Distributions and Convolutions

Spaces: $\mathcal{D} = C_c^\infty(\mathbb{R}^d)$ $\varphi_n \to \varphi$ in $\mathcal{D}$ means $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly for all α, and $\operatorname{supp} \varphi_n \subseteq K$ for some compact $K \subset \mathbb{R}^d$.

$\mathcal{E} = C^\infty(\mathbb{R}^d)$ $\varphi_n \to \varphi$ in $\mathcal{E}$ means $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly on every compact $K \subset \mathbb{R}^d$, for every α.

$\mathcal{D}'$, $\mathcal{E}'$ are the dual spaces of $\mathcal{D}, \mathcal{E}$. Convergence is weak convergence.

Alternative characterization of $\mathcal{D}'$:
A linear functional T on $\mathcal{D}$ is a distribution if, and only if, for each compact $K \subset \mathbb{R}^d$ there are constants k and C so that for all $\varphi \in \mathcal{D}$ with $\operatorname{supp} \varphi \subseteq K$: $|\langle T, \varphi \rangle| \le C \|\varphi\|_{C^k}$

Here, $\displaystyle \|\varphi\|_{C^k} = \max_{\substack{|\alpha| \le k \\ x \in \mathbb{R}^d}} |\partial^\alpha \varphi(x)|$.

Proof: Assume $K \subseteq \mathbb{R}^d$ is compact, but the stated definition does not hold. Then for each k, we can find φ_k with $\operatorname{supp} \varphi_k \subseteq K$, $\langle T, \varphi_k \rangle = 1$ and $\|\varphi_k\|_{C^k} \le 1/k$. But then $\varphi_k \to 0$ in $\mathcal{D}$, while $\langle T, \varphi_k \rangle \not\to 0$, so T is not continuous.

Conversely, if the stated condition holds, it is easy to see that T is a distribution. $\square$

Observation: The inclusion map $\mathcal{D} \to \mathcal{E}$ is a dense imbedding; i.e., it is continuous, injective, and the image is dense in $\mathcal{E}$.

Proof: The only part that is not obvious is the density. Let $\chi \in \mathcal{D}$ be so that $\chi = 1$ in a neighbourhood of 0. If $\psi \in \mathcal{E}$ then $\varphi_n(x) = \chi(x/n)\psi(x)$ belongs to $\mathcal{D}$, and $\varphi_n \to \psi$ in $\mathcal{E}$. Indeed for any given compact $K \subseteq \mathbb{R}^d$, $\varphi_n = \psi$ on K when n is big enough. $\square$

Corollary: The restriction map $\mathcal{E}' \to \mathcal{D}'$ $(\psi \mapsto \psi|_{\mathcal{D}})$ is a dense imbedding.

Proof. If $S, T \in \mathcal{E}'$ and $S|_{\mathcal{D}} = T|_{\mathcal{D}}$, then $S = T$ because $\mathcal{D}$ is dense and S, T are continuous. Thus the restriction map is injective. It is continuous, for if $T_n \to T$ in $\mathcal{E}'$ then $\langle T_n, \psi\rangle \to \langle T, \psi\rangle$ for all $\psi \in \mathcal{E}$, and hence for all $\psi \in \mathcal{D}$. It is dense, for with χ as defined above, if $T \in \mathcal{D}'$ we can define $T_n \in \mathcal{E}'$ by $\langle T_n, \psi\rangle = \langle T, \chi_n \psi\rangle$ where $\chi_n(x) = \chi(x/n)$. (The remainder of the proof is left to the reader. $\square$

CHARACTERIZATION OF $\mathcal{E}'$: Let $T \in \mathcal{D}'$. Then $T \in \mathcal{E}'$ if and only if $\operatorname{supp} T$ is compact.

Proof Assume $\operatorname{supp} T$ is <u>not</u> compact. Then there is some $\varphi_1 \in \mathcal{D}$ so that $\bar{B}_1(0) \cap \operatorname{supp} \varphi_1 = \emptyset$, yet $\langle T, \varphi_1\rangle = 1$. Let $\operatorname{supp} \varphi_1 \subseteq \bar{B}_{n_1}(0)$ for some integer n_1. There is some $\varphi_2 \in \mathcal{D}$ so that $\bar{B}_{n_1}(0) \cap \operatorname{supp} \varphi_2 = \emptyset$, yet $\langle T, \varphi_2\rangle = 1$. Continue in this way: Then $\varphi_n \to 0$ in $\mathcal{E}$, but $\langle T, \varphi_n\rangle \not\to 0$, so T is not continuous.

The converse is easy. $\square$

Convolution of test functions

If $\varphi, \psi \in \mathcal{D}$ then we have

$$\varphi * \psi(x) = \int_{\mathbb{R}^d} \varphi(y)\psi(x-y)\,dy = \int_{\mathbb{R}^d} \varphi(y)\,\tau_y\psi(x)\,dy,$$

and in fact we can write

$$\boxed{\varphi * \psi = \int_{\mathbb{R}^d} \varphi(y)\,\tau_y\psi\,dy}$$

where the integral is a limit <u>in $\mathcal{D}$</u> of the (d-dimensional) Riemann sums

$$\sum_j \varphi(y_j)\,\tau_{y_j}\psi\,\operatorname{vol}(R_j),$$

where a rectangular box containing $\operatorname{supp}\varphi$ is divided into rectangular boxes R_j, and $y_j \in R_j$, with the limit taken as the maximum size of the R_j goes to zero.

From the above we conclude that

$$\boxed{\langle T, \varphi * \psi\rangle = \int_{\mathbb{R}^d} \varphi(y)\,\langle T, \tau_y\psi\rangle\,dy}$$

for $T \in \mathcal{D}'$. Since the left hand side is unchanged when we swap φ and ψ, the same must be true of the right hand side.

CONVOLUTION: DISTRIBUTION * TEST FUNCTION

If $f \in L^1_{loc}$ and $\psi \in \mathcal{D}$, then

$$f * \psi(x) = \int_{\mathbb{R}^d} f(y)\psi(x-y)\,dy = \int_{\mathbb{R}^d} f(y)\tau_x\psi_\sigma(y)\,dy$$

and this motivates the

Definition: $\boxed{T * \psi(x) = \langle T, \tau_x \psi_\sigma\rangle}$

for $T \in \mathcal{D}'$, $\psi \in \mathcal{D}$. (This also works for $T \in \mathcal{E}'$, $\psi \in \mathcal{E}$.)

The formula on the bottom of the preceding page can now be written

$$\langle T, \varphi * \psi\rangle = \int_{\mathbb{R}^d} \varphi(y)\, T * \psi_\sigma(y)\,dy, \quad \text{or better:}$$

$$\boxed{\langle T, \varphi * \psi\rangle = \langle T * \psi_\sigma, \varphi\rangle.}$$

We can translate that into the language of convolutions:

$$\begin{aligned} T * (\varphi * \psi)(x) &= \langle T, \tau_x(\varphi * \psi)_\sigma\rangle = \langle T, \tau_x(\varphi_\sigma * \psi_\sigma)\rangle \\ &= \langle T, (\tau_x\varphi_\sigma) * \psi_\sigma\rangle = \langle T * \psi, \tau_x\varphi_\sigma\rangle \\ &= (T * \psi) * \varphi(x) \end{aligned}$$

Since $\varphi * \psi = \psi * \varphi$, we can write this as an associative law:

$$\boxed{T * (\psi * \varphi) = (T * \psi) * \varphi.}$$

CONVOLUTION: Distribution * distribution

We wish to define $S*T$ where S,T are distributions, so that

$$(S*T)*\varphi = S*(T*\varphi) \qquad (\varphi\in\mathcal{D}).$$

The right hand side is defined only if at least one of S,T has compact support.

To be specific, let us assume $S\in\mathcal{D}'$, $T\in\mathcal{E}'$.

Write the <u>desired</u> associative law above as

$$\langle S*T, \tau_x\varphi_\sigma\rangle = \langle S, \tau_x(T*\varphi)_\sigma\rangle$$
$$= \langle S, \tau_x T_\sigma * \varphi_\sigma\rangle$$

(Note that $(\tau_x T_\sigma)*\varphi_\sigma = \tau_x(T_\sigma*\varphi_\sigma)$.)

Put $x=0$ and replace φ by φ_σ, and get

$$\boxed{\langle S*T, \varphi\rangle = \langle S, T_\sigma*\varphi\rangle}$$

We take this as the <u>definition</u> of $S*T$.

In this definition, we can replace φ by $\tau_x\varphi_\sigma$, then run the above calculation in reverse, and conclude that the associative law (top of the page) <u>does</u> indeed hold.

We have just neglected one thing: $S*T$ is clearly well defined, but we have not shown that it is a <u>distribution</u>! (It is obviously linear.)

So pick any compact $K \subseteq \mathbb{R}^d$.
We need to estimate $|\langle S*T, \varphi\rangle|$
when $\operatorname{supp}\varphi \subseteq K$.

But $\operatorname{supp}(T_\sigma * \varphi) \subseteq \tilde{K} := K + \operatorname{supp} T_\sigma = K - \operatorname{supp} T$,
and $\tilde{K}$ is compact.

Let k_S, C_S be such that $|\langle S, \psi\rangle| \le C_S \|\psi\|_{C^{k_S}}$
whenever $\operatorname{supp}\psi \subseteq \tilde{K}$.

We must estimate $\|T_\sigma * \varphi\|_{C^{k_S}}$: But

$$|\partial^\alpha (T_\sigma * \varphi)(x)| = |T_\sigma * \partial^\alpha \varphi(x)| = |\langle T_\sigma, \tau_x \partial^\alpha \varphi_\sigma\rangle|$$
$$\le C_T \|\tau_x \partial^\alpha \varphi_\sigma\|_{C^{k_T}} \le C_T \|\varphi_\sigma\|_{C^{k_S+k_T}}$$

for $|\alpha| \le k_S$, where C_T, k_T are such that

$$|\langle T, \varphi\rangle| \le C_T \|\varphi\|_{C^{k_T}}$$

for all $\varphi \in \mathcal{D}$. (These constants exist because T has compact support, so we get this estimate independently of $\operatorname{supp}\varphi$.)

In summary, then, if $\operatorname{supp}\varphi \subseteq K$ then

$$|\langle S*T, \varphi\rangle| \le C_S C_T \|\varphi\|_{C^{k_S+k_T}}$$

and so $S*T$ is indeed a distribution!