

SOME CURIOUS RESULTS RELATED TO A CONJECTURE OF STROHMER AND BEAVER

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ABSTRACT. We study results related to a conjecture formulated by Thomas Strohmer and Scott Beaver about optimal Gaussian Gabor frame set-ups. Our attention will be restricted to the case of Gabor systems with standard Gaussian window and rectangular lattices of density 2. Although this case has been fully treated by Faulhuber and Steinerberger, the results in this work are new and quite curious. Indeed, the optimality of the square lattice for the Tolimieri and Orr bound already implies the optimality of the square lattice for the sharp lower frame bound. Our main tools include determinants of Laplace–Beltrami operators on tori as well as special functions from analytic number theory, in particular Eisenstein series, zeta functions, theta functions and Kronecker’s limit formula.

1. INTRODUCTION

The conjecture of Thomas Strohmer and Scott Beaver [29] relates to the problem of finding an optimal sampling strategy for Gaussian Gabor systems. They conjecture that a hexagonal pattern should be used in the time-frequency plane in order to minimize the condition number of the Gabor frame operator. This can also be formulated as an engineering problem about optimal orthogonal frequency division multiplexing (OFDM) systems and lattice-OFDM systems (see [29]). We will use the Gabor frame formulation of the problem.

Given a standard Gaussian window, we are looking for a lattice of given density (greater than 1) such that the condition number of the frame operator is minimal. For general lattices it is conjectured that the hexagonal lattice is optimal. If only separable (or rectangular) lattices are considered, the conjecture is that the square lattice provides the optimal solution. For even density, it was proven that the hexagonal lattice uniquely minimizes the upper frame bound [8] and in the separable case the square lattice minimizes the upper frame bound and maximizes the lower frame bound, implying that the condition number is minimal [9].

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From a qualitative point of view, the lower frame bound and the upper frame bound describe different aspects of the frame operator. Whereas the lower bound concerns the (bounded) invertibility of the frame operator, the upper bound gives the continuity of the operator since the frame operator is linear. Therefore, our main result can be considered to be quite curious.

Theorem 1.1 (Main Result). *Let $g_0(t) = 2^{1/4}e^{-\pi t^2}$ be the normalized standard Gaussian and $\Lambda_{(\alpha,\beta)} = \alpha\mathbb{Z} \times \beta\mathbb{Z}$. We denote the optimal lower and upper frame bound of the Gabor system $\mathcal{G}(g_0, \Lambda_{(\alpha,\beta)})$ by $A(\alpha, \beta)$ and $B(\alpha, \beta)$ respectively. By $\tilde{B}(\alpha, \beta)$ we denote the Tolimieri and Orr bound*

$$\tilde{B}(\alpha, \beta) = \sum_{\lambda^\circ \in \Lambda_{(\alpha,\beta)}^\circ} |V_{g_0}g_0(\lambda^\circ)|, \quad \Lambda_{(\alpha,\beta)}^\circ = \frac{1}{\beta}\mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z}.$$

Then, for $(\alpha\beta)^{-1} = 2$, we have the following implications

$$\begin{aligned} \tilde{B}\left(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right) \leq \tilde{B}(t\alpha, t\beta), \quad \forall t \in \mathbb{R}_+ &\implies A^4 B^2\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \geq A^4 B^2(\alpha, \beta) \\ &\implies A\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \geq A(\alpha, \beta). \end{aligned}$$

In each case, equality holds only if $\alpha = \beta = \frac{1}{\sqrt{2}}$.

This work is structured as follows.

- In Section 2 we introduce Gabor systems and frames for the Hilbert space $L^2(\mathbb{R})$. The concepts can easily be generalized to higher dimensions, but in this work we only consider the 1-dimensional case. At the end of the section we state 2 conjectures resulting from a conjecture of Strohmer and Beaver [29] and the results of Faulhuber [8] as well as Faulhuber and Steinerberger [9].
- In Section 3 we introduce some methods, going back to the work by Janssen [22], [23], which allow us to compute sharp frame bounds.
- In Section 4 we leave the field of time-frequency analysis and introduce some tools from analytic number theory. These tools include theta functions, zeta functions related to the Laplace–Beltrami operator on a torus and the Dedekind eta function. Equipped with these tools, we will prove our main result.
- In Section 5 we discuss some aspects of the proof.

2. TIME-FREQUENCY ANALYSIS

2.1. Gabor Systems. In this section we will set the notation and briefly recall the notion of a Gabor system. We are interested in Gabor systems for square integrable functions on the line. Hence, the Hilbert space of interest is $L^2(\mathbb{R})$. We note that the concepts can easily be generalized to higher dimensions or, more general, to locally compact Abelian groups (see e.g. [21]). For an introduction or more information on Gabor systems and frames we refer to some of the standard literature e.g. [3], [10], [11], [19], [20].

For the inner product in $L^2(\mathbb{R})$ we write

$$\langle f, g \rangle = \int_{\mathbb{R}} f \bar{g} d\mu,$$

where $d\mu$ denotes the Lebesgue measure and \bar{g} denotes the complex conjugate of g . For the Fourier transform we write

$$\mathcal{F}f(\omega) = \int_{\mathbb{R}} f(t)e^{-2\pi i\omega t} dt.$$

Next, we introduce the fundamental tool in time-frequency analysis, the short-time Fourier transform.

Definition 2.1 (STFT). For a function $f \in L^2(\mathbb{R})$ and a window $g \in L^2(\mathbb{R})$, the short-time Fourier transform of f with respect to g is defined as

$$V_g f(x, \omega) = \int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-2\pi i\omega t} dt, \quad x, \omega \in \mathbb{R}.$$

A Gabor system for $L^2(\mathbb{R})$ is generated by a (fixed, non-zero) window function $g \in L^2(\mathbb{R})$ and an index set $\Lambda \subset \mathbb{R} \times \widehat{\mathbb{R}}$ and is denoted by $\mathcal{G}(g, \Lambda)$. It consists of time-frequency shifted versions of g . By $\lambda = (x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}}$ we denote the generic point in the time-frequency plane and $\widehat{\mathbb{R}}$ is the group of characters or dual group of \mathbb{R} . We identify the time-frequency plane with \mathbb{R}^2 and for a time-frequency shift by λ we write

$$\pi(\lambda)g(t) = M_\omega T_x g(t) = e^{2\pi i\omega t}g(t-x), \quad x, \omega, t \in \mathbb{R}.$$

In general, time-frequency shifts do not commute, however they fulfill the commutation relations

$$M_\omega T_x = e^{2\pi i\omega x}T_x M_\omega.$$

For a window function g and an index set Λ the Gabor system is

$$\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g \mid \lambda \in \Lambda\}.$$

We note that we have $V_g f(\lambda) = \langle f, \pi(\lambda)g \rangle$. The time-frequency shifted versions of the window g are called atoms. In order to be a frame, $\mathcal{G}(g, \Lambda)$ has to satisfy the frame inequality

$$(2.1) \quad A \|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \leq B \|f\|_2^2, \quad \forall f \in L^2(\mathbb{R}),$$

for some positive constants $0 < A \leq B < \infty$ called frame constants or frame bounds. In this work, whenever we use the term frame bounds, we usually refer to the tightest possible (optimal) bounds in (2.1).

It is clear from the frame inequality (2.1) that the frame bounds crucially depend on the window g and the index set Λ and in general it is hard to calculate sharp frame bounds. However, for certain windows and certain index sets we have explicit formulas, as we will see later.

To the Gabor system $\mathcal{G}(g, \Lambda)$ we can associate the Gabor frame operator $S_{g, \Lambda}$ which acts on an element $f \in L^2(\mathbb{R})$ by the rule

$$S_{g, \Lambda} f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g.$$

The (optimal) frame bounds are connected to the frame operator in the following way

$$A^{-1} = \left\| S_{g, \Lambda}^{-1} \right\|_{op} \quad \text{and} \quad B = \|S_{g, \Lambda}\|_{op}.$$

The condition number of the frame operator is

$$\text{cond}(S_{g,\Lambda}) = B/A.$$

Hence, the frame bounds serve as a quantitative measure of how close or far the frame operator is from (a multiple of) the identity operator. However, as mentioned in the introduction, qualitatively the lower bound shows that the frame operator is boundedly invertible and the upper bound shows that the frame operator is continuous on $L^2(\mathbb{R})$.

2.2. Lattices. Throughout this work the index set $\Lambda \subset \mathbb{R}^2$ will be a lattice. A lattice Λ is generated by an invertible (non-unique) 2×2 matrix M , in the sense that

$$\Lambda = M\mathbb{Z}^2 = \{kv_1 + lv_2 \mid k, l \in \mathbb{Z}\},$$

where v_1 and v_2 are the columns of M . The parallelogram spanned by the vectors v_1 and v_2 is called the fundamental domain of Λ . The volume and the density of the lattice are given by

$$\text{vol}(\Lambda) = |\det(M)| \quad \text{and} \quad \delta(\Lambda) = \frac{1}{\text{vol}(\Lambda)},$$

respectively. The adjoint lattice Λ° is defined as

$$\Lambda^\circ = JM^{-T}\mathbb{Z}^2, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and M^{-T} denotes the transposed inverse of M . Hence, the adjoint lattice is just a 90 degrees rotated version of the dual lattice, usually denoted by Λ^\perp .

Furthermore, we will limit our attention to separable lattices. A lattice is called separable if the generating matrix can be represented by a diagonal matrix, which means that the lattice and its adjoint are given by

$$\Lambda_{(\alpha,\beta)} = \alpha\mathbb{Z} \times \beta\mathbb{Z} \quad \text{and} \quad \Lambda_{(\alpha,\beta)}^\circ = \frac{1}{\beta}\mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z}.$$

2.3. Conjectures Related to the Work of Strohmer and Beaver. Throughout this work we call the function $g_0(t) = 2^{1/4}e^{-\pi t^2}$ the standard Gaussian. In their article from 2003, Strohmer and Beaver conjecture that for any fixed density the hexagonal lattice minimizes the condition number B/A of the Gabor frame operator $S_{g_0,\Lambda}$. The following conjecture implies the conjecture of Strohmer and Beaver [29].

Conjecture 2.2. *For the standard Gaussian window and any fixed lattice density $\delta > 1$, among all lattices in the set*

$$\mathfrak{F}_\Lambda^\delta(g_0) = \{\Lambda \subset \mathbb{R}^2 \text{ a lattice} \mid \text{vol}^{-1}(\Lambda) = \delta\},$$

the hexagonal lattice is the unique maximizer for the lower frame bound and the unique minimizer for the upper frame bound.

Before the work of Strohmer and Beaver a conjecture by le Floch, Alard and Berrou (1995) [13] suggested that the square lattice could be optimal. This was disproved in [29], however, if we only consider the separable case we get the following conjecture.

Conjecture 2.3. *For the standard Gaussian window and any fixed lattice density $\delta > 1$, among all separable lattices in the set*

$$\mathfrak{F}_{(\alpha,\beta)}^\delta(g_0) = \{\alpha\mathbb{Z} \times \beta\mathbb{Z} \mid \alpha, \beta \in \mathbb{R}_+, (\alpha\beta)^{-1} = \delta\},$$

the square lattice is the unique maximizer for the lower frame bound and the unique minimizer for the upper frame bound.

For certain densities of the lattice, Conjecture 2.3 was proved by Faulhuber and Steinerberger in 2017 [9]. Also, for certain densities of the lattice it was shown that the hexagonal lattice minimizes the upper frame bound [8]. The general case for the lower frame bound is still open for all possible densities ¹.

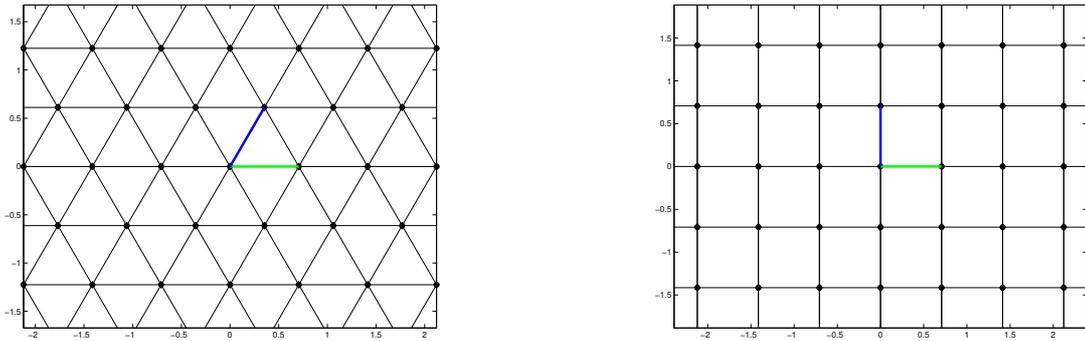


FIGURE 1. A hexagonal (sometimes called triangular) lattice and a square lattice of density 2. A possible basis is marked in both cases.

3. SHARP FRAME BOUNDS

3.1. Janssen's Work. Our starting point for proving Theorem 1.1 is the work of Janssen [22], [23]. Janssen observed that for integer density the frame bounds can be computed exactly by finding the minimum and the maximum of certain Fourier series related to the frame operator. This follows from the duality theory which he described in [22] and which allows to compute the spectrum of the frame operator via the spectrum of a related, bi-infinite matrix. For integer density of the lattice this bi-infinite matrix has a Laurent structure and, hence, by the general theory of Toeplitz and Laurent matrices and operators, the spectrum can be calculated by using Fourier series with the matrix entries as coefficients [23].

Janssen stated his results for $L^2(\mathbb{R})$ and separable lattices. However, since for $d = 1$ any lattice is symplectic, this actually already covers all cases since any Gabor system can be transformed into a Gabor system with separable lattice without losing any of its properties by using the proper metaplectic operator (see e.g. [5], [14], [17], [19]).

Let us now state the result in its original form, since we do not need a more general setting. A standard assumption is

$$\text{(Condition A)} \quad \sum_{\lambda^\circ \in \Lambda^\circ} |V_g g(\lambda^\circ)| < \infty.$$

As we will only need separable lattices in the sequel, we state the following result as given in [23].

¹Tolimieri and Orr [30] mention that, for even densities, in private correspondence with Janssen they found a lengthy argument solving the problem for the upper frame bound in Conjecture 2.3. It is very likely that the argument they found is similar to the proof in [9].

Proposition 3.1. *Let $g \in L^2(\mathbb{R})$, fulfilling (Condition A), and $\Lambda_{(\alpha,\beta)} = \alpha\mathbb{Z} \times \beta\mathbb{Z}$, $(\alpha\beta)^{-1} \in \mathbb{N}$. The optimal frame bounds of the Gabor system $\mathcal{G}(g, \Lambda_{(\alpha,\beta)})$ are given by*

$$A = \operatorname{ess\,inf}_{(x,\omega) \in \mathbb{R}^2} (\alpha\beta)^{-1} \sum_{k,l \in \mathbb{Z}} V_g g \left(\frac{k}{\beta}, \frac{l}{\alpha} \right) e^{2\pi i(kx+l\omega)}$$

$$B = \operatorname{ess\,sup}_{(x,\omega) \in \mathbb{R}^2} (\alpha\beta)^{-1} \sum_{k,l \in \mathbb{Z}} V_g g \left(\frac{k}{\beta}, \frac{l}{\alpha} \right) e^{2\pi i(kx+l\omega)}$$

A reformulation of Proposition 3.1, using symmetric time-frequency shifts and the ambiguity function, as well as a generalization to lattices in higher dimensions, can be found in [6].

3.2. A Remark on Condition A. We will now make a small excursion, showing that a good upper frame bound already implies a good lower frame bound and, hence, a good condition number. Recall that the frame operator acts on functions by the rule

$$S_{g,\Lambda} f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g.$$

The following result shows how to compute the distance between the identity operator I_{L^2} and the frame operator using (Condition A). We have

$$\|I_{L^2} - \operatorname{vol}(\Lambda)S_{g,\Lambda}\|_{op} \leq \sum_{\lambda^\circ \in \Lambda^\circ \setminus \{0\}} |V_g g(\lambda^\circ)|.$$

This follows from Janssen's representation of the Gabor frame operator

$$(3.1) \quad S_{g,\Lambda} = \operatorname{vol}(\Lambda)^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \langle g, \pi(\lambda^\circ)g \rangle \pi(\lambda^\circ),$$

which is basically a consequence of the so-called fundamental identity of Gabor analysis

$$\sum_{\lambda \in \Lambda} \langle f_1, \pi(\lambda)g_1 \rangle \langle \pi(\lambda)f_2, g_2 \rangle = \operatorname{vol}(\Lambda)^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \langle \pi(\lambda^\circ)f_1, g_2 \rangle \langle f_2, \pi(\lambda^\circ)g_1 \rangle.$$

For more details we refer to [12], [19], [22].

We will now assume that g is normalized, i.e., $\|g\|_2 = 1$. Also, and only for the rest of this section, we will normalize the frame inequality in the following sense;

$$(3.2) \quad A \|f\|_2^2 \leq \operatorname{vol}(\Lambda) \sum_{\lambda \in \Lambda} |V_g f(\lambda)|^2 \leq B \|f\|_2^2.$$

The bounds in (3.2) fulfill (see e.g. [21, Thm. 5.1.]

$$(3.3) \quad 0 \leq A \leq 1 \leq B < \infty.$$

Note that we included the case where the frame operator is not invertible. The upper bound B is finite as we assume (Condition A). In fact, the results in the work of Tolimiere and Orr [30] show that

$$\tilde{B} = \sum_{\lambda^\circ \in \Lambda^\circ} |V_g g(\lambda^\circ)|$$

always is an upper bound, not necessarily optimal, for the normalized frame inequality (3.2)².

²Hence, we have that $\operatorname{vol}(\Lambda)^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} |V_g g(\lambda^\circ)|$ always is an upper bound for the frame inequality (2.1).

We get the following results, which are a direct consequence of (3.1) and are also stated in [32, Chap. 3.1], where they are referred to as Janssen's test.

Proposition 3.2. *Assume that $g \in L^2(\mathbb{R})$ with $\|g\|_2 = 1$ and fulfilling (Condition A). If*

$$\tilde{B} < 2,$$

then $\mathcal{G}(g, \Lambda)$ is a frame. Also, if $\mathcal{G}(g, \Lambda)$ is not a frame, then

$$\tilde{B} \geq 2.$$

Proof. We use Janssen's representation of the Gabor frame operator and re-write it as

$$\text{vol}(\Lambda)S_{g,\Lambda} = \sum_{\lambda^\circ \in \Lambda^\circ} \langle g, \pi(\lambda^\circ)g \rangle \pi(\lambda^\circ) = \langle g, \pi(0)g \rangle \pi(0) + \sum_{\lambda^\circ \in \Lambda^\circ \setminus \{0\}} \langle g, \pi(\lambda^\circ)g \rangle \pi(\lambda^\circ).$$

Now, $\pi(0) = I_{L^2}$ and $\langle g, \pi(0)g \rangle = \|g\|_2^2 = 1$. Therefore,

$$\|\text{vol}(\Lambda)S_{g,\Lambda}\|_{op} \leq \|I_{L^2}\|_{op} + (\tilde{B} - 1).$$

It follows that, if $\tilde{B} < 2$, then

$$\|I_{L^2} - \text{vol}(\Lambda)S_{g,\Lambda}\|_{op} < \tilde{B} - 1 < 1,$$

hence, $\text{vol}(\Lambda)S_{g,\Lambda}$ is invertible and its inverse can be expressed by a Neumann series. The second statement now follows trivially. \square

Also, we get an estimate on the lower bound from (Condition A) (see also [32, Chap. 3.1]).

Proposition 3.3. *Assume that $g \in L^2(\mathbb{R})$ with $\|g\|_2 = 1$ and fulfilling (Condition A). Then we have the following estimates for the constants in (3.2).*

$$1 - A \leq \tilde{B} - 1 \quad \text{and} \quad B \leq \tilde{B}.$$

Proof. The statement that $B \leq \tilde{B}$ was proved by Tolimieri and Orr [30]. So, we only prove the statement for the lower frame bound.

If $\tilde{B} \geq 2$, then the statement is trivially true due to (3.3). Hence, assume $\tilde{B} < 2$, which already implies that we have a frame. This also means that

$$\|I_{L^2} - \text{vol}(\Lambda)S_{g,\Lambda}\|_{op} \leq \tilde{B} - 1 < 1.$$

Therefore, we have

$$A^{-1} = \left\| (\text{vol}(\Lambda)S_{g,\Lambda})^{-1} \right\|_{op} \leq \sum_{n \in \mathbb{N}} (\tilde{B} - 1)^n = \frac{1}{1 - (\tilde{B} - 1)}.$$

Hence, the result follows. \square

The previous result particularly shows that, if the bound \tilde{B} is close to 1, then the lower bound has to be close to 1 as well. This means, that a good upper frame bound has a good lower frame bound and a good condition number as a consequence. This already pinpoints into the direction of our main result, which is nonetheless stronger under its more restrictive assumptions.

3.3. Gaussian Windows. We will now use Proposition 3.1 to compute sharp frame bounds for the standard Gaussian window $g_0(t) = 2^{1/4}e^{-\pi t^2}$. It is a well-known fact (and also a nice exercise to verify) that

$$(3.4) \quad V_{g_0}g_0(x, \omega) = e^{-\pi i x \omega} e^{-\frac{\pi}{2}(x^2 + \omega^2)}.$$

By Proposition 3.1, it follows that the optimal frame bounds of the Gabor system $\mathcal{G}(g_0, \alpha\mathbb{Z} \times \beta\mathbb{Z})$, $(\alpha\beta)^{-1} \in \mathbb{N}$ are given by

$$A(\alpha, \beta) = (\alpha\beta)^{-1} \sum_{k, l \in \mathbb{Z}} (-1)^{\frac{kl}{\alpha\beta}} (-1)^{k+l} e^{-\frac{\pi}{2} \left(\frac{k^2}{\beta^2} + \frac{l^2}{\alpha^2} \right)},$$

$$B(\alpha, \beta) = (\alpha\beta)^{-1} \sum_{k, l \in \mathbb{Z}} (-1)^{\frac{kl}{\alpha\beta}} e^{-\frac{\pi}{2} \left(\frac{k^2}{\beta^2} + \frac{l^2}{\alpha^2} \right)}.$$

This was already computed by Janssen [23, Sec. 6].

We note that for $(\alpha\beta)^{-1} \in 2\mathbb{N}$ the alternating sign, given by $(-1)^{\frac{kl}{\alpha\beta}}$, equals $+1$ for all $k, l \in \mathbb{Z}$. We note that in this case the double series splits into a product of two simple series of same type;

$$A(\alpha, \beta) = (\alpha\beta)^{-1} \left(\sum_{k \in \mathbb{Z}} (-1)^k e^{-\frac{\pi}{2} \frac{k^2}{\beta^2}} \right) \left(\sum_{l \in \mathbb{Z}} (-1)^l e^{-\frac{\pi}{2} \frac{l^2}{\alpha^2}} \right),$$

$$B(\alpha, \beta) = (\alpha\beta)^{-1} \left(\sum_{k \in \mathbb{Z}} e^{-\frac{\pi}{2} \frac{k^2}{\beta^2}} \right) \left(\sum_{l \in \mathbb{Z}} e^{-\frac{\pi}{2} \frac{l^2}{\alpha^2}} \right).$$

The reader familiar with special functions will already have realized that we are dealing with the special cases of the classical Jacobi theta functions. We will introduce these functions in the upcoming section.

4. SOME TOOLS FROM ANALYTIC NUMBER THEORY

From here on, our analysis will be for arbitrary density of the lattice, i.e., $(\alpha\beta)^{-1} \in \mathbb{R}_+$. However, only if the density is an even, positive integer our formulas exactly describe the frame bounds of the Gaussian Gabor system. We will keep that fact in mind, even though we will not mention it explicitly.

4.1. Theta Functions. We will recall how Conjecture 2.3 was solved for even densities of the lattice by Faulhuber and Steinerberger [9]. First, we introduce the functions θ_2 , θ_3 and θ_4 . The literature on theta functions is extensive and any attempt to pick representative textbooks or surveys is doomed. We name the textbooks by Stein and Shakarchi [28] and Whittaker and Watson as [31] references.

As customary, we denote the complex plane by \mathbb{C} and the upper half plane by \mathbb{H} . A theta function is a function of two complex variables $(z, \tau) \in \mathbb{C} \times \mathbb{H}$ and the classical theta function is

$$\Theta(z, \tau) = \sum_{k \in \mathbb{Z}} e^{\pi i k^2 \tau} e^{2\pi i k z}.$$

In this work we will deal with the following list of theta functions which can be expressed by Θ in one way or another. We define

$$\begin{aligned} \vartheta_1(z, \tau) &= \sum_{k \in \mathbb{Z}} (-1)^{(k-1/2)} e^{\pi i(k+1/2)^2 \tau} e^{(2k+1)\pi i z}, & \vartheta_2(z, \tau) &= \sum_{k \in \mathbb{Z}} e^{\pi i(k+1/2)^2 \tau} e^{(2k+1)\pi i z}, \\ \vartheta_3(z, \tau) &= \sum_{k \in \mathbb{Z}} e^{\pi i k^2 \tau} e^{2k\pi i z}, & \vartheta_4(z, \tau) &= \sum_{k \in \mathbb{Z}} (-1)^k e^{\pi i k^2 \tau} e^{2k\pi i z}. \end{aligned}$$

These functions are Jacobi's classical theta functions.

For our purposes, we will use Jacobi's theta functions where we fix the first argument to be $z = 0$ (so called "Theta Nullwerte"). Hence, for $\tau \in \mathbb{H}$ we define the following theta functions.

$$\begin{aligned} \theta_2(\tau) &= \vartheta_2(0, \tau) = \sum_{k \in \mathbb{Z}} e^{\pi i \left(k + \frac{1}{2}\right)^2 \tau} \\ \theta_3(\tau) &= \vartheta_3(0, \tau) = \sum_{k \in \mathbb{Z}} e^{\pi i k^2 \tau} \\ \theta_4(\tau) &= \vartheta_4(0, \tau) = \sum_{k \in \mathbb{Z}} (-1)^k e^{\pi i k^2 \tau} \end{aligned}$$

We note that $\theta_1(\tau) = \vartheta_1(0, \tau) = 0$ for all $\tau \in \mathbb{H}$. Any of the above functions also has a product representation, called the Jacobi triple product representation;

$$\begin{aligned} \theta_2(\tau) &= 2 e^{-\frac{\pi}{4} \tau} \prod_{k \in \mathbb{N}} \left(1 - e^{2k\pi i \tau}\right) \left(1 + e^{2k\pi i \tau}\right)^2 \\ \theta_3(\tau) &= \prod_{k \in \mathbb{N}} \left(1 - e^{2k\pi i \tau}\right) \left(1 + e^{(2k-1)\pi i \tau}\right)^2 \\ \theta_4(\tau) &= \prod_{k \in \mathbb{N}} \left(1 - e^{2k\pi i \tau}\right) \left(1 - e^{(2k-1)\pi i \tau}\right)^2 \end{aligned}$$

Later, we will restrict our attention to purely imaginary τ and note, that the above functions are real-valued and positive in this case.

It is a well-known fact and easily proved by using the Poisson summation formula (see Appendix A) that

$$(4.1) \quad \theta_2(\tau) = \sqrt{\frac{i}{\tau}} \theta_4\left(-\frac{1}{\tau}\right) \quad \text{and} \quad \theta_4(\tau) = \sqrt{\frac{i}{\tau}} \theta_2\left(-\frac{1}{\tau}\right)$$

as well as

$$\theta_3(\tau) = \sqrt{\frac{i}{\tau}} \theta_3\left(-\frac{1}{\tau}\right).$$

We note that we can write the lower and upper bound of a Gaussian Gabor system as products of the above theta functions (θ_2 and θ_4 are always exchangeable by equation (4.1)).

$$\begin{aligned} A(\alpha, \beta) &= (\alpha\beta)^{-1} \theta_4\left(\frac{i}{2\alpha^2}\right) \theta_4\left(\frac{i}{2\beta^2}\right) \\ B(\alpha, \beta) &= (\alpha\beta)^{-1} \theta_3\left(\frac{i}{2\alpha^2}\right) \theta_3\left(\frac{i}{2\beta^2}\right) \end{aligned}$$

As already observed in [9], since the product $(\alpha\beta)^{-1}$ is fixed, the problem of finding extremal frame bounds can be reformulated as a problem of a (fixed) parameter and one variable. In [9], we find the following result.

Theorem 4.1. *For any fixed $r \in \mathbb{R}_+$, the function*

$$A_r(y) = \theta_4(ir y^{-1})\theta_4(ir y), \quad y \in \mathbb{R}_+$$

is maximal if and only if $y = 1$. Also, the function

$$B_r(y) = \theta_3(ir y^{-1})\theta_3(ir y), \quad y \in \mathbb{R}_+$$

is minimal if and only if $y = 1$.

This theorem is equivalent to saying that the Gabor frame bounds of the Gabor system $\mathcal{G}(g_0, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ are extremal if and only if $\alpha = \beta$.

A key ingredient in the work of Faulhuber and Steinerberger [9] is to formulate a problem independent of the parameter $r \in \mathbb{R}_+$, hence, independent of the density of the lattice. We note that, in [9], there is no hint in the proof of Theorem 4.1 that the minimality of B_r implies the maximality of A_r , not even for the case $r = 1$, which we essentially prove in Theorem 1.1.

4.2. Zeta Functions. For this section we will identify \mathbb{R}^2 with \mathbb{C} . The lattice in focus is then $\alpha\mathbb{Z} \times i\beta\mathbb{Z}$. Also, we will focus our attention on another class of special functions now, namely zeta functions related to Laplace–Beltrami operators on rectangular tori. A rectangular torus can be identified with a rectangular lattice;

$$\mathbb{T}_{(\alpha,\beta)}^2 = \mathbb{C}/(\alpha\mathbb{Z} \times i\beta\mathbb{Z}).$$

We will now mainly follow the work of Baernstein and Vinson [1] and the work of Osgood, Philips and Sarnak [26]. For the torus $\mathbb{T}_{(\alpha,\beta)}^2$ we denote its Laplace–Beltrami operator by $\Delta_{(\alpha,\beta)}$. We note that the eigenfunctions of $\Delta_{(\alpha,\beta)}$ are given by

$$f_{(\alpha,\beta)}^{k,l}(z) = e^{2\pi i \operatorname{Re}\left(z \cdot \left(\frac{k}{\alpha} + i\frac{l}{\beta}\right)\right)}$$

with eigenvalues³

$$\lambda_{(\alpha,\beta)}^{k,l} = \left(2\pi \left|\frac{k}{\alpha} + i\frac{l}{\beta}\right|\right)^2.$$

Consequently, for $t \in \mathbb{R}_+$, the heat kernel is then given by the formula

$$p_{(\alpha,\beta)}(z; t) = \sum_{k,l \in \mathbb{Z}} e^{-t\lambda_{(\alpha,\beta)}^{k,l}} f_{(\alpha,\beta)}^{k,l}(z).$$

The trace of the heat kernel associated to $\Delta_{(\alpha,\beta)}$ is given by

$$\operatorname{tr}(p_{(\alpha,\beta)})(t) = \sum_{k,l \in \mathbb{Z}} e^{-t\lambda_{(\alpha,\beta)}^{k,l}} = \sum_{k,l \in \mathbb{Z}} e^{-4\pi^2 t \left(\frac{k^2}{\alpha^2} + \frac{l^2}{\beta^2}\right)}$$

and its zeta function is given by

$$\begin{aligned} (4.2) \quad Z_{(\alpha,\beta)}(s) &= \sum'_{k,l \in \mathbb{Z}} \left(\lambda_{(\alpha,\beta)}^{k,l}\right)^{-s} = (2\pi)^{-2s} \sum'_{k,l \in \mathbb{Z}} \left|\frac{k}{\alpha} + i\frac{l}{\beta}\right|^{-2s} \\ &= (2\pi)^{-2s} (\alpha\beta)^s \sum'_{k,l \in \mathbb{Z}} \frac{\left(\frac{\alpha}{\beta}\right)^s}{\left|k + i\frac{\alpha}{\beta}l\right|^{2s}}, \end{aligned}$$

³We chose the sign of $\Delta_{(\alpha,\beta)}$ such that the eigenvalues are non-negative.

where the prime indicates that the sum excludes the origin. The determinant of the Laplace–Beltrami operator is formally given as the product of the non-zero eigenvalues,

$$\det \Delta_{(\alpha,\beta)} = \prod_{\lambda_{(\alpha,\beta)}^{k,l} \neq 0} \lambda_{(\alpha,\beta)}^{k,l}.$$

This product is not always meaningful and, usually, one uses the zeta regularized determinant of the Laplace–Beltrami operator $\Delta_{(\alpha,\beta)}$ instead, which is given by

$$\det' \Delta_{(\alpha,\beta)} = e^{-\frac{d}{ds} Z_{(\alpha,\beta)} \Big|_{s=0}}.$$

The problem under consideration is to find the pair (α, β) which maximizes the determinant $\det' \Delta_{(\alpha,\beta)}$ for fixed volume of the torus, i.e., $0 < \alpha\beta$ fixed. We note that the last line in equation (4.2) is, up to the factor in front, an Eisenstein series;

$$E(\tau, s) = \sum'_{k,l \in \mathbb{Z}} \frac{y^s}{|k + l\tau|^{2s}}, \quad \tau = x + iy \in \mathbb{H}.$$

In order to compute the regularized determinant $\det' \Delta_{(\alpha,\beta)}$, Osgood, Philips and Sarnak employ Kronecker’s limit formula and get [26, eq. (4.4)]

$$\begin{aligned} E(\tau, 0) &= -1 \\ \frac{\partial}{\partial s} E \Big|_{s=0} &= -2 \log \left(2\pi y^{1/2} |\eta(\tau)|^2 \right). \end{aligned}$$

Here, η is the Dedekind eta function which we will study in more detail in the upcoming section.

By the last results, it follows that

$$\begin{aligned} \frac{d}{ds} Z_{(\alpha,\beta)} \Big|_{s=0} &= -2 \log \left(\frac{2\pi}{(\alpha\beta)^{1/2}} \right) E \left(i \frac{\alpha}{\beta}, 0 \right) + \frac{\partial}{\partial s} E \Big|_{s=0} \\ &= -\log(\alpha\beta) - \log \left(\left(\frac{\alpha}{\beta} \right) \Big| \eta \left(i \frac{\alpha}{\beta} \right) \Big|^4 \right). \end{aligned}$$

Therefore, the determinant is given by

$$\det' \Delta_{(\alpha,\beta)} = (\alpha\beta) \left(\frac{\beta}{\alpha} \right) \Big| \eta \left(i \frac{\beta}{\alpha} \right) \Big|^4.$$

Since we fixed the product $\alpha\beta$, which expresses the surface area of the torus, we see that the problem of maximizing the determinant is invariant under scaling and we may therefore assume that $\alpha\beta = 1$.

For the rest of this work, we choose the following normalization and notation for our torus;

$$\mathbb{T}_\alpha^2 = \mathbb{C} / \left(\alpha^{1/2} \mathbb{Z} \times i \alpha^{-1/2} \mathbb{Z} \right).$$

Likewise, in the sequel we will use the index α for the index pair $(\alpha^{1/2}, \alpha^{-1/2})$, appearing in the Laplace–Beltrami operator, the heat kernel, the determinant and the zeta function.

For general tori, it was shown in [26] that the hexagonal torus is the unique maximizer of the determinant by using a combined proof of analytic and numerical methods. For rectangular tori, it was shown in [7] that the square torus is the unique maximizer by purely analytic means. However, it also follows from the results in Montgomery’s work [25] that the hexagonal torus maximizes the determinant among all tori of given surface area. That was also remarked in the work of Baernstein and Vinson [1]. Likewise, it follows from one of the results of

Faulhuber and Steinerberger [9] that the square torus is the maximizer of the determinant among all rectangular tori of given surface area.

We will shortly sketch the arguments used in [1] and [25]. For brevity, we set

$$W_\alpha(t) = \text{tr}(p_\alpha)(t) = \sum_{k,l \in \mathbb{Z}} e^{-4\pi^2 t \left(\frac{k^2}{\alpha} + \alpha l^2 \right)}.$$

The results in [9] show that

$$W_\alpha(t) \geq W_1(t), \quad \forall t \in \mathbb{R}_+$$

with equality if and only if $\alpha = 1$.

We will show how this result already implies the result for the determinant. Now, the zeta function also has an integral representation and it is connected to the heat kernel via the Mellin transform [24].

$$(2\pi)^s \Gamma(s) Z_\alpha(s) = \int_{\mathbb{R}_+} (W_\alpha(t) - 1) t^{s-1} dt, \quad s \in \mathbb{C}, \text{Re}(s) > 1,$$

where Γ denotes the usual Gamma function. In [25], Montgomery uses the regularized version

$$(4.3) \quad (2\pi)^s \Gamma(s) Z_\alpha(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty (W_\alpha(t) - 1) (t^{s-1} + t^{-s}) dt,$$

which is well defined for $s \in \mathbb{C}$, except for the points 0 and 1. Also, we have the functional equation

$$(4.4) \quad (2\pi)^s \Gamma(s) Z_\alpha(s) = (2\pi)^{1-s} \Gamma(1-s) Z_\alpha(1-s).$$

Since $W_\alpha(t)$ does not depend on s , by differentiating both sides of (4.3), it follows that

$$Z'_\alpha(s) \geq Z'_1(s), \quad s > 1,$$

where the prime indicates differentiation with respect to s . By taking the limit $s \rightarrow 1$ and using the functional equation (4.4), we get

$$Z'_\alpha(0) \geq Z'_1(0)$$

and, since $\det' \Delta_\alpha = e^{-Z'(0)}$,

$$\det' \Delta_\alpha \leq \det' \Delta_1, \quad \forall \alpha \in \mathbb{R}_+$$

with equality if and only if $\alpha = 1$.

4.3. The Dedekind Eta Function. The last function we investigate is the Dedekind eta function. We start with defining the function by its product representation;

$$(4.5) \quad \eta(\tau) = e^{\pi i \tau / 12} \prod_{k \in \mathbb{N}} (1 - e^{2k\pi i \tau}), \quad \tau \in \mathbb{H}.$$

We note that it can be expressed as a product of theta functions in the following way (see e.g. [2, Chap. 3], [4, Chap. 4] or [31, (21.41) p. 463, Ex. 1 p. 466])⁴;

$$2\eta(\tau)^3 = \frac{\partial}{\partial z} \vartheta_1(z, \tau) \Big|_{z=0} = \vartheta_2(0, \tau) \vartheta_3(0, \tau) \vartheta_4(0, \tau).$$

Since we are interested in rectangular tori, we are only interested in purely imaginary arguments $\tau = iy$, $y \in \mathbb{R}_+$. We note that in this case it follows from the formula (4.5) defining

⁴The function $\frac{\partial \vartheta_1}{\partial z}(0, \tau)$ is often denoted by $\theta'_1(\tau)$.

η , that also $\eta(iy) \in \mathbb{R}_+$. Also, we have the following connection to the theta functions defined in Section 4.1;

$$(4.6) \quad 2\eta(iy)^3 = \theta_2(iy)\theta_3(iy)\theta_4(iy).$$

4.4. Proof of Theorem 1.1. We collected all the material to prove our main result. We recall that the optimal Gabor frame bounds of the Gaussian Gabor system $\mathcal{G}\left(g_0, \frac{1}{\sqrt{2}}\Lambda_y\right)$ with $\Lambda_y = (y^{1/2}\mathbb{Z} \times y^{-1/2}\mathbb{Z})$ are given by

$$\begin{aligned} A(\Lambda_y) &= 2\theta_4(iy^{-1})\theta_4(iy) = 2y^{1/2}\theta_2(iy)\theta_4(iy) \\ B(\Lambda_y) &= 2\theta_3(iy^{-1})\theta_3(iy) = 2y^{1/2}\theta_3(iy)\theta_3(iy). \end{aligned}$$

We start now by multiplying both sides of equation (4.6) with y^3 and take both sides to the power 4, which gives

$$2^4(\det'\Delta_y)^3 = 2^4y^3\eta(iy)^{12} = y^3\theta_2(iy)^4\theta_3(iy)^4\theta_4(iy)^4.$$

By multiplying both sides of the last equation with 2^6 , it follows that

$$2^{10}(\det'\Delta_y)^3 = \left(2y^{1/2}\theta_2(iy)\theta_4(iy)\right)^4 \left(2y^{1/2}\theta_3(iy)^2\right)^2 = A(\Lambda_y)^4B(\Lambda_y)^2.$$

Also, we note that for all $t \in \mathbb{R}_+$ we have

$$W_y\left(\frac{t}{4\pi}\right) = \sum_{k,l \in \mathbb{Z}} e^{-\pi t\left(\frac{k^2}{y} + yl^2\right)} = \tilde{B}\left(\frac{t}{\sqrt{2}}\Lambda_y\right) = \theta_3(it y^{-1})\theta_3(it y) = y^{1/2}\theta_3(it y)^2,$$

which by Theorem 4.1 is minimal if and only if $y = 1$. By the results from Section 4.2, we have the following chain of implications

$$\begin{aligned} W_y(t) \geq W_1(t), \quad \forall t \in \mathbb{R}_+ &\implies \det'\Delta_y \leq \det'\Delta_1 \\ &\implies y^{1/2}\theta_2(iy)\theta_4(iy) \leq \theta_2(i)\theta_4(i), \end{aligned}$$

as we have $(\det'\Delta_y)^3 = y^3\theta_2(iy)^4W_y\left(\frac{1}{4\pi}\right)^2\theta_4(iy)^4$. Consequently, we get the following chain of implications for the Gabor system $\mathcal{G}\left(g_0, \frac{1}{\sqrt{2}}\Lambda_y\right)$;

$$\begin{aligned} \tilde{B}\left(\sqrt{\frac{t}{2}}\Lambda_y\right) \geq \tilde{B}\left(\sqrt{\frac{t}{2}}\Lambda_1\right), \quad \forall t \in \mathbb{R}_+ &\implies A\left(\frac{1}{\sqrt{2}}\Lambda_y\right)^4 B\left(\frac{1}{\sqrt{2}}\Lambda_y\right)^2 \leq A\left(\frac{1}{\sqrt{2}}\Lambda_1\right)^4 B\left(\frac{1}{\sqrt{2}}\Lambda_1\right)^2 \\ &\implies A\left(\frac{1}{\sqrt{2}}\Lambda_y\right) \leq A\left(\frac{1}{\sqrt{2}}\Lambda_1\right). \end{aligned}$$

For the first implication we used the fact that for $t = 1$ we have $\tilde{B}\left(\frac{1}{\sqrt{2}}\Lambda_y\right) = B\left(\frac{1}{\sqrt{2}}\Lambda_y\right)$. This completes the proof of Theorem 1.1. \square

5. THE DIFFERENT ASPECTS OF THE PROOF

We note that the proof relies on several curious facts. We will discuss the different aspects in this section and start with an overview.

First of all, the results of Faulhuber and Steinerberger [9] and of Montgomery [25] are results about periodized Gaussian functions on a lattice. The parameter appearing in those works should be interpreted as the density of the lattice. However, the results can also be interpreted as problems for a family of heat kernels on a torus with varying metric. Only for

very special cases the results describe the exact behavior of Gaussian Gabor frame bounds. Also, we note that we needed the optimality of the Tolimieri and Orr bound for the square lattice for all densities to establish the optimality of the lower frame bound for the square lattice of density 2. All attempts of the author to show that the results are independent of the parameter t , hence holding for arbitrary (even) density, ended up to be equivalent to Theorem 4.1. However, then the optimality of the lower frame bound follows independently from the optimality of the upper frame bound. Hence, it would be nice to find an argument not relying on [9], which shows that the frame bound problem is invariant under scaling.

Another curiosity is that Jacobi's ϑ_3 function as well as the Dedekind eta function are actually modular functions of positive weight with respect to $\tau \in \mathbb{H}$. The frame bounds of a Gaussian Gabor frame are also modular forms, however, of weight 0. It is the separability of the problem which makes the proof of our Theorem 1.1 work. At the moment, it seems as if the techniques used in this work, will not work for non-separable lattices.

But before going into detail, we will shortly explain why density 2 is such a special case.

5.1. Why Density 2? We will now shortly describe one key aspect, why our result was only established for density 2. Recall, that the sharp frame bounds are derived by sampling $V_{g_0}g_0$ on the adjoint time-frequency lattice and that we have, according to formula (3.4),

$$V_{g_0}g_0(x, \omega) = e^{-\pi i x \omega} e^{-\frac{\pi}{2}(x^2 + \omega^2)}.$$

The adjoint of the square lattice of density 2, $\Lambda = \frac{1}{\sqrt{2}}\mathbb{Z}^2$, is given by $\Lambda^\circ = \sqrt{2}\mathbb{Z}^2$. This means that for $(k, l) \in \mathbb{Z}^2$ we have

$$V_{g_0}g_0(\sqrt{2}k, \sqrt{2}l) = e^{-\pi(k^2 + l^2)}.$$

We have that $e^{-\pi(x^2 + \omega^2)}$ is a fixed point of the (symplectic) Fourier transform. Hence, Poisson's summation formula is particularly nice in this case and, also, the above formula allows us to easily connect to Jacobi's theta functions and the Dedekind eta function with purely imaginary arguments.

5.2. A Problem for the Heat Kernel. The results of Faulhuber and Steinerberger [9] and of Montgomery [25] provide solutions about a problem concerning the temperature on a torus. We identify the torus with a fundamental domain of the lattice Λ , $\text{vol}(\Lambda)$ fixed, and denote it by

$$\mathbb{T}_\Lambda^2 = \mathbb{R}^2 / \Lambda.$$

The corresponding heat kernel is given by

$$p_\Lambda(z; t) = \frac{1}{4\pi t} \sum_{\lambda \in \Lambda} e^{-\frac{\pi}{4\pi t} |\lambda + z|^2}, \quad z \in \mathbb{T}_\Lambda^2.$$

By the symplectic version of Poisson's summation formula (see Appendix A) we have the alternative representation

$$\tilde{p}_\Lambda(z; t) = \text{vol}(\Lambda)^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} e^{-\pi(4\pi t) |\lambda^\circ|^2} e^{2\pi i \sigma(\lambda^\circ, z)}.$$

The two formulas describe, of course, the same temperature distribution, i.e.,

$$p_\Lambda(z; t) = \tilde{p}_\Lambda(z; t).$$

For time $t > 0$, we denote the minimal and the maximal temperature of the heat kernel by

$$m_\Lambda(t) = \min_{z \in \mathbb{T}_\Lambda^2} \tilde{p}_\Lambda(z; t) \quad \text{and} \quad M_\Lambda(t) = \max_{z \in \mathbb{T}_\Lambda^2} \tilde{p}_\Lambda(z; t).$$

By using the triangle inequality, we easily conclude that

$$\tilde{p}_\Lambda(z; t) \leq \tilde{p}(0; t),$$

which shows that $M_\Lambda(t)$ is taken at the origin and, due to periodicity, at any other lattice point of Λ . Also, $M_\Lambda(t)$ equals the trace $W_\Lambda(t)$ of the heat kernel. Locating the minimum is not as easy and the author is not aware of a closed expression for this problem. For separable lattices and for the hexagonal lattice, the minimum is achieved in a “deep hole”. This is where one would naturally expect the minimum as this is the point with the largest distance to its closest neighbors. It is also the point which is covered last if one places disks of radius ε at the lattice points and blows them up until they cover the whole plane. However, the generic non-separable case seems to be that the minimum is achieved at some point close to a “deep hole” and that the location also depends on the parameter t . Similar observations are described in the article by Baernstein and Vinson [1].

Montgomery’s result [25] shows that

$$M_{\Lambda_h}(t) \leq M_\Lambda(t), \quad \forall t \in \mathbb{R}_+,$$

where Λ_h is the hexagonal lattice. The results of Faulhuber and Steinerberger [9] show that

$$m_1(t) \geq m_\alpha(t) \quad \text{and} \quad M_1(t) \leq M_\alpha(t), \quad \forall t \in \mathbb{R}_+,$$

where we used the notation for separable tori from Section 4.2. The problem remaining open is whether

$$m_{\Lambda_h}(t) \stackrel{(?)}{\geq} m_\Lambda(t), \quad \forall t \in \mathbb{R}_+,$$

which is closely related to Conjecture 2.2.

We note the close connection of the above problems to the Strohmer and Beaver Conjecture. The key observation is that the time variable t can also be used to describe the volume of the lattice. Consider the Gaussian Gabor system

$$\mathcal{G}(g_0, (8\pi t)^{-1/2} \Lambda)$$

with $\text{vol}(\Lambda) = 1$. This means that the time-frequency shifts are carried out along a scaled version of the lattice Λ with density $8\pi t$. We note that we only have a frame if $8\pi t > 1$ and for $4\pi t = n \in \mathbb{N}$, the sharp Gaussian Gabor frame bounds are given by ⁵

$$A = 8\pi t m_\Lambda(t) \Big|_{t=\frac{n}{4\pi}} = 2n m_\Lambda\left(\frac{n}{4\pi}\right) \quad \text{and} \quad B = 8\pi t M_\Lambda(t) \Big|_{t=\frac{n}{4\pi}} = 2n M_\Lambda\left(\frac{n}{4\pi}\right)$$

More general, $M_\Lambda\left(\frac{t}{8\pi}\right) = \tilde{B}_{(8\pi t)^{-1/2} \Lambda}$ gives the Tolimieri and Orr bound for the Gaussian Gabor system with lattice density $8\pi t$, i.e.,

$$B_{(8\pi t)^{-1/2} \Lambda} \leq 8\pi t \tilde{B}_{(8\pi t)^{-1/2} \Lambda} = 8\pi t M_\Lambda\left(\frac{t}{8\pi}\right).$$

We have equality whenever $4\pi t \in \mathbb{N}$.

One curious part of the proof is now that we lose the parameter t when moving from the trace of the heat kernel to the zeta function, as we integrate over $t \in \mathbb{R}_+$. Actually, we need

⁵For $t = \frac{n}{8\pi}$, $n \in \mathbb{N}$, the time-frequency lattice has density n , which means that $4\pi t \in \mathbb{N}$ gives a time-frequency lattice of even density.

the information that the minimal trace of the heat kernel stays minimal for all t in order to establish our main result, which only holds for $t = \frac{1}{4\pi}$, which corresponds to the Gabor case of density 2. Lastly, we note that the (real) parameter t is usually interpreted as “time” in the case of the heat kernel, whereas the density of the lattice (which equals the surface area of the torus) is fixed. However, it can also be interpreted as the density of the lattice for fixed time or as the product of both, density and time.

5.3. The Modular Aspect of the Proof. We will only give some basic details on modular functions and refer to the textbooks of Serre [27] or Stein and Shakarchi [28]. A modular function f of weight k satisfies

$$f(\tau) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{\pm I\}.$$

We note that \mathbb{Z}^2 is invariant under the action of the group $SL(2, \mathbb{Z})$, which consists of determinant 1 matrices with integer entries. Expressed differently, we only change the basis of the lattice under the action of this group. Since, in general, the matrices S and $-S$ generate the same lattice, they are identified which is why we only consider $PSL(2, \mathbb{Z})$, the modular group.

Next, we note that a 2-dimensional lattice can be identified (modulo rotation) with a complex number $\tau = x + iy \in \mathbb{H}$. The generating matrix then has the form

$$S = y^{-1/2} \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}.$$

We set

$$a(\tau; t) = m_\Lambda\left(\frac{t}{4\pi}\right) \quad \text{and} \quad b(\tau; t) = M_\Lambda\left(\frac{t}{4\pi}\right),$$

where we identify $\tau = x + iy$ with Λ . Then, it follows that $a(\tau; t)$ and $b(\tau; t)$ are modular forms of weight 0 with respect to τ . After all, choosing a different basis for the lattice just changes the order of summation. However, the values do not depend on the order of summation as the sums defining the heat kernel are absolutely convergent. In the separable case, τ is purely imaginary, i.e. $\tau = iy$ with $y \in \mathbb{R}_+$, and the values of $a(iy; t)$ and $b(iy; t)$ split into products of theta functions of the form

$$a(iy; t) = \theta_2(it y^{-1}) \theta_4(it y) \quad \text{and} \quad b(iy; t) = \theta_3(it y^{-1}) \theta_3(it y).$$

In the non-separable case, the author is not aware of a possibility to write $a(\tau; t)$ and $b(\tau; t)$ as just a product of Jacobi’s theta functions. However, the possibility to write a and b as products of theta functions lastly gives the connection to the Dedekind eta function, which is a modular form of weight 1/2.

Recall that the determinant of the Laplace–Beltrami operator is given by

$$\det' \Delta_\tau = \text{Im}(\tau) \eta(\tau)^2 \overline{\eta(\tau)}^2.$$

The behavior of the eta function under the modular group is given by

$$\eta(\tau) = (c\tau + d)^{-1/2} \eta\left(\frac{a\tau + b}{c\tau + d}\right)$$

and the imaginary part of τ transforms as

$$\text{Im}(\tau) = |c\tau + d|^2 \text{Im}\left(\frac{a\tau + b}{c\tau + d}\right).$$

Hence, we have the result

$$\det' \Delta_\tau = \det' \Delta_{\frac{a\tau+b}{c\tau+d}}.$$

It follows that the determinant of the Laplace–Beltrami operator on a two dimensional torus is a modular form of weight 0 with respect to the metric on the torus.

However, by using the symplectic version of Poisson’s summation formula we find that the determinant also behaves like a modular form with respect to the volume of the lattice (also see Appendix A)

$$\det' \Delta_\Lambda = \text{vol}(\Lambda)^{-1} \det' \Delta_{\Lambda^\circ} = \text{vol}(\Lambda)^{-1} \det' \Delta_{\text{vol}(\Lambda)^{-1}\Lambda}.$$

However, the volume is a positive number and the behavior of the determinant is therefore comparable to the behavior of the Dedekind eta function for purely imaginary arguments. As already explained, the parameter t describes the volume of the lattice. It is for this reason that we cannot really distinguish between the modular parameter $\tau \in \mathbb{H}$ and the real parameter t for separable tori which causes one of the curiosities mentioned at the beginning of this section.

APPENDIX A. THE SYMPLECTIC POISSON SUMMATION FORMULA

When working in the time-frequency plane or phase space, it is often advantageous to exploit the symplectic structure of \mathbb{R}^{2d} . The concept of phase space allows for a simultaneous treatise of two aspects of a function, coupled via the Fourier transform, such as position and momentum or, in the case of Gabor analysis, time and frequency. For more details on phase space methods and symplectic geometry we refer to the textbooks of de Gosson [15, 16].

For two points $z = (x, \omega)$ and $z' = (x', \omega')$ in phase space, the symplectic form is given by

$$\sigma(z, z') = x\omega' - \omega x' = z \cdot Jz', \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where $z \cdot z'$ denotes the standard Euclidean inner product of two vectors z and z' in \mathbb{R}^d . A matrix S is symplectic if and only if

$$\sigma(Sz, Sz') = \sigma(z, z'), \quad \text{or equivalently,} \quad S^T JS = J.$$

We note that $\sigma(z, z') = -\sigma(z', z)$ and, hence, $\sigma(z, z) = 0$ for all z . The group of all symplectic matrices is denoted by $Sp(d)$ and is a proper subgroup of $SL(2d, \mathbb{R})$, except for the case $d = 1$, where we have $Sp(1) = SL(2, \mathbb{R})$.

The classical version of Poisson’s summation formula for lattices in \mathbb{R}^d is

$$\sum_{\lambda \in \Lambda} f(\lambda + z) = \text{vol}(\Lambda)^{-1} \sum_{\lambda^\perp \in \Lambda^\perp} \mathcal{F}f(\lambda^\perp) e^{2\pi i \lambda^\perp \cdot z}, \quad z \in \mathbb{R}^d.$$

The dual of the lattice $\Lambda = M\mathbb{Z}^d$ is given by $\Lambda^\perp = M^{-T}\mathbb{Z}^d$, $M \in GL(d, \mathbb{R})$. The details when this formula holds pointwise have been worked out in [18].

However, we want to work in phase space which is always even dimensional. For a lattice $\Lambda = M\mathbb{Z}^{2d}$ we can replace the dual lattice by the adjoint lattice $\Lambda^\circ = JM^{-T}\mathbb{Z}^{2d}$, the Fourier transform by the symplectic Fourier transform and the Euclidean inner product in the exponent is replaced by the symplectic form (see also [6]). The symplectic version of Poisson’s summation formula is given by

$$\sum_{\lambda \in \Lambda} f(\lambda + z) = \text{vol}(\Lambda)^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \mathcal{F}_\sigma f(\lambda^\circ) e^{2\pi i \sigma(\lambda^\circ, z)}, \quad z \in \mathbb{R}^{2d},$$

where \mathcal{F}_σ is the symplectic Fourier transform of a function of $2d$ variables, given by

$$\mathcal{F}_\sigma f(z) = \int_{\mathbb{R}^{2d}} f(z') e^{-2\pi i \sigma(z, z')} dz'.$$

We call a lattice symplectic if its generating matrix is a multiple of a symplectic matrix. From the definition of a symplectic matrix, it follows that in this case $\Lambda^\circ = \text{vol}(\Lambda)^{-1/d} \Lambda$, as by definition $JS^{-T}J^{-1}\mathbb{Z}^{2d} = S\mathbb{Z}^{2d}$ for $S \in Sp(d)$ and $J^{-1}\mathbb{Z}^{2d} = \mathbb{Z}^{2d}$. We note that any 2-dimensional lattice is symplectic.

Also, using the symplectic Fourier transform has the advantage that all 2-dimensional, normalized Gaussians are eigenfunctions of the symplectic Fourier transform \mathcal{F}_σ with eigenvalue 1 [6]. Hence, for 2-dimensional Gaussians and 2-dimensional lattices all formulas become particularly easy.

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