1. Chapter 1.3-1.4 Fourier series: Convergence in L^p, Regularity

1.1. **Convergence in $L^p(\mathbb{T})$.**

First, we rewrite partial sums of the Fourier series and Cesáro sums as convolutions.

$$S_nf = f * D_n, \quad D_n(x) = \sum_{-n}^{n} e(kx) = \frac{\sin(2n+1)\pi x}{\sin \pi x},$$

$$\sigma_n f = f * K_n, \quad K_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} D_k(x) = \frac{1}{n}\left(\frac{\sin \pi n x}{\sin \pi x}\right)^2.$$

These functions are called Dirichlet and Fejér kernels. We have the following useful estimates for them

$$|D_n(x)| \le C\min\{n, |x|^{-1}\}, \ 0 \le K_n(x) \le C\min\{n, n^{-1}|x|^{-2}\}, \quad |x| \le 1/2.$$

It is proved in Section 1.2 that $\sigma_n(f) \to f$ in L^p when $1 \le p < \infty$ and $\sigma_n(f) \to f$ uniformly when $f \in C(\mathbb{T})$ and it follows from the fact that Fejér kernel is an approximate identity. The main question for the first several chapters is if $S_n f \to f$ in L^p when $1 < p < \infty$.

A simple remark (specially if you know the uniform boundedness principle from functional analysis) is that $S_n f \to f$ in L^p if and only if

$$\sup_n \|S_n\|_{L^p \to L^p} < \infty.$$

The last inequality fails when $p = 1$ or $p = \infty$ since $\|D_n\|_1 \approx \log n$.

1.2. **Bernstein's inequality.**

The classical Bernstein's inequality from 1912 bounds the maximum of the derivative of a trigonometric polynomial over the unit circle by the degree times the maximum of the polynomial itself. We will prove the corresponding inequality in L^p.

Lemma 1. $\|K_n'\|_1 \le Cn$

Proof. K_n is a real-valued trigonometric polynomial of degree $(n-1)$ and $\|K_n'\|_1$ is the total variation of K_n. We note that K_n has $(n-1)$ double zeros on the unit circle at points j/n for $j = 1, ..., n-1$. Then K_n' has simple zeros at these $(n-1)$ points and at least one zero between each pair of neighboring zeros (whereK_n attains its maximum). Bur since it is a trigonometric polynomial of degree $n-1$ it can not have more than $2(n-1)$ zeros. Thus

$$\|K_n'\| = 2K_n(0) + 2\sum_{j=1}^{n-1} \max_{j<nx<j+1} K_n(x) \le Cn + 4n\sum_{j=1}^{[} n/2]j^{-2} \le Cn$$

□

Now, to prove the Bernstein's inequality, we construct a new kernel that preserves the polynomials of order n. We take $V_n(x) = K_n(x)(1+e(-nx)+e(nx))$. It is called de la Vallée Poussin kernel, it has $\widehat{V_n}(k) = 1$ when $|k| \leq n$. Clearly,

$$V_n'(x) = K_n'(x)(1 + e(-nx) + e(nx)) + \pi i n K_n(x)(e(nx) - e(-nx))$$

and $\|V_n'\| \leq Cn$.

Proposition 1. *(Bernstein's inequality) If P is a trigonometric polynomial of degree n than $\|f'\|_p \leq Cn\|f\|_p$ for $1 \leq p \leq \infty$.*

Proof. We have $P = P * V_n$ then from the Young inequality we get $\|P'\|_p \leq \|V_n'\|_1 \|P\|_p \leq Cn\|P\|_p$. □

For $p = \infty$ the precise value of the constant (with our normalization is $C = 2\pi$.

1.3. **Sobolev spaces and Hölder continuity.** We follow the book to show that $C^\alpha \subset H^s$ when $\alpha > s$. The idea is to handle dyadic blocks of the Fourier series using shifted versions of the de la Vallée Poissin kernels $\phi_n = V_n(e(nx) + e(-nx))$. This implies that $C^\alpha \subset \mathbb{A}$ for $\alpha > 1/2$, since $H^s \subset \mathbb{A}$ when $s > 1/2$ by the Cauchy-Schwarz inequality.

The last part is to construct an example of a function $f \in C^{1/2}$ which does not belong to $\mathbb{A}$. It is done using the Rudin-Shapiro polynomials.