

Whittaker-Kotelnikov-Shannon.

Basic qn: { Given a continuous signal $f(t)$ transform it into a sequence and then reconstruct.

Bandlimited functions

Def f is bandlimited to $[-\Omega, \Omega]$
if $\text{supp} \hat{f} \subset [-\Omega, \Omega]$, i.e.

$$f(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}(\omega) e^{i\omega t} d\omega$$

Theorem (WKS) ~~fixed~~,

$f \in L^2(\mathbb{R})$, f -bandlimited to $[-\Omega, \Omega]$

$$\Rightarrow f(t) = \sum_k f(k \frac{\pi}{\Omega}) \frac{\sin(k\pi - \Omega t)}{k\pi - \Omega t}$$

$$\|f\|_{L^2(\mathbb{R})}^2 = \sum_k |f(k \frac{\pi}{\Omega})|^2$$

Check if I need
a constant ~~term~~

Proof

See Appendix

3.2.

Comments

• \Rightarrow Reconstruction of f
through $\{f(k\frac{\pi}{2})\}$

: \Rightarrow stability of reconstruction

Comment

~~sticker~~ bigger spectrum -
more often we have to sample.

Notion "Sampling theorem".

Definition Paley-Wiener space ~~$L^2(\Omega)$~~

$PW_{\Omega} = \{f(t) \text{ of the form}$

$$f(t) = \int_{-\Omega}^{\Omega} \varphi(\omega) e^{i\omega t} d\omega, \varphi \in L^2(-\Omega, \Omega)$$

Theorem (Paley-Wiener) (with no proof)

$$f \in PW_{\Omega} \Leftrightarrow f \in L^2(-\infty, \infty)$$

f is holomorphic in \mathbb{C}

$$\Omega |y|$$

$$|f(x+iy)| \leq \text{Const } \ell$$

Exercise Prove \Rightarrow

Comment Functions ~~sketch~~

$\left\{ \frac{\sin(k\pi - \omega t)}{k\pi - \omega t} \right\}_k$ is an orthonormal basis in $PW_{\mathbb{R}}$

Just because it is the Fourier image of an orthonormal basis in $L^2(-\pi, \pi)$

Comment WKS theorem as a Lagrange interpolation type formula.

Exercise Let $f \in PW_{\mathbb{R}}$ & prove that

$$f(k\frac{\pi}{\omega}) = \left\langle f, \frac{\sin(k\pi - \omega t)}{k\pi - \omega t} \right\rangle_{L^2(\mathbb{R})}$$

(if need be assume that f growth a bit slower, this may ease technicalities).

Irregular sampling:

3.4.

$$A = \sum \lambda_k z_k; \quad |\lambda_k - k \frac{\pi}{\Delta}| < \varepsilon < \frac{1}{4} \frac{\pi}{\Delta}$$

$\Rightarrow f \in PW_{\Delta}$ can be reconstructed via
 $\{f(\lambda_k)\}$ and for some $A, B > 0$

~~and~~ $A \|f\|_2^2 \leq \sum |f(\lambda_k)|^2 \leq B \|f\|^2$ (Kotets '4 theorem).

↑ stability

(replacement for Parseval)
 equality

Riesz basis

H -Hilbert space $\{x_n\} \subset H$ is a
Riesz basis if each $x \in H$ admits unique
 expansion

$$x = \sum c_n x_n$$

and

$$\|x\|_H^2 \asymp \sum |c_n|^2$$

↑ New notation it means

$$A \|x\|^2 \leq \sum |c_n|^2 \leq B \|x\|^2$$

for all $x \in H$.

Exercise $\{x_k\}$ -Riesz basis in $H \Leftrightarrow \exists \{e_k\} \subset H$ -orthonormal basis and bounded invertible linear operator $L: H \rightarrow H$ such that $L e_k = x_k$.

Non-harmonic Fourier series

Exercise* Prove the following:

$$|\lambda_k - \frac{k\pi}{\Omega}| < \varepsilon < \frac{1}{4}\frac{\pi}{\Omega} \Rightarrow \{e^{i\lambda_k w}\}_k \text{ is}$$

a Riesz basis in $L^2(-\Omega, \Omega)$.

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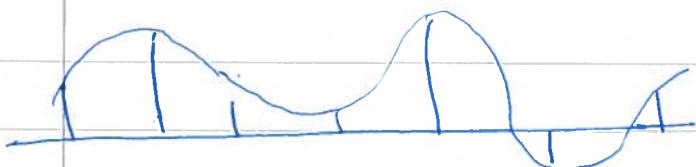
Non-bandlimited functions

$f(t)$ - "good" signal (smooth, decaying).

$$\tilde{T} > 0$$

"Uniform sampling of f with period T "

$$f_d(t) := \sum_n f(nT) \delta(t-nT)$$



3.6.

Fact: $\hat{f}_d(\omega) = \frac{1}{T} \sum_k \hat{f}\left(\omega - \frac{2k\pi}{T}\right)$

Exercises: 1) Prove this yourself
 (Poisson summation)

- 2). Look how does textbook do it
 (and learn to play with δ -functions
 and, in general, distributions)
- 3) Obtain WKS theorem from this fact.

Further discussion:

1. Aliasing
2. Numerical implementation
3. Relation between size of the spectra
 and density of the sampling points
 Nyquist density.

Lecture 3 Appendix 1

Proof of WKS theorem

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega$$

$$\hat{f}(\omega) = \sum_k c_k e^{i\frac{\pi}{\Omega} k \omega} \Rightarrow$$

$$\Rightarrow f(t) = \frac{1}{2\pi} \sum_k c_k \underbrace{\int_{-\infty}^{\infty} e^{i\frac{\pi}{\Omega} k \omega + i\omega t} d\omega}_{= 2\pi \frac{\sin(k\pi + \Omega t)}{k\pi + \Omega t}} \Rightarrow$$

$$\Rightarrow f(t) = \frac{1}{2\pi} \sum_k c_k \cdot 2\pi \frac{\sin(k\pi + \Omega t)}{k\pi + \Omega t};$$

$$c_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{-i\frac{\pi}{\Omega} k \omega} d\omega = \frac{2\pi}{2\pi} f(-\frac{\pi}{\Omega} k)$$

Finally $f(t) = \sum_k f(k\frac{\pi}{\Omega}) \frac{\sin(k\pi - \Omega t)}{k\pi - \Omega t}$

I replaced
k by -k in
summation