The Porous Media Equation

$$u_t = \Delta u^m \qquad m > 1 \qquad \text{(PME)}$$

The PME is a nonlinear diffusion equation:

$$u_t = \operatorname{div}(m u^{m-1} \nabla u)$$

Note that the diffusion coefficient $m u^{m-1}$ vanishes when $u = 0$, making the equation degenerate. Therefore, the PME is sometimes called the slow diffusion equation (SDE).

In contrast, if $0 < m < 1$, the diffusion coefficient goes to ∞ when $u \to 0$.
This is <u>fast</u> diffusion, which we shall not consider here.

If $m = 1$, we just get the standard heat equation. From now on, we assume $m > 1$.

MODELLING

Let u be the density of gas in a porous medium: Then

$$u_t + \operatorname{div}(uv) = 0$$

where v is the flow rate. Darcy's law:

$$v = -k \nabla p$$

And the gas law: $p = cu^\gamma$ ($\gamma = 1$ for an ideal gas).

Thus we get

$$u_t = ck \operatorname{div}(u \nabla u^\gamma) = ck\gamma \operatorname{div}(u^\gamma \nabla u) = \frac{ck\gamma}{\gamma+1} \Delta u^{\gamma+1}$$

The constant is easily scaled away.

The equation also models the expansion of the fireball in a nuclear explotion. The "porous medium" is the air. It does not have time to move. Heat transfer is by radiation, more efficient at higher temperature. (I have not studied the details.)

SELF-SIMILAR SOLUTIONS: THE BARENBLATT SOLUTION

Look for solutions on the form $u(t,x) = t^{-\alpha} f(x t^{-\beta})$.

Conservation of mass $\Leftrightarrow$ $\alpha = \beta d$. Also $\alpha = \dfrac{d}{(m-1)d+2}$

Barenblatt: $$\boxed{u(t,x) = \left(A - \frac{\beta(m-1)}{2m} \frac{x^2}{t^\beta}\right)_+^{\frac{1}{m-1}}}$$

The Generalised PME

(PME) only makes sense for $u \geq 0$.
But we can extend it: $u_t = \Delta(|u|^m \operatorname{sgn} u)$.
This way, the diffusion coefficient $m|u|^{m-1}$ is nonnegative for all u.
(A negative diffusion coefficient leads to behavior like the backward heat equation, which is not well behaved.)
More generally, we consider

$$u_t = \Delta(\varphi(u)) \qquad \text{(gPME)}$$

where φ is a strictly increasing function $\mathbb{R} \to \mathbb{R}$ with $\varphi(0) = 0$. We shall always assume that φ is locally Lipschitz. By Rademacher's theorem, then φ is differentiable almost everywhere.
(In one dimension, this is easy, since a Lipschitz function is clearly absolutely continuous.)
A fancier way of saying this is $\varphi \in W^{1,\infty}_{loc}(\mathbb{R})$.

SOME PROPERTIES OF THE gPME

We skip some technical details in this section.

Prop Let u and v be solutions of (gPME). Then $\int_{\mathbb{R}^d} (u-v)_+ \, dx$ is a (non-strictly) decreasing function of t.

Proof sketch Let $w = \varphi(u) - \varphi(v)$, so $(u-v)_t = \Delta w$.

Let $p: \mathbb{R} \to \mathbb{R}$ be smooth with $p' \geq 0$ and $0 \leq p \leq 1$. Then

$$\begin{aligned}\int_{\mathbb{R}^d} p(w)\,(u-v)_t \, dx &= \int_{\mathbb{R}^d} p(w)\,\Delta w \, dx \\ &\overset{*}{=} -\int_{\mathbb{R}^d} \nabla(p(w)) \cdot \nabla w \, dx \\ &= -\int_{\mathbb{R}^d} p'(w)\,|\nabla w|^2 \, dx \leq 0\end{aligned}$$

* assuming reasonably fast decay at infinity, so the boundary term vanishes

Now let $p(w) \to [w>0]$ (Iverson bracket).

Because φ is strictly increasing, $[w>0] = [u-v>0]$ and $[u-v>0](u-v)_t = \partial_t (u-v)_+$, leading to

$\partial_t \int_{\mathbb{R}^d} (u-v)_+ \, dx \leq 0$ $\square$

COROLLARY (MONOTONICITY)

Let u, v be solutions of (gPME) with initial data u_0, v_0. If $u_0 \le v_0$ then $u \le v$.

Pf. $\int_{\mathbb{R}^d} (u-v)_+ \, dx = 0$ for $t=0$, so it remains zero $\square$

COROLLARY (MAXIMUM PRINCIPLE)

If $a \le u_0 \le b$ then $a \le u \le b$. Also, $\|u\|_\infty \le \|u_0\|_\infty$

Pf. The constant functions are solutions. $\square$

COROLLARY (L^1 CONTRACTION) $\|u(t)-v(t)\|_1 \le \|u_0 - v_0\|_1$

Pf. Both $\int (u-v)_+ \, dx$ and $\int (v-u)_+ \, dx$ are decreasing. Add the two together. $\square$

COROLLARY (CONSERVATION) $\int_{\mathbb{R}^d} u \, dx$ is constant.

Pf. As the proof of L^1 contraction, with $v=0$. But subtract instead of adding. $\square$

CONSTRUCTING A SOLUTION

Assume given the INITIAL VALUE PROBLEM:

$$\begin{cases} u_t = \Delta(\varphi \circ u) & x \in \mathbb{R}^d,\ t > 0 \\ u(0,x) = u_0(x) \end{cases}$$

We assume that φ is increasing and locally Lipschitz, and $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. We seek a weak solution $u \in L^\infty([0,T] \times \mathbb{R}^d) \cap C_b([0,T]; L^1(\mathbb{R}^d))$.

It should satisfy the IVP in the distributional sense:

$$\boxed{\int_0^T \int_{\mathbb{R}^d} (u \psi_t + \varphi(u) \Delta \psi) \, dx \, dt + \int_{\mathbb{R}^d} u_0(x) \psi(0,x) \, dx = 0}$$

for all $\varphi \in \mathcal{D}((-\infty, T) \times \mathbb{R}^d)$.

FOR SIMPLICITY, PUT $d=1$. Everything still works the same for $d > 1$.

We choose a time step k and look for approximations $u^n(x) \approx u(nk, x)$ for all k.

We DO NOT DISCRETIZE IN SPACE, but we still employ the discrete Laplacian, and put

$$\frac{u^{n+1}(x) - u^n(x)}{k} = \Delta_h(\varphi \circ u) := \frac{\varphi^n(x+h) - 2\varphi^n(x) + \varphi^n(x-h)}{h^2}$$

where we write $\varphi^n(x) = \varphi(u^n(x))$ for brevity.

Introduce $\lambda = \frac{k}{h^2}$, and write the scheme as

$$\boxed{\begin{aligned} u^{n+1}(x) &= u^n(x) - 2\lambda\varphi(u^n(x)) \\ &\quad + \lambda\big(\varphi(u^n(x+h)) + \varphi(u^n(x-h))\big) \end{aligned}}$$

We want the scheme to be monotone, meaning the RHS should be an increasing function of $u^n(x)$, $u^n(x+h)$ and $u^n(x-h)$.

The latter two are easy since φ is increasing.

The first requires $\boxed{2\lambda K \leq 1}$, where $K = \sup \varphi'(s)$.

To be precise, we should take the supremum over $[a,b] = [\inf_x u_0(x), \sup_x u_0(x)]$

We should use ess sup here...

Then u^n will take values in this interval for all n.

Our upper limit on λ can be written as $k \leq \frac{h^2}{2K}$.

NOTATION Write our scheme on the form $u^{n+1} = G[u^n]$.

So long as $a \leq u \leq v \leq b$ $(u, v \in L^\infty(\mathbb{R}))$, also $G[a] \leq G[u] \leq G[v] \leq G[b]$.

From this monotonicity, we get desirable properties of our scheme, resembling properties of (gPME).

LEMMA $\int_{\mathbb{R}} u^n(x)\,dx$ is independent of x.

Proof. Trivial, because $\int_{\mathbb{R}} \Delta_h(\varphi \circ u)\,dx = 0$ $\square$

LEMMA Let u^n, v^n be solutions of our scheme. Then $\int_{\mathbb{R}} (u^n - v^n)_+\,dx$ is a (non-strictly) decreasing function of n.

Proof We want to prove

$$\int_{\mathbb{R}} (G[u]-G[v])_+\,dx \leq \int_{\mathbb{R}} (u-v)_+\,dx$$

Consider the pointwise maximum $u \vee v$.

Clearly $u \leq u \vee v$, so $G[u] \leq G[u \vee v]$.

Therefore $G[u] - G[v] \leq G[u \vee v] - G[v]$

But then also $(G[u]-G[v])_+ \leq G[u \vee v] - G[v]$

since RHS ≥ 0 everywhere. Therefore

$$\begin{aligned}\int_{\mathbb{R}} (G[u]-G[v])_+\,dx &\leq \int_{\mathbb{R}} (G[u \vee v] - G[v])\,dx \\ &= \int_{\mathbb{R}} ([u \vee v] - [v])\,dx \\ &= \int_{\mathbb{R}} (u-v)_+\,dx\end{aligned}$$

$\square$

Corollary Our scheme is L^1-contractive: i.e., $\|u^{n+1} - v^{n+1}\|_1 \le \|u^n - v^n\|$. $\square$

Translation: Write $\tau_\delta u(x) = u(x-\delta)$.
Our scheme is translation invariant, so $(\tau_\delta u^n)$ is a solution of the scheme with initial value $\tau_\delta u^0$. From L^1 contraction we get

$$\|u^n - \tau_\delta u^n\|_1 \le \|u^0 - \tau_\delta u^0\|_1$$

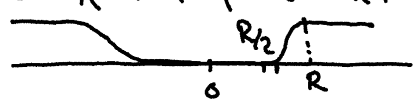
Put

$$\omega(\varepsilon) = \sup_{|\delta| \le \varepsilon} \|u^0 - \tau_\delta u^0\|_1 .$$

It is well known that $\lim_{\varepsilon \downarrow 0} \omega(\varepsilon) = 0$,
and we have just seen that

$$\|u^n - \tau_\delta u^n\|_1 \le \omega(\varepsilon) \quad \text{for all } n \text{ and } |\delta| \le \varepsilon.$$

Tightness: Let $\chi \in C^\infty(\mathbb{R})$ with $\chi(x) = 0$ for $|x| < \frac{1}{2}$, $\chi(x) = 1$ for $|x| > 1$, and $0 \le \chi(x) \le 1$ for all x.
Let $\chi_R(x) = \chi(x/R)$



Then

$$\int \chi_R u^{n+1} dx - \int \chi_R u^n dx = k \int \chi_R \Delta_h (\varphi \circ u^n) dx = k \int (\chi_R - 1) \Delta_h (\varphi \circ u^n) dx$$

$$= k \int \Delta_h \chi_R (\varphi \circ u^n) dx = k\, O(R^{-2}) \text{ as } R \to \infty .$$

By induction, $\int \chi_R u^n dx - \int \chi_R u^0 dx = kn\, O(R^{-2}) = O(TR^{-2})$.

L^1 compactness:

By Kolmogorov, the set $\{u^n \mid n \in \mathbb{N}, kn \leq T\}$ is precompact in $L^1(\mathbb{R})$.

A small but necessary digression:

ESTIMATES FOR $\|\Delta_h u\|_1$, where $u \in L^1(\mathbb{R})$.

Assume first u is smooth: Then

$u(x+h) - u(x) = \int_0^h u'(x+t)\,dt$. So

$$u(x+h) - 2u(x) + u(x-h) = (u(x+h) - u(x)) - (u(x) - u(x-h))$$
$$= \int_0^h (u'(x+t) - u'(x-h+t))\,dt$$
$$= \int_0^h \int_{-h}^0 u''(x+s)\,ds\,dt \quad \text{and so}$$
$$\|\Delta_h u\|_1 = h^{-2} \int_{\mathbb{R}} \left| \int_0^h \int_{-h}^0 u''(x+s)\,ds\,dt \right| dx$$
$$\leq h^{-2} \int_0^h \int_{-h}^0 \int_{\mathbb{R}} |u''(x+s)|\,dx\,ds\,dt = \|u''\|_1$$

Next, let ρ be a standard mollifier, $\rho_\varepsilon(x) = \varepsilon^{-1} \rho(x/\varepsilon)$.

Now $\rho_\varepsilon''(x) = \varepsilon^{-3} \rho''(x/\varepsilon)$, $\|\rho_\varepsilon''\|_1 = \varepsilon^{-2} \|\rho''\|_1$

If $u \in L^1(\mathbb{R})$ then $\Delta_h(u * \rho_\varepsilon) = u * \Delta_h(\rho_\varepsilon)$ and

$$\|\Delta_h(u * \rho_\varepsilon)\|_1 \leq \|u\|_1 \cdot \|\Delta_h \rho_\varepsilon\|_1 \leq \varepsilon^{-2} \|\rho''\|_1 \cdot \|u\|_1 .$$

Approximation by mollifier:

Write $u = u^n$. Then $u(x) - u*\rho_\varepsilon(x) = \int_{-\varepsilon}^{\varepsilon} (u(x) - u(x-y))\rho_\varepsilon(y)\,dy$

and therefore

$$\|u - u*\rho_\varepsilon\|_1 \le \int_{-\varepsilon}^{\varepsilon} \underbrace{\int_{\mathbb{R}} |u(x) - u(x-y)|\,dx}_{\le \omega(\varepsilon)}\, \rho_\varepsilon(y)\,dy \le \omega(\varepsilon).$$

Write $u_\varepsilon^n = u^n * \rho_\varepsilon$. Then u_ε^n also satisfies the scheme! So $u_\varepsilon^{n+1} - u_\varepsilon^n = k\,\Delta_h u_\varepsilon^n$, and therefore

$$\|u_\varepsilon^{n+1} - u_\varepsilon^n\|_1 = k\,\|\Delta_h u_\varepsilon^n\|_1 \le \varepsilon^{-2} k\, c\, \|u^0\|_1 \qquad (c = \|\rho''\|_1).$$

By induction, then $\|u_\varepsilon^{n+j} - u_\varepsilon^n\|_1 \le \varepsilon^{-2} C jk \quad (C = c\|u^0\|_1)$

Thus we get

$$\|u^{n+j} - u^n\|_1 \le \underbrace{\|u^{n+j} - u_\varepsilon^{n+j}\|_1}_{\le \omega(\varepsilon)} + \underbrace{\|u_\varepsilon^{n+j} - u_\varepsilon^n\|_1}_{\le \varepsilon^{-2} C jk} + \underbrace{\|u_\varepsilon^n - u^n\|_1}_{\le \omega(\varepsilon)}$$

Thus

$$\|u^{n+j} - u^n\|_1 \le \bar{\omega}(jk) := \inf_{\varepsilon > 0} \left(2\omega(\varepsilon) + \varepsilon^{-2} C jk\right)$$

We note that $\lim_{s \downarrow 0} \bar{\omega}(s) = 0$:

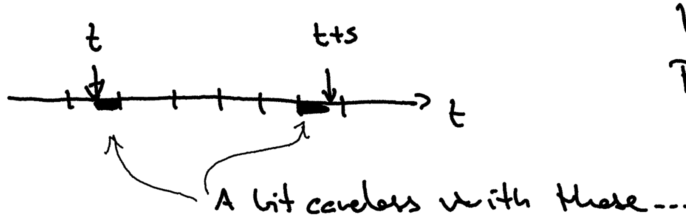
(Given any $\eta > 0$, pick ε so $\omega(\varepsilon) < \eta$, then note that $\varepsilon^{-2} C s < \eta \Leftrightarrow s < \varepsilon^2 \eta / C$.)

Define an approximate solution $\tilde{u}(t,x)$ by linear interpolation: With $t^n = nk$, put

$$\tilde{u} = \frac{t^{n+1}-t}{k} u^n + \frac{t-t^n}{k} u^{n+1} \quad \text{for } t \in [t^n, t^{n+1}].$$

Then we get $\|\tilde{u}(t+s) - \tilde{u}(t)\|_1 \le \underbrace{\tilde{\omega}(s)}$

a bit sloppy here: Disregarding partial intervals...



We can now call upon the Arzelà–Ascoli theorem: Writing our approximate solution $\tilde{u}$ as $\tilde{u}(h,k) : [0,T] \to L^1(\mathbb{R})$, there is a sequence $(h_i, k_i) \to (0,0)$ with $k_i \le \frac{h_i^2}{2K}$ which converges uniformly to some $u \in C([0,T]; L^1(\mathbb{R}))$.
Since φ is Lipschitz (constant K), also
$\|\varphi(\tilde{u}(h,k)) - \varphi(u)\|_1 \le K \|\tilde{u}(h,k) - u\|_1 \to 0$ uniformly.
Thus the limit u is a weak solution.

We have proved **EXISTENCE**:

Assume $\varphi : \mathbb{R} \to \mathbb{R}$ is strictly increasing and locally Lipschitz,

Then the initial value problem

$$\begin{cases} u_t = \Delta(\varphi \circ u) & x \in \mathbb{R}^d, \ t > 0 \\ u(0,x) = u_0(x) \end{cases}$$

has a solution

$$u \in L^\infty([0,T] \times \mathbb{R}^d) \cap C([0,T]; L^1(\mathbb{R}^d)),$$

in the weak sense:

$$\int_0^T \int_{\mathbb{R}^d} (u \psi_t + \varphi(u) \Delta \psi) \, dx \, dt + \int_{\mathbb{R}^d} u_0(x) \psi(0,x) \, dx = 0$$

for all $\psi \in \mathcal{D}((-\infty, T) \times \mathbb{R}^d)$.

Assuming we can prove that only one such solution exists, i.e., **uniqueness**, then we must get the same limit for every choice of $(h_i, k_i) \to (0,0)$ (satisfying $k_i \leq \frac{h_i^2}{2K}$),

and so we conclude that $\tilde{u}(h,k) \to u$ when $(h,k) \to (0,0)$ with $k \leq \frac{h^2}{2K}$.