1. Chapter 1.1-1.2 Fourier series: the basics, Dirichlet and Fejer kernels

1.1. **Classical inequalities.** Let X and Y be measure spaces equipped with some measures. The following inequalities are used extensively in the course

- **Hölder's inequality** Let $f \in L^p(X)$, $g \in L^q(X)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1(X)$ and
$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$
- **Generalized Hölder's inequality** Suppose that $p_1, ..., p_k > 1$ such that $\sum_j p_j^{-1} = 1$ and $f = f_1 ... f_k$ then
$$\|f\|_1 \leq \prod_{j=1}^{k} \|f_j\|_{p_j}.$$
- **Minkovski's integral inequality** Suppose that $1 \leq p < \infty$ and $h(x, y)$ is a measurable function on $X \times Y$ such that $h(\cdot, y) \in L^p(X)$ for a.e. $y \in Y$ and $F(y) = \|h(\cdot), y)\|_p$ is in $L^1(X)$. Then $f(x, \cdot) \in L^1(Y)$ for a.e. x in X and the function $G(x) = \int h(x, y)dy$ satisfies
$$\|G\|_p \leq \int \|h(\cdot, y)\|_p dy.$$

The Minkovski inequality above is the integral version of the triangle inequality for L^p-norms.

1.2. **Inequalities for convolution.** As usual we define the convolution on X that is $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{T}$ or $\mathbb{T}^n$ (one can think more generally about a locally compact (abelian) group with Haar measure, we will come back to it later in the course),
$$f * g = \int f(y)g(x - y)dy.$$
We have the following inequalities for convolutions

- **L^1-norm** If $f, g \in L^1(X)$ then $f*g \in L^1(X)$ and $\|f*g\|_1 \leq \|f\|_1 \|g\|_1$. It follows immediately from Fubini's theorem.
- **L^∞-norm** Further, if $f \in L^p(X)$ and $g \in L^q(X)$ where $p^{-1} + q^{-1} = 1$ then by the Hölder inequality $|f * g(x)| \leq \|f\|_p \|g\|_q$ (the inequality holds for each $x \in X$). Moreover, $f * g$ is uniformly continuous. Since
$$|f * g(x + \delta) - f * g(x)| = |(\tau_\delta f - f) * g(x)| \leq \|\tau_\delta f - f\|_p \|g\|_q$$
and the first factor goes to zero as $\delta \to 0$.
- **Young's inequality** If $f \in L^1(X)$ and $g \in L^p(X)$, $1 \leq p \leq \infty$ then $f * g \in L^p(X)$ and
$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

For $p = \infty$ the inequality is trivial. For $1 \le p < \infty$ we apply the Minkovski inequality to $h(x,y) = f(y)g(x-y)$ then

$$\|f * g\|_p \le \int |f(y)| \|g(\cdot - y)\|_p dy = \|f\|_1 \|g\|_p.$$

- **Generalized Young's inequality** Let $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $1 \le p, q, r \le \infty$. If $f \in L^p$, $g \in L^q$ then

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

We first fix x and apply the generalized Hölder's inequality to functions $f_1 = |f(y)|^{1-p/r}$, $f_2 = |g(x-y)|^{1-q/r}$ and $f_3 = |f(y)|^{p/r}|g(x-y)|^{q/r}$ and $p_1 = pr/(r-p)$, $p_2 = qr/(r-q)$ and $p_3 = r$ to obtain

$$||f * g(x)|^r \le \|f\|_p^{r-p} \|g\|_q^{r-q} \int |f(y)|^p |g(x-y)|^q dy.$$

Then by Fubini's theorem, $\|f * g\|_r \le \|f\|_p \|g\|_q$.

1.3. **Wiener algebra.** I would recommend to read Section 1.2 before looking at Exercise 1.2.

Let μ be a measure on $\mathbb{T}$ such that $\sum_n |\hat{\mu}(n)| < +\infty$. Then we consider the function

$$f(x) = \sum_n \hat{\mu}(n) e(nx).$$

Clearly f is well-defined and moreover f is continuous on $\mathbb{T}$. Then $\hat{f}(n) = \hat{\mu}(n)$ it means that the measures $\mu(dx)$ and fdx coincide on trigonometric polynomials which are dense in the space of continuous functions (it follows for example from the next section of the book, where convolution with the Fejer kernel is considered). Thus $\mu(dx) = fdx$. The space of all functions f on $\mathbb{T}$ such that

$$\sum_n |\hat{f}(n)| < \infty$$

is denoted by $\mathbb{A}(\mathbb{T})$ and called the Wiener algebra with the norm

$$\|f\|_{\mathbb{A}} = \sum_n |\hat{f}(n)|.$$

It is easy to see that if $f, g \in \mathbb{A}(\mathbb{T})$ then $fg \in \mathbb{A}(\mathbb{T})$ and $\|fg\|_{\mathbb{A}} \le \|f\|_{\mathbb{A}} \|g\|_A$ and if $f, g \in L^2(\mathbb{T})$ then $f * g \in \mathbb{A}(\mathbb{T})$,

$$\|f * g\|_{\mathbb{A}(\mathbb{T})} = \sum_n |\widehat{f * g}(n)| = \sum_n |\hat{f}(n)||\hat{g}(n)| \le \|f\|_2 \|g\|_2.$$