Lecture 1 28.08.2017

I. Remainder about Fourier integral

- Spaces $L^1(\mathbb{R})$, $L^2(\mathbb{R})$ – norm, inner product.

Comments:
- Lebesgue integrability
  No values at a single point
  ...
- L^2-norm serves as "energy".

- Definition of Fourier integral in L^1

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

Comments:
- $f \in L^1 \Rightarrow \hat{f}$ is well-defined,
- $f \in L^1 \Rightarrow \hat{f}$ is continuous.
- Riemann-Lebesgue lemma:

$$f \in L^1 \Longrightarrow \hat{f}(\omega) \to 0, \quad \omega \to \pm\infty$$

- $f \in L^1$ not necessarily $\hat{f} \in L^1$

Example  make calculation!

- Discussion: smoothness f vs decay $\hat{f}$

• Inversion formula

Fact:

$$f \in L^1,\ \hat{f} \in L^1 \Rightarrow f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega$$

Comment - this is a part of Fourier analysis course, we ~~will prove~~ WILL prove it later, it will be for free.

• Main operators

Translation : $T_\tau : f(t) \mapsto f(t-\tau)$

Modulation $M_\omega : f(t) \mapsto e^{i\omega t} f(t)$

Scaling $A_s : f(t) \mapsto f\left(\frac{t}{s}\right),\quad s > 0$

This is a non-standard notation, I do not know standard one.

• Convolution

$$f, g \in L^1 \Rightarrow (f * g)(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

Comment

- Integral is convergent for almost all $t \in \mathbb{R}$
- $f * g \in L^1,\ \|f * g\|_1 \leq \|f\|_1 \|g\|_1$
- $(f * g)^\wedge(\omega) = \hat{f}(\omega)\, \hat{g}(\omega)$.

– Example: Low pass filter

• Other properties

Homework #1 Read the properties of the Fourier transform table 2.1 p. 38 of the textbook and prove them all yourself (except, perhaps inversion formula 2.15)

Comment: – we will use these properties in future.

Fourier integral in $L^2(\mathbb{R})$

General pattern of the theory:

• Let $f, g, \hat{f}, \hat{g}$ decay fast everything is in $L^2 \cap L^1$

Then:

Plancherel $\int_{-\infty}^{\infty} f(t)\overline{h(t)}\,dt = \frac{1}{2\pi}\int \hat{f}(\omega)\overline{\hat{g}(\omega)}\,d\omega$

Parseval $\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega$

Comment Calculation of energy in two different ways.

- General case $f, g \in L^2$

- <u>Idea</u> - Approximate arbitrary functions in L^2 by functions in $L^2 \cap L^1$
  - Write the desired relations for the approximants and then pass to the limit.

<u>Machinery</u>: ~~Hilbert space technique~~ Cauchy sequences in L^2

<u>Realization</u> - Truncation
- Remind Cauchy inequality

<u>Concrete realization</u>

- $f \in L^2 \Rightarrow f_N(t) = \begin{cases} f(t), & |t| < N \\ 0 & |t| > 0 \end{cases} \in L^1 \cap L^2$

- $\| f_N - f \|_{L^2} \to 0, \quad N \to \infty \Rightarrow$

$$\Rightarrow \| \hat{f}_N - \hat{f} \|_{L^2} \to 0, \quad N \to \infty$$

- $\hat{f}(\omega) := \underset{N \to \infty}{\text{l.i.m.}} \int_{-N}^{N} f(t) e^{i\omega t} \, dt$

- We write $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} \, dt$ meaning this

Examples:

- Low pass filter,

Notation $\operatorname{sinc} x$

- Digression: + Convolution of functions in L^2 and in L^1

+ Triangle inequality for convolution

- Gaussian:

$$f(t) = e^{-t^2} \Rightarrow \hat{f}(\omega) = \sqrt{\pi}\, e^{-\omega^2/4}$$

Proof: a) Integration by parts $\Rightarrow$

$$\Rightarrow 2\hat{f}'(\omega) + \omega \hat{f}(\omega) = 0 \Rightarrow$$

$$\Rightarrow \hat{f}(\omega) = K e^{-\omega^2/4}$$

b) $K = \hat{f}(0) = \int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}$

Homework. Find a such that $f(t) = e^{-at^2}$ is an eigenfunction of the Fourier transform with eigenvalue 1, i.e.

$$(\mathcal{F} f)(\omega) = e^{-a\omega^2}$$

↑ another notation for Fourier transform.

- Gaussian chirp

$$f(t) = e^{-(a-ib)t^2}, \quad a>0$$

- Explain why this is a "chirp"

- **Homework** Prove

$$\hat{f}(\omega) = \sqrt{\frac{\pi}{a-ib}}\, e^{-\frac{(a+ib)\omega^2}{4(a^2+b^2)}}$$

Digression Hermite functions

$$H_n(t) = \frac{1}{\pi^{1/4}} \frac{(-1)^n}{\sqrt{2^n n!}} e^{\frac{t^2}{2}} \left(e^{-t^2}\right)^{(n)}$$

Examples $H_0(t) =$ $\qquad$ $H_1(t) =$

General form $\underbrace{p_n(t)}_{\text{Hermite polynomial (of deg } n)} e^{-t^2/2}$

(I do not remember standard notation)

Homework · $\langle H_n, H_m \rangle = \delta_{mn}$ ← Kronecker delta.

- $(\mathcal{F} H_n)(\omega) = i^n H(\omega)$

Homework* (Star means that it is not in pensum)

• Functions $H_n(t)$ form an ~~orthogo~~ orthonormal basis in $L^2(\mathbb{R})$ i.e. each function $f \in L^2(\mathbb{R})$ admits an expansion

$$f(t) = \sum_0^\infty a_n H_n(t)$$

with $\sum_0^\infty |a_n|^2 = \|f\|_{L^2}^2$

Corollary Denote ~~$\mathcal{Y}_k = \mathrm{Clos}(\mathrm{Span}\{H$~~

$$\mathcal{Y}_k = \mathrm{Clos}\left(\mathrm{Span}\{H_{4l+k}\}_{l=0,1,\dots}\right), \quad k=0,1,2,3$$

Then each $\mathcal{Y}_k$ is an eigenspace of the Fourier transform with eigenvalue i^k i.e.

$$f \in \mathcal{Y}_k \Rightarrow \mathcal{F}f = i^k f$$

- The whole $L^2(\mathbb{R})$ is the orthogonal sum of eigenspaces

$$L^2(\mathbb{R}) = \mathcal{Y}_0 \oplus \mathcal{Y}_1 \oplus \mathcal{Y}_2 \oplus \mathcal{Y}_3$$