y)|^2}{(\sin \pi(x-y))^2} dx dy \le C \sum_{n \ne 0} n |\hat{f}(n)|^2$$

Solution We rewrite the integral using the Fubini theorem for positive functions (also called Tonelli's theorem) and apply the Parseval identity

$$\int\int \frac{|f(x) - f(y)|^2}{(\sin \pi(x-y))^2} dx dy = \int_0^1 \frac{1}{(\sin \pi t)^2} \int_0^1 |f(x+t) - f(x)|^2 dx dt$$

$$= \int_0^1 \frac{1}{(\sin \pi t)^2} \sum_n |\hat{f}(n)|^2 |1 - e(nt)|^2 dt = \sum_n |\hat{f}(n)|^2 \int_0^1 \frac{(\sin \pi n t)^2}{(\sin \pi t)^2} dt.$$

The last integral is equal to n (it is a multiple of the Fejer kernel).

For other $s \in (0,1)$ we have

$$\int\int \frac{|f(x)-f(y)|^2}{(\sin \pi(x-y))^{1+2s}} dxdy = \sum_n |\hat{f}(n)|^2 \int_0^1 \frac{(\sin \pi n t)^2}{(\sin \pi t)^{1+2s}} dt$$

To estimate the last integral we use the inequality $2|x| \le |\sin \pi x| \le \pi|x|$ when $0 \le |x| \le 1/2$. Then

$$\int_{-1/2}^{1/2} \frac{(\sin \pi n t)^2}{(\sin \pi t)^{1+2s}} dt \ge \int_{|t|<1/2n} \frac{4n^2 t^2}{(\pi t)^{1+2s}} dt = c_s n^{2s}.$$

On the other hand,

$$\int_{-1/2}^{1/2} \frac{(\sin \pi n t)^2}{(\sin \pi t)^{1+2s}} dt \le \int_{|t|<1/2n} \frac{\pi^2 n^2 t^2}{(2t)^{1+2s}} dt + \int_{1/2n<|t|<1/2} \frac{1}{(2t)^{1+2s}} dt = C_s n^{2s}.$$

1.4. **Problem 1.7.** *Use the previous two problems to prove the following theorem of Pal and Bohr: for any real function $f \in C(\mathbb{T})$ there exists a homeomorphism $\phi : \mathbb{T} \to \mathbb{T}$ such that $S_n(f \circ \phi) \to f \circ \phi$ uniformly.*
Hint: Without loss of generality let $f > 0$. Consider the domain defined in terms of polar coordinates by means of $r(\theta) = f(\theta)$. Then apply the Riemann mapping theorem to the unit disk.

Solution We will need the following fact about conformal mappings. Let $g : \mathbb{D} \to \Omega$ be a conformal map and let Ω be bounded domain. For each $r < 1$ consider $\Omega_r = g(r\mathbb{D})$ with boundary $\gamma_r = \partial\Omega_r = g(r\mathbb{T})$. Then

$$2iArea(\Omega_r) = \int_{\gamma_r} \bar{z} dz = 2\pi i \int_0^1 \overline{g(re(\theta))} g'(re(\theta)) re(\theta) d\theta.$$

Now if $g(z) = \sum_{n=0}^{\infty} a_n z^n$ then $zg'(z) = \sum_{n=0}^{\infty} n a_n z^n$ and

$$Area(\Omega_r) = \pi \sum_n n|a_n|^2 r^{2n}.$$

If we now let r go to 1 we get $\sum_n n|a_n|^2 \le Area(\Omega) < \infty$.

Now we do as in the hint. Assume that $f > 0$ by adding a constant, f is also bounded. Then there exists a conformal map g continuous on $\overline{\mathbb{D}}$ such that $|g(\theta)| = f(\alpha)$ where $\text{Arg}(g(\theta)) = \alpha$. Let $\phi(\theta) = \text{Arg}(g(\theta)$, then $\phi : \mathbb{T} \to \mathbb{T}$ is a homeomorphism and $f \circ \phi = |g|$. Now we know that $g \in H^{1/2}$ by the area computation above. Then by problem 1.6 we see that $|g| \in H^{1/2}$. It is also continuous and Problem 1.5 implies the statement.

1.5. **Problem 1.9.** *Sjhow that $\|f * g\|_2^2 \le \|f * f\|_2^2 \|g * G\|_2^2$ for all $f, g \in L^2(\mathbb{T})$.*

Solution It follows easily from the Parseval's identity and the Cauchy-Schwarz inequality.

1.6. **Problem 1.12.** *Given a function $\psi : \mathbb{Z}_+ \to \mathbb{R}_+$ such that $\psi(n) \to 0$ as $n \to \infty$. Show that one can find a measurable set $E \subset \mathbb{T}$ for which*

$$\limsup \frac{|\widehat{\chi_E}(n)|}{\psi(n)} = +\infty$$

Solution We will construct E as the union of sets E_k, $E = \cup_k E_k$, where $E_k \subset (2^{-k}, 2^{-k+1})$ and $E_k = \cup_{j=1}^{N_k} I_{j,k}$ is a finite union of some intervals with

$$I_{j,k} = (m_{j,k} n_k^{-1}, (m_{j,k} + 1/2) n_k^{-1}), \quad 2^{-k} n_k \le m_{k,j} \le 2^{-k+1} n_k - 1.$$

We let $N_k = 4^{-k} n_k$. and choose an increasing sequence $n_k = 2^{l_k}$ which will depend on ψ and such that $l_{k+1} > l_k + 5$.

First we estimate the Fourier coefficients of χ_{E_k}. We have

$$|\widehat{\chi_{E_j}}(n_j)| = \left| \int_{E_k} e(-n_k \theta) d\theta \right| = 4^{-j} \pi^{-1}.$$

For $n_j < n_k$ we have

$$|\widehat{\chi_{E_k}}(n_j)| = \left| \int_{E_k} e(-n_j \theta) d\theta \right| \le 4^{-k}/2.$$

and for $n_j > n_k$

$$|\widehat{\chi_{E_k}}(n_j)| = \left| \int_{E_k} e(-n_j \theta) d\theta \right| \le \pi^{-1} N_k n_j^{-1}.$$

Then using that $\sum_{k<j} N_k \le N_j/4$ we have

$$|\widehat{\chi_E}(n_j)| \ge 4^{-j}\pi^{-1} - 4^{-j}/6 - 4^{-j}(4\pi)^{-1} > 4^{-j-3}.$$

Finally, we can choose l_k such that $\phi(2^{l_k}) < 8^{-k}$ because $\psi(n) \to 0$ as $n \to \infty$.