Chapter 5
Uncertainty Quantification for conservation laws

kapittel3

It is not certain that everything is uncertain.

Blaise Pascal, 1857 (find better!)

Whenever a (partial) differential equation arises as a model of something outside mathematics, it is dependent on certain external components. Typical examples are boundary or initial values and coefficients in the PDE, such as heat conductivity, equation of state, the speed of light in a vacuum, Boltzmann's constant etc. Very often, these quantities are obtained from measurements, and cannot be known exactly. In other words, they are uncertain, but whose uncertainty can be quantified. In this book, uncertainty quantification is the attempt to determine how this uncertainty influences and manifests itself in an uncertainty in solutions of the (partial) differential equation.

If we want to study uncertainty in solutions to hyperbolic conservation laws we must resort to studying uncertainty in numerical approximations. If we have error estimates we can try to glean information on how uncertain the exact solutions are by computing a lot of "typical" numerical solutions and various statistical estimates on these. Ideally, we also would like to have statistical bounds on how accurate these statistical estimates are.

To model uncertainty mathematically, it is common to use so-called random variables

5.1 A quick introduction to Random variables and the Monte-Carlo method

seq:rvmc_intro

"Random" means unpredictable, with no specific pattern.

Chat GPT 3.5

To define a random variable we start by having a set Ω of all possible outcomes (of an experiment), and a σ-algebra $\mathcal{B}$ of subsets of Ω. Members of

$\mathcal{B}$ are then assigned a probability, this means that we have a measure $\mathbb{P}$ on the measure space $(\Omega, \mathcal{B})$ such that $\mathbb{P}(D) \geq 0$ for all $D \in \mathcal{B}$ and $\mathbb{P}(\Omega) = 1$. The measure $\mathbb{P}$ is called a probability measure.

♢ ***Example***

The simplest and perhaps most familiar situation where one can model randomness using these concepts is a single toss of a coin. Let "H" and "T" denote the outcome that the coin comes up heads or tails respectively. This means that

$$\Omega = \{\mathrm{H},\, \mathrm{T}\}, \quad \mathcal{B} = \{\emptyset, \{\mathrm{H}\}, \{\mathrm{T}\}, \{\mathrm{H},\mathrm{T}\}\}.$$

Given two nonnegative numbers p_H and p_T such that $p_H + p_T = 1$, we can define a probability measure μ by

$$\mathbb{P}(D) = \begin{cases} p_H & D = \{\mathrm{H}\}, \\ p_T & D = \{\mathrm{T}\}. \end{cases}$$

If we want to model N tosses of the same coin, we can let Ω consist of all sequences of N letters where each letter is "H" or "T". The σ-algebra $\mathcal{B}$ will then consist of all subsets of Ω, and the probability of an elementary even is

$$\mathbb{P}(l_1 l_2 \dots l_N) = p_H^{(\#\ \mathrm{H})} p_T^{(\#\ \mathrm{T})},$$

where "# H" denotes the number of "H"s among the l_is. ♢ □

However, we very rarely need the concrete description of Ω, it is simpler to view Ω as an abstract set containing the elementary outcomes of an experiment.

A real valued *random variable* is defined as a measurable function $X : \Omega \to \mathbb{R}$, where $\mathbb{R}$ is equipped with the Lebesgue measure, so that the set $X^{-1}(A) \in \mathcal{B}$ for all Lebesgue measurable $A \subset \mathbb{R}$. We then define the *expectation* (also called *mean*) of X as the integral of X over Ω with respect to $\mathbb{P}$, i.e.,

$$\mathbb{E}[X] = \int_\Omega X(\omega)\, d\mathbb{P}(\omega).$$

Once we have a notion of an integral, we can define the spaces $L^p(\Omega; \mathbb{R})$ in the usual manner, and Cauchy-Schwartz's, Hölder's and Minkowski's inequalities hold. In particular Jenssen's inequality holds and in this formulation reads

$$\Theta(\mathbb{E}[X]) \leq \mathbb{E}[\Theta(X)],$$

for any convex function Θ. Let $p \geq q$ and use Jenssen's inequality for the convex function $\Theta(u) = |u|^{p/q}$ to find that

$$\|X\|^p_{L^p(\Omega;\mathbb{R})} = \mathbb{E}[\Theta\left(|X|^q\right)] \geq \Theta\left(\mathbb{E}[|X|^q]\right) = \|X\|^p_{L^q(\Omega;\mathbb{R})}.$$

Thus for $p \geq q$, $L^p(\Omega;\mathbb{R}) \subset L^q(\Omega;\mathbb{R})$. If a random variable is in $L^2(\Omega;\mathbb{R})$ we define its *variance* as

$$\mathrm{Var}[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

For any random variable X we can define a probability measure μ_X on $\mathbb{R}$ by the relation

$$\mu_X(A) = \mathbb{P}(X^{-1}(A)).$$

the measure μ_X is called the *law* of X, and intuitively $\mu_X(A)$ is the probability that $X(\omega)$ takes values in A. We shall use the notation $\mathrm{Prob}(X \in A) = \mathbb{P}(X^{-1}(A)) = \mu_X(A)$. We can also integrate (in $\mathbb{R}$) with respect to μ_X using the definition

$$\int_{\mathbb{R}} f(x)\, d\mu_X(x) = \int_{\Omega} f(X(\omega))\, d\mathbb{P}(\omega).$$

A real valued random variable is called *Gaussian* if its probability density μ_X is

$$d\mu_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}\, dx,$$

for a positive constant σ, and it is called *standard* if $\sigma = 1$. We define the *Fourier transform* of a real random variable by the formula

$$\hat{X}(\xi) := \mathbb{E}\left[e^{-iX\xi}\right] = \int_{\mathbb{R}} e^{-ix\xi}\, d\mu_X(x),$$

where $i = \sqrt{-1}$. A random variable X is standard Gaussian if and only if $\hat{X}$ is given by

$$\hat{X}(\xi) = e^{-\xi^2/2}.$$

The following inequalities give bounds on the probability that random variables are "large".

lem:chebyshev

Lemma 5.2 (Chebyshev's inequalities) *Let X be a random variable.*

- *If $\mathbb{E}[X] < \infty$, then for any $t > 0$*

$$\mathrm{Prob}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}.$$

- *If $\mathrm{Var}[X] < \infty$, then for any $t > 0$*

$$\mathrm{Prob}(|X - \mathbb{E}[X]| \geq t) \leq \frac{\mathrm{Var}[X]}{t^2}.$$

Proof Observe that for $\omega \in X^{-1}([t,\infty))$

$$1 = \chi_{X^{-1}([t,\infty))}(\omega) \leq \frac{X(\omega)}{t}$$

where χ_A denotes the indicator function of the set A. Hence

$$\text{Prob}(X \geq t) = \mathbb{E}\left[\chi_{X^{-1}([t,\infty))}\right] \leq \mathbb{E}\left[\frac{X}{t}\right] = \frac{\mathbb{E}[X]}{t}.$$

This proves the first inequality. To prove the second, observe that

$$\text{Prob}(|X - \mathbb{E}[X]| \geq t) = \text{Prob}(|X - \mathbb{E}[X]|^2 \geq t^2),$$

and apply the first inequality to the random variable $|X - \mathbb{E}[X]|^2$.

If we have a collection $X_1, \ldots, X_N$ of random variables defined on $(\Omega, \mathcal{B}, \mathbb{P})$ we say that these are independent if

$$\text{Prob}\left(X_1 \in A_1, X_2 \in A_2, \ldots, X_N \in A_N\right) = \prod_{i=1}^{N} \text{Prob}\left(X_i \in A_i\right) \tag{5.1}$$

eq:independentdef

for all measurable sets $A_i \subset \mathbb{R}$. It follows immediately that if $f_1, \ldots, f_N$ are real measurable functions and $X_1, \ldots, X_N$ are independent, then so are the random variables $f_1(X_1), \ldots, f_N(X_N)$. Conversely, if $f_1(X_1), \ldots, f_N(X_N)$ are independent for any collection of measurable functions $f_1, \ldots, f_N$, then $X_1, \ldots, X_N$ are independent.

Regarding the expectation of products of independent random variables we have the very useful lemma:

lem:independentrv

Lemma 5.3 *Let X and Y be random variables defined on the probability space $(\Omega, \mathcal{B}, \mathbb{P})$, then X and Y are independent if and only if*

$$\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\,\mathbb{E}[g(Y)],$$

for all real measurable functions f and g.

Proof Assume that X and Y are independent. We prove the equality for simple functions X and Y, the general result then follows since simple functions are dense. If X and Y are simple we have that

$$X(\omega) = \sum_i x_i \chi_{A_i}(\omega) \quad \text{and} \quad Y(\omega) = \sum_j y_j \chi_{B_j}(\omega),$$

where A_i and B_j are in $\mathcal{B}$. Then

$$\begin{aligned}
\mathbb{E}[XY] &= \int_\Omega X(\omega)Y(\omega)\, d\mathbb{P}(\omega) \\
&= \sum_{i,j} x_i y_j \int_\Omega \chi_{A_i}(\omega)\chi_{B_j}(\omega)\, d\mathbb{P}(\omega) \\
&= \sum_{i,j} x_i y_j \,\text{Prob}\left(X \in A_i, Y \in B_j\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j} x_i y_j \,\mathrm{Prob}\,(X \in A_i)\,\mathrm{Prob}\,(Y \in B_j) \\
&= \Big(\sum_i x_i \int_\Omega \chi_{A_i}(\omega)\, d\mathbb{P}(\omega)\Big)\Big(\sum_j y_j \int_\Omega \chi_{B_j}(\omega)\, d\mathbb{P}(\omega)\Big) \\
&= \mathbb{E}[X]\,\mathbb{E}[Y].
\end{aligned}
$$

Assume now that the equality holds and choose $f = \chi_A$ and $g = \chi_B$ for measurable sets A and B in $\mathbb{R}$. Then

$$
\begin{aligned}
\mathrm{Prob}(X \in A, Y \in B) &= \int_\Omega f(X(\omega))g(Y(\omega))\, d\mathbb{P}(\omega) = \mathbb{E}[f(X)g(Y)] \\
&= E[f(X)]\,\mathbb{E}[g(y)] = \int_\Omega f(X(\omega))\, d\mathbb{P}(\omega) \int_\Omega g(Y(\omega))\, d\mathbb{P}(\omega) \\
&= \mathrm{Prob}(X \in A)\,\mathrm{Prob}(Y \in B).
\end{aligned}
$$

If we have infinitely many random variables $\{X_i\}_{i=1}^{\infty}$ we say that these are independent if $\{X_{j_k}\}_{k=1}^{N}$ are independent for all finite N and for all sequences $j_1 < j_2 < \cdots < j_N$.

5.1.1 The Monte-Carlo method

subsec:mc_intro

Regression to the mean has an explanation but does not have a cause.

Daniel Kahneman, Thinking, fast and slow

In our experience, especially outside academia, we rarely encounter expectations (though others seem to have them of us), but we see averages all the time. As an example, one often reads statements like "the average sugar consumption was 26 kilograms per Norwegian in 2020"[1] When encountering a Norwegian, are we wrong to assume that this person ate 26 kilograms of sugar in 2020? In other words, can we replace the (mathematical) mean with the average? And if we do so, what error do we make?

To answer this we must first define the average. To this end, let X be a random variable on the probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and define the average as a random variable on the product probability space $(\Omega, \mathcal{B}, \mathbb{P})^N$

$$
E_N(\omega_1, \ldots, \omega_N) = \frac{1}{N} \sum_{i=1}^{N} X(\omega_i).
$$

The random variable E_N is called the *average* of X. To determine the relation between the mean and the average we define random variables $\tilde{X}_i$ on the

[1] Or even "the average Norwegian consumed 26 kg of sugar in 2020"!

product probability space by

$$\tilde{X}_i(\omega_1, \ldots, \omega_N) = X(\omega_i) \quad \text{for } i = 1, \ldots, N.$$

Observe that using this notation

$$E_N = \frac{1}{N}\left(\tilde{X}_1 + \cdots + \tilde{X}_N\right).$$

We claim that for $i \neq j$, $\tilde{X}_i$ and $\tilde{X}_j$ are independent. To see this, first calculate

$$\begin{aligned}\mathbb{E}[\tilde{X}_i] &= \int_{\Omega^N} \tilde{X}_i(\omega_1, \ldots, \omega_N)\, d\mathbb{P}(\omega_1) \cdots d\mathbb{P}(\omega_N) \\ &= \int_{\Omega^N} X(\omega_i)\, d\mathbb{P}(\omega_1) \cdots d\mathbb{P}(\omega_N) = \mathbb{E}[X],\end{aligned}$$

and then

$$\begin{aligned}\mathbb{E}[\tilde{X}_i\tilde{X}_j] &= \int_{\Omega^N} \tilde{X}_i(\omega_1, \ldots, \omega_N)\tilde{X}_j(\omega_1, \ldots, \omega_N)\, d\mathbb{P}(\omega_1) \cdots d\mathbb{P}(\omega_N) \\ &= \int_{\Omega^N} X(\omega_i)X(\omega_j)\, d\mathbb{P}(\omega_1) \cdots d\mathbb{P}(\omega_N) \\ &= \mathbb{E}[X]^2 = \mathbb{E}[\tilde{X}_i]\,\mathbb{E}[\tilde{X}_j].\end{aligned}$$

For $i = j$, a similar calculation reveals that $\mathbb{E}[\tilde{X}_i^2] = \mathbb{E}[X^2]$. Incidentally, the above calculations imply that $\mathbb{E}[E_N] = \mathbb{E}[X]$, so we can expect that the average equals the expectation[2]. However, we can quantify the discrepancy further by calculating $\|E_N - \mathbb{E}[X]\|_{L^2(\Omega^N)}$,

$$\begin{aligned}\mathbb{E}\left[(E_N - \mathbb{E}[X])^2\right] &= \mathbb{E}\Big[\frac{1}{N^2}\Big(\sum_{i=1}^{N}\left(\tilde{X}_i - \mathbb{E}[X]\right)\Big)^2\Big] \\ &= \frac{1}{N^2}\mathbb{E}\Big[\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\tilde{X}_i - \mathbb{E}[X]\right)\left(\tilde{X}_j - \mathbb{E}[X]\right)\Big] \\ &= \frac{1}{N^2}\mathbb{E}\Big[\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\tilde{X}_i\tilde{X}_j - \mathbb{E}[X]\left(\tilde{X}_i + \tilde{X}_j\right) + \mathbb{E}[X]^2\right)\Big] \\ &= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\mathbb{E}[\tilde{X}_i\tilde{X}_j] - 2\mathbb{E}[X]^2 + \mathbb{E}[X]^2\right) \\ &= \frac{1}{N^2}\left(N\mathbb{E}[X^2] + N(N-1)\mathbb{E}[X]^2 - N^2\mathbb{E}[X]^2\right) \\ &= \frac{1}{N}\left(\mathbb{E}[X^2] - \mathbb{E}[X]^2\right) = \frac{\mathrm{Var}[X]}{N}. \qquad (5.2)\end{aligned}$$

eq:mcerrorest

[2] Whatever this means.

If $\mathrm{Var}[X] < \infty$ we see that E_N converges to $\mathbb{E}[X]$ in some sense.

In order to facilitate the formulation of convergence of random variables, it is convenient to have X and E_N defined on the same probability space. To this end, let

$$\omega = (\omega_0, \omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots) \in \Omega \times \Omega \times \cdots \times \Omega \times \cdots =: \Omega^\infty,$$

and define $\mathcal{B}^\infty = \mathcal{B} \otimes \mathcal{B} \otimes \cdots \otimes \mathcal{B} \otimes \cdots$ and $\mathbb{P}^\infty = \mathbb{P} \otimes \mathbb{P} \otimes \cdots \otimes \mathbb{P} \otimes \cdots$. Then we define X^∞ and E_N^∞ as random variables on the probability space $(\Omega^\infty, \mathcal{B}^\infty, \mathbb{P}^\infty)$ by

$$X^\infty(\omega) = X(\omega_0) \quad \text{and} \quad E_N^\infty(\omega) = E_N(\omega_1, \ldots, \omega_N).$$

To confuse matters we shall drop the superscript ∞, this saves us some typing, but also makes the expectations appearing in the calculations above the same. So with this new notation (with the invisible ∞), $\{E_N\}_{N=1}^\infty$ is a sequence of random variables in the probability space $(\Omega, \mathcal{B}, \mathbb{P})$.

Let $\{X_N\}_{N=1}^\infty$ be a sequence of real random variables defined on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$. This sequence can converge to a random variable X in different ways. We say that X_N converges to X in L^p if

$$\lim_{N\to\infty} \mathbb{E}[|X_N - X|^p] = 0.$$

We say that X_N converges to X *almost surely* if there is a set $\mathcal{N} \subset \Omega$ with $\mathbb{P}(\mathcal{N}) = 0$, such that for $\omega \notin \mathcal{N}$,

$$X_N(\omega) \to X(\omega).$$

We say that X_N converges to X *in probability* if for any $\delta > 0$,

$$\lim_{N\to\infty} \mathrm{Prob}\,(|X_N - X| > \delta) = 0.$$

We say that X_N converges to X *in law* if μ_{X_N} converges weakly to μ_X, meaning that

$$\lim_{N\to\infty} \int_{\mathbb{R}} f(x)\, d\mu_{X_N}(x) = \int_{\mathbb{R}} f(x)\, d\mu_X(x),$$

for all bounded continuous functions f. We write these types of convergence as

$$X_N \xrightarrow{L^p} X, \quad X_N \xrightarrow{\text{a. s.}} X, \quad X_N \xrightarrow{\mathbb{P}} X \quad \text{and} \quad X_N \xrightarrow{\mu} X$$

respectively. These modes of convergence are not independent, and we have

$$\left.\begin{array}{l} \xrightarrow{L^p} \\ \xrightarrow{\text{a. s.}} \end{array}\right\} \Rightarrow \xrightarrow{\mathbb{P}} \Rightarrow \xrightarrow{\mu}. \tag{5.3}$$

eq:rvconvergs

Furthermore, if $X_N \xrightarrow{\mathbb{P}} X$, then there exists a subsequence $\{N_k\}_{k=1}^{\infty}$ such that $X_{N_k} \xrightarrow{\text{a. s.}} X$.

Applying this to E_N, we see that $E_N \xrightarrow{L^2} \mathbb{E}[X]$, and that there is a subsequence such that $E_{N_k} \xrightarrow{\text{a. s.}} \mathbb{E}[X]$. In practice, one has to stop at some finite N, one would also like to be fairly sure that the estimate E_N is close to $\mathbb{E}[X]$. So, given $\varepsilon > 0$ and $\delta > 0$, find N such that

$$\text{Prob}\left(|E_N - \mathbb{E}[X]| > \delta\right) < \varepsilon.$$

We can use Chebyshev's second inequality for the random variable $Y = E_N - \mathbb{E}[X]$, we have $\mathbb{E}[Y] = 0$ and therefore

$$\text{Prob}\left(|Y| > \delta\right) \leq \frac{\text{Var}[Y]}{\delta^2} = \frac{\mathbb{E}[|E_N - \mathbb{E}[X]|^2]}{\delta^2} \leq \frac{\text{Var}[X]}{\delta^2 N}.$$

To ensure that this is less than ε we can choose $N > \text{Var}[X]/(\varepsilon\delta^2)$. As an example, to bee 99% sure that the error is less than 1% ($\delta = \varepsilon = 10^{-2}$), we can choose $N > 10^6 \,\text{Var}[X]$.

♢ ***Example***

The Monte-Carlo method is often used to numerically calculate integrals over high dimensional domains. We shall use it to estimate the volume of the unit ball B_d in $\mathbb{R}^d$. This volume is then written as an integral

$$V_d = 2^d \int_{[0,1]^d} \chi_{B_d}(x)\, dx$$

To recast this as an expectation, let $\Omega = [0,1]^d$, and let $\mathbb{P}$ be the d-dimensional Lebesgue measure and $\mathcal{B}$ the corresponding σ-algebra. Set $f(x) = \chi_{B_d}(x)$. If X is a random variable in this probability space, then $2^d\mathbb{E}[f(X)] = V_d$. Note that we can overestimate $\text{Var}[f(X)]$ by $\left(\max_x f(x) - \min_x f(x)\right)^2 = 1$. We want to approximate V_d using E_N. If we are to be 99% sure that the error is less than 1% for any d, we need to take $N > 2^d \times 10^6$. For large d this is impractical, so we relax our demands; we want to be 90% sure that the error is less than 10%. This is then ensured if $N > 1000 \times 2^d$. There is also a recursive formula for V_d,

$$V_d = \begin{cases} 1 & d = 0, \\ 2 & d = 1, \\ \frac{2\pi}{d} V_{d-2} & d > 1, \end{cases}$$

so we can compare E_N with the exact integral. In Figure 5.1 we plotted the relative errors; $100 \times |2^d E_N - V_d|/V_d$, for $d = 1, \ldots, 17$. Most of the computed errors are well under the 10% mark, but for $d = 17$ the error is almost 16%. This is not surprising, since we can only bound the probability

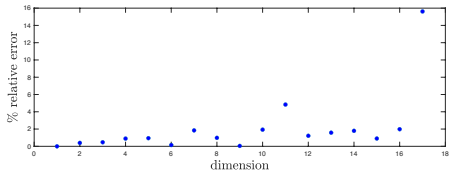

Fig. 5.1: The relative error when using the Monte-Carlo method to compute the volume of the unit ball in d dimensions.

fig:unitball

that the error is large. The Monte-Carlo estimator E_N is a random variable, and in applications we must accept whatever value it gives us.

A possible source of errors is that the "random" numbers given to us by a computer[3] are not truly random. In order not to stray too far from the topic of this book, we shall henceforth blissfully ignore this.

♢ □

5.1.2 Random variables in Banach spaces

subsec:rvbanach

In order to study how uncertainty manifests itself in solutions of partial differential equations, in particular in weak solutions of conservation laws, we need to work with random variables taking values in Banach spaces.

As before let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space, and let X be a function $X : \Omega \to B$, where B is a Banach space. We first need to define $\int_\Omega X$. The generic method of defining the integral is via simple functions. We say that $X : \Omega \to B$ is simple if $X(\omega)$ can be written as

$$X(\omega) = \sum_{i=1}^{N} x_i \chi_{A_i}(\omega),$$

where $A_i \in \mathcal{B}$ and $x_i \in B$ for $i = 1, \ldots, N$. For a simple function we define the integral

$$\int_\Omega X(\omega)\, d\mathbb{P}(\omega) = \sum_{i=1}^{N} x_i \mathbb{P}(A_i).$$

Note that for simple functions the triangle inequality holds,

$$\left\| \int_\Omega X \, d\mathbb{P} \right\|_B \leq \int_\Omega \|X\|_B \, d\mathbb{P},$$

[3] Except perhaps by a quantum computer.

where $\|\cdot\|_B$ denotes the norm in B.

For Banach space valued functions there are several types of measurability.

A function X is said to be *strongly measurable* if there exists a sequence of simple functions $\{X_j\}$ such that for all $\omega \in \Omega$

$$\lim_{j\to\infty} X_j(\omega) = X(\omega),$$

i.e., $X_j \to X$ pointwise.

The norm in B induces a topology on B. Let $\mathcal{L}(B)$ denote the smallest σ-algebra of subsets of B containing all the open sets in this topology. A function X is called *Borel measurable* if

$$X^{-1}(D) \in \mathcal{B} \text{ for all } D \in \mathcal{L}(B).$$

A function $X : \Omega \to B$ is *weakly measurable* if the real valued random variable $\omega \mapsto \xi(X(\omega))$ is measurable for all $\xi \in B^*$ (the dual of B).

Recall that by the Hahn-Banach extension theorem

$$\|x\|_B = \sup_{\|\xi\|_{B^*}\le 1} |\langle x, \xi\rangle|,$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between B and B^*. A linear subspace F of B^* is called *norming* if we have

$$\|x\|_B = \sup_{\substack{\|\xi\|_{B^*}\le 1\\ \xi\in F}} |\langle x, \xi\rangle|,$$

for all $\xi \in B^*$.

In order to establish whether a function $X : \Omega \to B$ is strongly measurable we need the next lemma.

lem:prepetti

Lemma 5.5 *If B_0 is a separable[4] linear subspace of B and F is a norming linear subspace of B^*, then F contains a sequence of unit vectors which is norming for B_0.*

Proof Choose a dense sequence $\{x_i\}_{i=1}^{\infty}$ in B_0, and choose a sequence of unit vectors $\{\xi_i\}_{i=1}^{\infty}$ in F such that

$$|\langle x_i, \xi_i\rangle| \ge (1-\varepsilon_i)\|x_i\|_B,$$

where the numbers ε_i are positive and $1 > \varepsilon_i \to 0$ as $i \to \infty$. We claim that $\{\xi_i\}$ is norming for B_0. To verify this choose $x \in B_0$, a number $\delta > 0$ and an integer k such that $\|x - x_k\|_B \le \delta$ and $0 < \varepsilon_k \le \delta$. Now we have

$$\begin{aligned}(1-\delta)\|x\|_B &\le (1-\varepsilon_k)\|x\|_B \le (1-\varepsilon_k)\|x_k\|_B + (1-\varepsilon_k)\|x - x_k\|_B\\ &\le |\langle x_k, \xi_k\rangle| + (1-\varepsilon_k)\,\delta\end{aligned}$$

[4] B_0 contains a dense countable subset.

$$\begin{aligned} &\leq |\langle x, \xi_k \rangle| + |\langle x_k - x, \xi_k \rangle| + \delta \\ &\leq |\langle x, \xi_k \rangle| + 2\delta. \end{aligned}$$

Since we can choose δ arbitrary small, we have that

$$\|x\|_B \leq \sup_i |\langle x, \xi_k \rangle| ,$$

and the sequence $\{\xi_i\}$ is norming.

We call a function $X : \Omega \to B$ *separately valued* if it takes its values in a closed separable subspace $B_0 \subset B$.

thm:petti1

Theorem 5.6 (Petti's measurability theorem, I) *Let F be a norming subspace of B^*. For a function $X : \Omega \to B$ the following are equivalent:*

1. *X is strongly measurable;*
2. *X is separately valued and $\langle X, \xi \rangle$ is measurable for all $\xi \in B^*$;*
3. *X is separately valued and $\langle X, \xi \rangle$ is measurable for all $\xi \in F$.*

Proof Let $\{X_i\}$ be a sequence of simple functions converging pointwise to X, and let B_0 be the linear subspace of B spanned by this sequence. Then B_0 is separable and X takes its values in B_0. Furthermore the function $f(\omega) = \langle X(\omega), \xi \rangle$ is $\mathbb{P}$ measurable for all $\xi \in B^*$ since it is the pointwise limit of the simple functions $f_i(\omega) = \langle X(\omega), \xi \rangle$. This shows that $1 \Rightarrow 2$. It is obvious that $2 \Rightarrow 3$, and it remains to show that $3 \Rightarrow 1$.

We shall construct a sequence of simple functions approximating X and to this end we use Lemma 5.5 and choose a sequence $\{\xi_i\}$ of unit vectors in F that is norming for a separable subspace B_0. We also have that X takes its values in B_0. Since for each $\xi \in F$ the map $\omega \mapsto \langle X, \xi \rangle$ is measurable, also the map

$$\omega \mapsto \langle X(\omega) - x, \xi_i \rangle$$

is $\mathbb{P}$-measurable for all $x \in B_0$. Let $\{x_i\}_{i=1}^{\infty}$ be a dense sequence in B_0. For each $y \in B_0$ define $k = k(i, y)$ to be the smallest integer such that

$$\|y - x_k\|_B = \min_{1 \leq j \leq i} \|y - x_j\|_B \quad \text{and define} \quad z_i(y) = x_{k(i,y)}.$$

Since $\{x_i\}$ is dense, $\|z_i(y) - y\|_B \to 0$ as $i \to \infty$, in other words, $\{z_i(y)\}$ is a subsequence of $\{x_i\}$ which approximates y. Now define

$$X_i(\omega) = z_i \left(X(\omega) \right).$$

It is clear that X_i approximates X, so we must show that X_i is simple. It is also clear that X_i takes values in the finite set $\{x_1, \ldots, x_i\}$ and it remains to show that for $1 \leq j \leq i$ the set $\left\{ \omega \;\middle|\; X_i(\omega) = x_k \right\}$ is $\mathbb{P}$-measurable. We have that $X_i = x_k$ if the distance from x_k to X is smaller or equal to $\min_{1 \leq j \leq i} \|X - x_j\|_B$, and there is no l smaller than k such that $\|X - x_l\|_B \leq \|X - x_k\|_B$. Thus

$$\left\{\omega \;\middle|\; X_i(\omega) = x_k\right\} = \left\{\omega \;\middle|\; \|X(\omega) - x_j\|_B = \min_{1\le j\le i} \|X(\omega) - x_j\|_B\right\}$$
$$\bigcap_{l=1}^{k-1} \left\{\omega \;\middle|\; \|X(\omega) - x_l\|_B > \min_{1\le j\le i} \|X(\omega) - x_j\|_B\right\}.$$

Since the real functions $\|X(\omega) - x_j\|_B$ are $\mathbb{P}$-measurable, $\left\{\omega \;\middle|\; X_i(\omega) = x_k\right\} \in \mathcal{B}$ and hence $3 \Rightarrow 1$.

cor:petti11

Corollary 5.7 *The pointwise limit of a sequence of strongly measurable functions is strongly measurable.*

Proof Each strongly measurable function X_i takes values in a separable subspace $B_{0,i}$ of B. This means that X takes its values in the closure of the linear span of these spaces, which is separable. Now for all $\xi \in B^*$ the real functions $f(\omega) = \langle X(\omega), \xi\rangle$ are measurable since each of is the pointwise limit of the measurable functions $f_i(\omega) = \langle X(\omega), \xi\rangle$.

cor:petti12

Corollary 5.8 *Let $X : \Omega \to B$ be a strongly measurable function, G a Banach space and $f : B \to G$ a continuous function. Then the composite mapping $\omega \mapsto f(X(\omega))$ is strongly measurable.*

Proof Choose a sequence of simple functions X_i converging to X pointwise, then $\{f(X_i)\}$ is a sequence of simple functions converging to $f(X)$ which is strongly measurable by Corollary 5.7.

prop:petti11

Proposition 5.9 *For a function $X : \Omega \to B$ the following is equivalent:*

1. *X is strongly measurable;*
2. *X is separately valued and for all $D \in \mathcal{L}(B)$ we have $X^{-1}(D) \in \mathcal{B}$.*

Proof Assume that X is strongly measurable, and thus separately valued. To prove that 2 holds, it suffices to prove that $X^{-1}(D) \in \mathcal{B}$ for all open sets D. Assume that D is open and let $\{X_i\}$ be a sequence of simple functions converging to X. For $x \in D$, let $D_r = \left\{x \in D \;\middle|\; \operatorname{dist}(x, D^C) > r\right\}$. Since X_i is simple $X_i^{-1}(D_r) \in \mathcal{B}$, in particular

$$X_i^{-1}(D_{1/j}) \in \mathcal{B} \quad \text{for all } i \text{ and } j > 0.$$

After some pondering, we realized that

$$X^{-1}(D) = \bigcup_{j\ge 1} \bigcup_{i\ge 1} \bigcap_{k\ge i} X_k^{-1}(D_{1/j}).$$

That $2 \Rightarrow 1$ follows from Petti's measurability theorem 5.6.

Now we can extend the definition of the integral to strongly measurable functions with values in B. A function is called *Bochner integrable* is there exists a sequence of simple functions $\{X_i\}_{i=1}^{\infty}$ such that

- $\lim_{i\to\infty} X_i(\omega) = X(\omega)$ for all $\omega \notin \mathcal{N}$, where $\mathbb{P}(\mathcal{N}) = 0$;
- $\lim_{i\to\infty} \int_\Omega \|X - X_i\|_B \, d\mathbb{P} = 0$.

Observe that this makes sense since $\|X_i - X\|_B$ is measurable by Corollary 5.8. It is also straightforward to show that the integral does not depend on the approximating sequence and that the triangle inequality holds for the Bochner integral. Furthermore we have the identity

$$\left\langle \int_\Omega X \, d\mathbb{P}, \xi \right\rangle = \int_\Omega \langle X, \xi \rangle \, d\mathbb{P},$$

for all $\xi \in B^*$.

prop:bochnerintegrable

Proposition 5.10 *A strongly measurable function $X : \Omega \to B$ is Bochner integrable if and only if*

$$\int_\Omega \|X\|_B \, d\mathbb{P} < \infty.$$

We remark that a "dominated convergence theorem" holds: If $\{X_i\}$ is a sequence of Bochner integrable functions such that $X_i \to X$ almost everywhere and Y is another Bochner integrable function with $\|X_i\|_B \leq \|Y\|_B$ for all i, then X is Bochner integrable.

Equipped with this integral, we can define the expectation of the Banach space valued random variable X,

$$\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P}.$$

As with real random variables, Banach space valued random variables induces a measure μ_X on B by the formula

$$\mu_X(D) = \mathbb{P}(X^{-1}(D))$$

for $D \in \mathcal{L}(B)$. The measure μ_X is called the *distribution* of X. If two random variables have the same distribution they are said to be *identically distributed*. With this definition two identically distributed random variables can be defined on different probability spaces. A random variable is called *symmetric* if X and $-X$ are identically distributed.

Independence of random variables is defined as in the real case, cf. 5.1. The different notions of convergence of sequences of random variables are also defined as for real random variables, see the definitions above (5.3), with $|\cdot|$ replaced by $\|\cdot\|_B$

Proposition 5.11 *Let $X_i : \Omega_i \to B_i$ be random variables, then $X_1, \ldots, X_N$ are independent if and only if*

$$\mu_{(X_1,\ldots,X_N)} = \mu_{X_1} \times \cdots \times \mu_{X_N}.$$

Remark (The Monte-Carlo method for Hilbert spaces) If B is a separable Hilbert space H (with inner product $\langle \cdot, \cdot \rangle$) and X is a H valued random variable, it is straightforward to obtain an error estimate for the Monte-Carlo method. With the necessary modifications, we can repeat the calculations leading up to (5.2) to get

$$\mathbb{E}\left[\|E_N - \mathbb{E}[X]\|_H^2\right] = \frac{1}{N}\left(\mathbb{E}\left[\|X\|_H^2\right] - \|\mathbb{E}[X]\|_H^2\right). \tag{5.4}$$

eq:hilmcerrest

In the context of conservation laws, B is often an L^1 space (which is not Hilbert!), and we must find a replacement for this error estimate. □

To bound the error of the Monte-Carlo algorithm, we must know how to estimate sums of independent random variables taking values in a Banach space.

thm:ito-nisio

Theorem 5.13 (Itô-Nisio theorem) *Let $X_i : \Omega \to B$, $i \geq 1$, be independent symmetric random variables and let $S : \Omega \to B$ be a random variable. Define the partial sums*

$$S_n = \sum_{i=1}^{n} X_i.$$

Then the following statements are equivalent:

1. *for all $\xi \in B^*$ we have $\langle S_n, \xi \rangle \xrightarrow{\text{a. s.}} \langle S, \xi \rangle$ as $n \to \infty$;*
2. *for all $\xi \in B^*$ we have $\langle S_n, \xi \rangle \xrightarrow{\mathbb{P}} \langle S, \xi \rangle$ as $n \to \infty$;*
3. *$S_n \xrightarrow{\text{a. s.}} S$ as $n \to \infty$;*
4. *$S_n \xrightarrow{\mathbb{P}} S$ as $n \to \infty$.*

If these equivalent statements hold and if $\mathbb{E}[\|S\|_B^p] < \infty$ for some $1 \leq p < \infty$ then

$$\lim_{n\to\infty} \mathbb{E}[\|S_n - S\|_B^p] = 0.$$

For a proof of this, see [**?**].

We now show some estimates on sums of random variables, and start with the so-called Kahane contraction principle:

thm:kahanecontract

Theorem 5.14 *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent, symmetric random variables. Then for all $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$ and all $1 \leq p < \infty$*

$$\mathbb{E}\Big[\Big\|\sum_{i=1}^{N} a_i X_i\Big\|_B^p\Big] \leq \max_i |a_i| \, \mathbb{E}\Big[\Big\|\sum_{i=1}^{N} X_i\Big\|_B^p\Big].$$

Proof First we assume that $a_i \in \{-1, 1\}$ for all i. Since X_i is symmetric, $\mathbb{E}[a_i X_i] = \mathbb{E}[X_i]$ and therefore

$$\mathbb{E}\Big[\Big\|\sum_{i=1}^{N} a_i X_i\Big\|_B^p\Big] = \mathbb{E}\Big[\Big\|\sum_{i=1}^{N} X_i\Big\|_B^p\Big].$$

Secondly, we observe that to prove the general case suffices to consider the case when $\max_i |a_i| \leq 1$, in which case a is in the hypercube in $\mathbb{R}^N$ with 2^N corners $\{c_j\}_{j=1}^{2^N}$ with components $c_{ji} \in \{-1, 1\}$. Therefore a is a convex combination of the vectors c_j, i.e.,

$$a = \sum_{j=1}^{2^N} \alpha_j c_j,$$

where $\alpha_j \geq 0$ and $\sum_j \alpha_j = 1$. Then we calculate

$$\begin{aligned}
\mathbb{E}\Big[\Big\|\sum_{i=1}^{N} a_i X_i\Big\|_B^p\Big] &= \mathbb{E}\Big[\Big\|\sum_{i=1}^{N}\sum_{j=1}^{2^N} \alpha_j c_{ji} X_i\Big\|_B^p\Big] \\
&= \mathbb{E}\Big[\Big\|\sum_{j=1}^{2^N} \alpha_j \sum_{i=1}^{N} c_{ji} X_i\Big\|_B^p\Big] \\
&\leq \mathbb{E}\Big[\Big(\sum_{j=1}^{2^N} \alpha_j \Big\|\sum_{i=1}^{N} c_{ji} X_i\Big\|_B\Big)^p\Big] \\
&\leq \mathbb{E}\Big[\sum_{j=1}^{2^N} \alpha_j \Big\|\sum_{i=1}^{N} c_{ji} X_i\Big\|_B^p\Big] \\
&= \sum_{j=1}^{2^N} \alpha_j \mathbb{E}\Big[\Big\|\sum_{i=1}^{N} c_{ji} X_i\Big\|_B^p\Big] = \mathbb{E}\Big[\Big\|\sum_{i=1}^{N} X_i\Big\|_B^p\Big],
\end{aligned}$$

where we used Jenssen's inequality to get from the third line to the fourth.

We shall have use for so-called *Rademacher* (random) variables. A $\{-1, 1\}$ valued random variable r is called a Rademacher variable if

$$\text{Prob}\,\{r = 1\} = \text{Prob}\,\{r = -1\} = \frac{1}{2}.$$

A Rademacher sequence is a sequence $\{r_i\}_i$ of independent Rademacher variables. If $\{\varphi_i\}_{j=1}^{\infty}$ is an independent sequence of symmetric real random variables, and $\{r_i\}_{i=1}^{\infty}$ is a Rademacher sequence which is independent of $\{\varphi_i\}_i$ then we claim that the sequences $\{r_i\,|\varphi_i|\}_{j=1}^{\infty}$ and $\{\varphi_i\}_{j=1}^{\infty}$ are identically distributed. To show this claim we use independence to compute

$$\begin{aligned}
\text{Prob}\,\{r\,|\varphi| \in A\} &= \text{Prob}\,\{r = 1,\ |\varphi| \in A\} + \text{Prob}\,\{r = -1,\ |\varphi| \in -A\} \\
&= \frac{1}{2}\,\text{Prob}\,\{|\varphi| \in A\} + \frac{1}{2}\,\text{Prob}\,\{|\varphi| \in -A\} \\
&= \frac{1}{2}\,\text{Prob}\,\{\varphi \geq 0,\ \varphi \in A\} + \frac{1}{2}\,\text{Prob}\,\{\varphi < 0,\ \varphi \in -A\}
\end{aligned}$$

$$+\frac{1}{2}\,\mathrm{Prob}\,\{\varphi \geq 0,\ \varphi \in -A\} + \frac{1}{2}\,\mathrm{Prob}\,\{\varphi < 0,\ \varphi \in A\}$$
$$= \frac{1}{2}\,\mathrm{Prob}\,\{\varphi \in A\} + \frac{1}{2}\,\mathrm{Prob}\,\{\varphi \in -A\} = \mathrm{Prob}\,\{\varphi \in A\},$$

since φ is symmetric. With the use of this claim we can prove the following theorem.

thm:comparison

Theorem 5.15 *Let $\{r_i\}_{i=1}^{\infty}$ be a Rademacher sequence, and let $\{\varphi_i\}_{i=1}^{\infty}$ be a sequence of real symmetric random variables with $\mathbb{E}[|\varphi_i|] \geq 1$ for all i. If $\{\varphi_i\}$ and $\{r_i\}$ are independent then for all $x_1, \ldots, x_N \in B$ and all $1 \leq p < \infty$ we have*

$$\mathbb{E}\Big[\Big\|\sum_{i=1}^{N} r_i x_i\Big\|_B^p\Big] \leq \mathbb{E}\Big[\Big\|\sum_{i=1}^{N} \varphi_i x_i\Big\|_B^p\Big].$$

Proof Since r_i and φ_i are independent, we can think of these as defined on different probability spaces, and for clarity we denote by $\mathbb{E}_r$ and $\mathbb{E}_\varphi$ the expectations on these different probability spaces.

Set $Y_i = \mathbb{E}_\varphi[|\varphi_i|] r_i x_i$ and $a_i = 1/\mathbb{E}_\varphi[|\varphi_i|]$ in the Kahane comparison principle to get

$$\begin{aligned}
\mathbb{E}_r\Big[\Big\|\sum_i r_i x_i\Big\|_B^p\Big] &= \mathbb{E}_r\Big[\Big\|\sum_i a_i Y_i\Big\|_B^p\Big] \leq \max_i |a_i|\, \mathbb{E}_r\Big[\Big\|\sum_i Y_i\Big\|_B^p\Big] \\
&= \frac{1}{\min_i \mathbb{E}_\varphi[|\varphi_i|]}\mathbb{E}_r\Big[\Big\|\sum_i \mathbb{E}_\varphi[|\varphi_i|] r_i x_i\Big\|_B^p\Big] \\
&\leq \mathbb{E}_r\Big[\Big\|\mathbb{E}_\varphi\Big[\sum_i |\varphi_i| r_i x_i\Big]\Big\|_B^p\Big] \\
&\leq \mathbb{E}_r\Big[\mathbb{E}_\varphi\Big[\Big\|\sum_i r_i\,|\varphi_i|\,x_i\Big\|_B^p\Big]\Big] && \text{(Jenssen's ineq.)} \\
&= \mathbb{E}_\varphi\Big[\Big\|\sum_i \varphi_i x_i\Big\|_B^p\Big] && \text{(above claim).}
\end{aligned}$$

To prove the convergence of the Monte-Carlo method in Banach spaces, it is useful to be able to "measure" how far a Banach space is from being a Hilbert space. This is indicated by its *type*, which is defined as follows:

Definition 5.16 We say that a Banach space B has type q, where $1 \leq q \leq 2$, if there is a constant κ such that for any finite sequence $\{x_i\}_{i=1}^N \subset B$ and any Rademacher sequence $\{r_i\}$, the inequality

$$\mathbb{E}\Big[\Big\|\sum_{i=1}^{N} r_i x_i\Big\|_B^q\Big] \leq \kappa^q \sum_{i=1}^{N} \|x_i\|_B^q\,. \tag{5.5}$$

eq:typedef

The smallest constant κ such that (5.5) holds is called the *the type constant*, and we shall henceforth refer to the type constant as κ. Every Banach space

has type 1, and every Hilbert space has Banach space type 2. Moreover, L^p spaces have type $q = \min\{2, p\}$. For Lebesgue spaces we have the following lemma:

lem:lebtype **Lemma 5.17** *Let $\{\Omega, \mathcal{B}\}$ be a measure space, $1 \le r \le \infty$ and B a Banach space of type q. Then $L^r(\Omega, B)$ is a Banach space of type* $\min\{r, q\}$.

Observe that every Hilbert space has type $q = 2$, and crucially for our application $L^1(\mathbb{R})$ has type 1.

In order to deduce some consequences of B having a type, we prove a useful inequality. Let F be a real convex function on $[0, \infty)$ be B valued zero mean random variables, and in addition assume that Y is symmetric. Now

$$\begin{aligned}\mathbb{E}[F(\|X\|_B)] &= \mathbb{E}[H((X+Y)/2 + (X-Y)/2)] \\ &\le \frac{1}{2}\mathbb{E}[H(X+Y)] + \frac{1}{2}\mathbb{E}[H(X-Y)] \\ &= \mathbb{E}[H(X+Y)] = \mathbb{E}[F(\|X+Y\|_B)].\end{aligned}$$

With this observation we can prove the following lemma:

lem:sumsconvex **Lemma 5.18** *Let $F : [0, \infty) \to [0, \infty$ be a convex function, and let $\{r_i\}$ be a Rademacher sequence. Then for any finite sequence $\{X_i\}_{i=1}^N$ of independent zero mean random variables such that $\mathbb{E}[F(\|X_i\|_B)] < \infty$ for all i,*

$$\mathbb{E}\Big[F\Big(\frac{1}{2}\Big\|\sum_{i=1}^N r_i X_i\Big\|_B\Big)\Big] \le \mathbb{E}\Big[F\Big(\Big\|\sum_{i=1}^N X_i\Big\|_B\Big)\Big] \le \mathbb{E}\Big[F\Big(2\Big\|\sum_{i=1}^N r_i X_i\Big\|_B\Big)\Big].$$

Proof Let $\tilde{X}_i = X_i - X_i'$, where X_i' is a copy of X_i defined on Ω'. By the above observation we have

$$\begin{aligned}\mathbb{E}\Big[F\Big(\Big\|\sum_{i=1}^N X_i\Big\|_B\Big)\Big] &\le \mathbb{E}\Big[F\Big(\Big\|\sum_{i=1}^N \tilde{X}_i\Big\|_B\Big)\Big] \\ &= \mathbb{E}\Big[F\Big(\Big\|\sum_{i=1}^N r_i\tilde{X}_i\Big\|_B\Big)\Big] \le \mathbb{E}\Big[F\Big(2\Big\|\sum_{i=1}^N r_i X_i\Big\|_B\Big)\Big],\end{aligned}$$

and conversely

$$\begin{aligned}\mathbb{E}\Big[F\Big(\frac{1}{2}\Big\|\sum_{i=1}^N r_i X_i\Big\|_B\Big)\Big] &\le \mathbb{E}\Big[F\Big(\frac{1}{2}\Big\|\sum_{i=1}^N r_i\tilde{X}_i\Big\|_B\Big)\Big] \\ &= \mathbb{E}\Big[F\Big(\frac{1}{2}\Big\|\sum_{i=1}^N \tilde{X}_i\Big\|_B\Big)\Big] \le \mathbb{E}\Big[F\Big(\Big\|\sum_{i=1}^N r_i X_i\Big\|_B\Big)\Big].\end{aligned}$$

Now we are ready to prove the important result by which we can gain information about the error of the Monte-Carlo method.

lem:MMbanach

Lemma 5.19 *Let B be a Banach space of type q with type constant κ. Then for every finite sequence $\{X_i\}_{i=1}^N$ of B valued independent and zero mean random variables in $L^q(\Omega; B)$*

$$\mathbb{E}\Big[\Big\|\sum_{i=1}^N X_i\Big\|_B^q\Big] \le (2\kappa)^q \sum_{i=1}^N \mathbb{E}[\|X_i\|_B^q].$$

Proof Using Lemma 5.18 and the definition of the type

$$\mathbb{E}\Big[\Big\|\sum_{i=1}^N X_i\Big\|_B^q\Big] \le 2^q \mathbb{E}\Big[\Big\|\sum_{i=1}^N r_i X_i\Big\|_B^q\Big] \le (2\kappa)^q \sum_{i=1}^N \mathbb{E}[\|X_i\|_B^q].$$

Now we are ready to estimate how close the average E_N is to the mean $\mathbb{E}[X]$. Let X_i, $i = 1, \dots, N$ be N independent copies of X, set $Y_i = (X_i - \mathbb{E}[X])/N$. Then $\{Y_i\}_{i=1}^N$ are independent zero mean random variables, and

$$E_N - \mathbb{E}[X] = \sum_{i=1}^N Y_i.$$

Lemma 5.19 says that

$$\begin{aligned} \mathbb{E}\left[\|E_N - \mathbb{E}[X]\|_B^q\right] &\le (2\kappa)^q N^{-q} \sum_{i=1}^N \mathbb{E}\left[\|X_i - \mathbb{E}[X]\|_B^q\right] \\ &= (2\kappa)^q N^{1-q} \mathbb{E}\left[\|X - \mathbb{E}[X]\|_B^q\right], \end{aligned} \tag{5.6}$$

eq:MCbanach1

where q is the type of B. Unfortunately, if $B = L^1(\mathbb{R}^d)$, which has type 1, this estimate fails to imply that E_N is close to $\mathbb{E}[X]$ for large N.

In the special case where $X \in L^r(\Omega; L^p(\mathbb{R}^d))$ we can partially circumvent this difficulty, and we have the following lemma:

lem:mclpbnd

Lemma 5.20 *Assume that $1 \le r, p \le \infty$ and let $X \in L^r(\Omega; L^p(\mathbb{R}^d))$. For $q = \min\{2, r, p\}$ we have the bound*

$$\|\mathbb{E}[X] - E_N[X]\|_{L^q(\Omega; L^p(\mathbb{R}^d))} \le C N^{(1-q)/q} \|X\|_{L^q(\Omega; L^p(\mathbb{R}^d))},$$

where the constant C depends on q and κ.

Proof In general, $L^p(\mathbb{R}^d)$ is a Banach space of type $\min\{2, p\}$, thus $L^q(\Omega; L^p(\mathbb{R}^d))$ is of type q. By (5.6) and Jenssen's inequality

$$\begin{aligned} \mathbb{E}\left[\|\mathbb{E}[X] - E_N[X]\|_{L^p(\mathbb{R}^d)}^q\right] &\le (2\kappa)^q N^{1-q} \mathbb{E}\left[\|\mathbb{E}[X] - X\|_{L^p(\mathbb{R}^d)}^q\right] \\ &\le C(2\kappa)^q N^{1-q} \mathbb{E}\left[\left(\|\mathbb{E}[X]\|_{L^p(\mathbb{R}^d)} + \|X\|_{L^p(\mathbb{R}^d)}\right)^q\right] \\ &\le C 2^{q+1} \kappa^q N^{1-q} \left(\|\mathbb{E}[X]\|_{L^p(\mathbb{R}^d)}^q + \mathbb{E}\left[\|X\|_{L^p(\mathbb{R}^d)}^q\right]\right) \end{aligned}$$

$$\leq CN^{1-q} \|X\|^q_{L^q(\Omega;L^p(\mathbb{R}^d))}.$$

5.2 Random solutions of conservation laws

sec:randcons

A random entropy solution of the scalar conservation law

$$u_t + \operatorname{div} \boldsymbol{f}(u) = 0,$$

is a random variable u, defined on some probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and taking values in $C([0,T]; L^1(\mathbb{R}^d))$ or some subset thereof, such that u is almost surely an entropy solution, i.e., for all convex functions η the inequality

$$\eta(u)_t + \operatorname{div} \boldsymbol{q}(u) \leq 0, \quad \text{holds weakly for } \omega \in \Omega \setminus \mathcal{N},$$

where
$P(\mathcal{N}) = 0$. Here $\boldsymbol{q}$ is the entropy flux associated to η, defined by $q_i'(u) = \eta'(u) f_i'(u)$ for $i = 1, \ldots, d$. There are many different ways to construct random solutions depending on what one wants to model. We shall focus on two examples; random initial data and random flux function.

5.2.1 Existence of random entropy solutions

subsec:randexist

In order to show the existence of random entropy solutions we have to show that the map $\omega \mapsto u(\cdot, t, \omega)$ is strongly measurable.

Random initial data

subsec:randominit

One can envisage many situations in which the time evolution, i.e., the conservation law, is known but the initial data is uncertain. The uncertainty is modeled by a random variable u_0. In order to keep matters simple, we assume that u_0 takes values in the set

$$V = \left\{ v \in L^1(\mathbb{R}^d) \;\middle|\; |v|_{BV(\mathbb{R}^d)} < \infty \right\},$$

and that

$$\|u_0\|_{L^r(\Omega;L^\infty(\mathbb{R}^n))} < \infty,$$

for some $r > 1$. Define $u : \mathbb{R}^d \times [0,T] \times \Omega \to \mathbb{R}$ to be the entropy solution of

$$\begin{cases} u_t(x,t,\omega) + \operatorname{div} \boldsymbol{f}(u(x,t,\omega)) = 0, & (x,t) \in \mathbb{R}^d \times (0,T], \\ u(x,0,\omega) = u_0(x,\omega). \end{cases} \tag{5.7}$$

eq:randominit_esol

We see that $\{u(\cdot,\cdot,\omega)\}_{\omega\in\Omega}$ is a family of entropy solutions to the scalar conservation law (5.7), in particular the continuity with respect to the initial data estimate (4.26) holds. In the present context this reads;

$$\|u(\cdot,t,\omega) - u(\cdot,t,\omega')\|_{L^1(\mathbb{R}^d)} \le \|u_0(\cdot,\omega) - u_0(\cdot,\omega')\|_{L^1(\mathbb{R}^d)},$$

for ω and ω' in Ω and $t \ge 0$. This means that the composite map

$$\Omega \ni \omega \mapsto u_0(\cdot,\omega) \mapsto u(\cdot,t,\omega) \in L^1(\mathbb{R}^d)$$

is the composition of a continuous and a measurable map. Therefore, by Corollary 5.8 this composite map $\omega \mapsto u(\cdot,t,\omega)$ is strongly measurable. Furthermore, (4.24) states that $|u(\cdot,t,\omega)|_{BV(\mathbb{R}^d)} \le |u_0(\cdot,\omega)|_{BV(\mathbb{R}^d)} < \infty$. Hence $u(\cdot,t,\cdot)$ is a (well defined) random variable taking values in V.

Random flux functions

subsubsec:randflux

For some phenomena described by conservation laws the flux function is uncertain. One may know $f(0)$ and $f(1)$, but not $f(u)$ for u between 0 and 1. Similarly, it happens that one only knows the "shape" of the flux function, such as monotonicity, convexity and so on. In such cases it is natural to describe the flux function as a random variable. For simplicity, in this section we assume that $d = 1$, the extension to $d > 1$ is straightforward.

Let f $(= f(u,\omega))$ be a random variable taking values in the Sobolev space $W^{1,\infty}$. Since there is no loss of generality in assuming $f(0,\cdot) = 0$, we can assume that f takes values in

$$V = \left\{ g \in W^{1,\infty}(\mathbb{R}) \;\middle|\; g(0) = 0 \right\}$$

Since the initial data is not random, by scaling if suffices to consider $u \in [-1,1]$. Therefore we can use

$$\|g\|_V = \sup_{\substack{u,v\in[-1,1] \\ u\neq v}} \frac{|g(u) - g(v)|}{|u-v|} = \|g\|_{\mathrm{Lip}},$$

as a norm for V. We shall also assume that $\|u_0\|_{L1(\mathbb{R})} \le M$ and $|u_0|_{BV(\mathbb{R})} \le M$, which clearly implies that $\|u_0\|_{L^\infty(\mathbb{R})} < \infty$.

In this case we define a random solution $u = u(x,t,\omega)$ as the entropy solution to

$$\begin{cases} u_t(x,t,\omega) + f_x(u(x,t,\omega),\omega) = 0, & (x,t) \in \mathbb{R} \times (0,T], \\ u(x,0,\omega) = u_0(x). \end{cases} \tag{5.8}$$

eq:randomflux_esol

Since $\{u(\cdot,\cdot,\omega)\}_{\omega\in\Omega}$ is a parameterised family of entropy solutions, we can invoke the stability estimate (4.26) for each pair (ω,ω') to get

$$\|u(\cdot,t,\omega) - u(\cdot,t,\omega')\|_{L^1(\mathbb{R})} \leq tM\|f(\cdot,\omega) - f(\cdot,\omega')\|_{\mathrm{Lip}}.$$

Thus the map from Ω to $L^1(\mathbb{R})$ defined by $\omega \mapsto u(\cdot,t,\omega)$ is the composition of the measurable map from Ω to V; $\omega \mapsto f(\cdot,\omega)$, and the continuous map from V to $L^1(\mathbb{R})$ given by $f \mapsto u(\cdot,t)$. Hence by Corollary 5.8 the random variable u is strongly measurable.

5.2.2 Computing approximations to the mean

subsec:approxmean

Se start by bounding the difference between the mean and the average. For simplicity we hereafter assume that $d = 1$. We shall also assume that u_0 has bounded support or is periodic, either of which implies that there exists a finite interval D such that all relevant norms are in $L^p(D)$ rather than $L^p(\mathbb{R})$. In addition we assume that

$$\|u_0\|_{L^r(\Omega;L^\infty(D))} < \infty, \tag{5.9}$$

eq:linftyass

for some $1 < r \leq \infty$. This implies that also

$$\|u(\cdot,t)\|_{L^r(\Omega;L^\infty(D))} < \infty \quad \text{and} \quad \|u_h(\cdot,t)\|_{L^r(\Omega;L^\infty(D))} < \infty$$

as well. We shall use this simplifying assumption as follows: Assume that $u_0 \in L^r(\Omega;L^\infty(D))$ for some $r > 1$. For any $1 \leq p \leq \infty$ we have

$$\begin{aligned} \|u(\cdot,t,\cdot)\|^r_{L^r(\Omega;L^p(D)} &= \int_\Omega \|u(\cdot,t,\omega)\|^r_{L^p(D)}\, d\mathbb{P}(\omega) \\ &\leq |D|^{r/p} \int_\Omega \|u(\cdot,t,\omega)\|^r_{L^\infty(D)}\, d\mathbb{P}(\omega) \\ &\leq |D|^{r/p} \int_\Omega \|u_0(\cdot,\omega)\|^r_{L^\infty(D)}\, d\mathbb{P}(\omega) \\ &= |D|^{r/p} \|u_0\|^r_{L^r(\Omega;L^\infty(D))}. \end{aligned} \tag{5.10}$$

eq:simplibnd

Now we can prove that the Monte-Carlo estimate approximates the mean.

lem:mcerror

Lemma 5.21 *For each $1 \leq p \leq \infty$ and $0 \leq t \leq T < \infty$ and for $q = \min\{2,r\} > 1$, we have that*

$$\|\mathbb{E}[u] - E_N[u]\|_{L^q(\Omega;L^p(D))} \leq CN^{\frac{1-q}{q}} \|u_0\|_{L^r(\Omega;L^\infty(D))},$$

for some constant C depending on D, p, q and T.

Proof First consider the case $p \geq q$. By (5.10) and Lemma 5.20

$$\begin{aligned}\|\mathbb{E}[u] - E_N[u]\|_{L^q(\Omega;L^p(D))} &\leq CN^{\frac{1-q}{q}} \|u(\cdot,t)\|_{L^q(\Omega;L^p(D))} \\ &\leq CN^{\frac{1-q}{q}} \|u(\cdot,t)\|_{L^r(\Omega;L^p(D))} \\ &\leq CN^{\frac{1-q}{q}} \|u_0\|_{L^r(\Omega;L^\infty(D))}.\end{aligned}$$

If $p < q$ we can use Hölder's inequality to get

$$\|\mathbb{E}[u] - E_N[u]\|_{L^q(\Omega;L^p(D))} \leq C \|\mathbb{E}[u] - E_N[u]\|_{L^q(\Omega;L^q(D))}.$$

Since $q \leq r$

$$\|u(\cdot,t)\|_{L^q(\Omega;L^q(D))} \leq C \|u_0\|_{L^q(\Omega;L^\infty(D))} \leq C \|u_0\|_{L^r(\Omega;L^\infty(D))},$$

and thus $u(\cdot,t) \in L^q(\Omega;L^q(D))$ and we can invoke Lemma 5.20 to get

$$\begin{aligned}\|\mathbb{E}[u] - E_N[u]\|_{L^q(\Omega;L^q(D))} &\leq CN^{\frac{1-q}{q}} \|u(\cdot,t)\|_{L^q(\Omega;L^q(D))} \\ &\leq CN^{\frac{1-q}{q}} \|u_0\|_{L^q(\Omega;L^\infty(D))}.\end{aligned}$$

Hence we have proved the lemma for any $1 \leq p \leq \infty$.

In general we cannot compute the entropy solutions to conservation laws, therefore we do not know the averages $E_N[u]$. What we can compute are approximations to entropy solutions and we can use the approximations to approximate the Monte-Carlo averages $E_N[u]$.

To generate approximations to random entropy solutions we can use any convergent numerical method, we shall consider front tracking and finite volume methods. Let u_δ and $u_{\Delta t}$ denote the front tracking approximation and the finite volume approximation respectively. These depend on the initial data u_0 and the flux functions f. See Section 2.3 regarding how to compute u_δ. For convenience we give a brief summary of how to construct $u_{\Delta t}$.

Let Δt and Δx be (small) positive numbers, $t^n = n\Delta t$ for $n = 0, 1, 2, \ldots$, $x_j = j\Delta x$ for $j \in \mathbb{Z}/2 \cap D/\Delta x$, $I^n = [t^n, t^{n+1})$, $I_j = [x_{j-1/2}, x_{j+1/2})$ and define the piecewise constant function

$$u_{\Delta t}(x,t) = \sum_{j,n} u_j^n \chi_{I_j}(x) \chi_{I^n}(t),$$

where the sum is over integers j such that $j\Delta x \in D$ and n such that $t^n \leq T$. The numbers u_j^n are computed by a conservative, consistent and monotone finite volume method

$$u_j^{n+1} = u_j^n - \lambda \left(f_{j+1/2}^n - f_{j-1/2}^n \right), \quad n \geq 0,$$
$$u_j^0 = \frac{1}{\Delta x} \int_{I_j} u_0(x)\, dx,$$

where $\lambda = \Delta t / \Delta x$, and recall that in order for the method to be monotone, $\lambda \leq C$ for some constant C depending on $\|f\|_{\text{Lip}}$. Then we can apply Kuznetsov's lemma (Theorem 3.14) and the calculations leading up to (3.71) to get the error estimate

$$\|u(\cdot,t) - u_{\Delta t}(\cdot,t)\|_{L^1(D)} \leq C\, |u_0|_{BV(D)} \left(1 + \|f\|_{\text{Lip}}\right) \sqrt{\Delta t}, \tag{5.11}$$

eq:detailedrate

where the constant C depends on T but not on the initial data u_0 or on the flux function f. Recall that this estimate also holds for the front tracking approximation with δ replacing $\sqrt{\Delta t}$, viz.

$$\|u(\cdot,t) - u_\delta(\cdot,t)\|_{L^1(D)} \leq C\, |u_0|_{BV(D)} \left(1 + \|f\|_{\text{Lip}}\right) \delta.$$

In order to unify the presentation we assume that we have a family of approximations $\{u_h\}_{h>0}$ satisfying the error estimate

$$\|u(\cdot,t,\omega) - u_h(\cdot,t,\omega)\|_{L^1(D)} \leq C h^s, \tag{5.12}$$

eq:detstability

where $h = \Delta t$ and $s = 1/2$ for the finite volume approximation, and $h = \delta$ and $s = 1$ for the front tracking approximation.

Now we can bound how well we can estimate how well we can approximate averages of exact solutions by averages of numerical approximations.

lem:averagebnd

Lemma 5.22 *For each $1 \leq p \leq \infty$ and $0 \leq t \leq T < \infty$ and for $q = \min\{2, r\} > 1$, we have that*

$$\|E_N[u] - E_N[u_h]\|_{L^q(\Omega;L^p(D))} \leq C h^{\frac{s}{p}} \|u_0\|_{L^r(\Omega;L^\infty(D))},$$

for some constant C depending on D, p, q and T.

Proof Let $u^i = u(\cdot,\cdot,\omega_i)$ and $u_h^i = u_h(\cdot,\cdot,\omega_i)$ where $\{\omega_i\}_{i=1}^\infty$ are independent and identically distributed. By definition

$$\begin{aligned}\|E[u] - E_N[u_h]\|_{L^q(\Omega;L^p(D))} &= \frac{1}{N} \Big\| \sum_{i=1}^N u^i - u_h^i \Big\|_{L^q(\Omega;L^p(D))} \\ &\leq \frac{1}{N} \sum_{i=1}^N \|u - u_h\|_{L^q(\Omega;L^p(D))} = \|u - u_h\|_{L^q(\Omega;L^p(D))}.\end{aligned}$$

By the simplifying assumption (5.10), both $u(\cdot,t,\omega)$ and $u_h(\cdot,t,\omega)$ are in $L^\infty(D) \cap L^1(D)$ ω almost surely. We can then use L^1/L^∞ interpolation, which says that if $g \in L^p \cap L^\infty$ then

$$\|g\|_{L^p} \leq \|g\|_{L^1}^{\frac{1}{p}} \, \|g\|_{L^\infty}^{1-\frac{1}{p}},$$

for any $1 \leq p \leq \infty$. This inequality gives us

$$\begin{aligned}
\|u - u_h\|_{L^q(\Omega;L^p(D))} &= \left(\int_\Omega \|u - u_h\|_{L^p(D)}^q \, d\mathbb{P} \right)^{\frac{1}{q}} \\
&\leq \left(\int_\Omega \|u - u_h\|_{L^1(D)}^{\frac{q}{p}} \, \|u - u_h\|_{L^\infty(D)}^{q(1-\frac{1}{p})} \, d\mathbb{P} \right)^{\frac{1}{q}} \\
&\leq \|u - u_h\|_{L^q(\Omega;L^1(D))}^{\frac{1}{p}} \, \|u - u_h\|_{L^q(\Omega;L^\infty(D))}^{1-\frac{1}{p}} \\
&\leq C h^{\frac{s}{p}} \, \|u_0\|_{L^r(\Omega;L^\infty(D))},
\end{aligned}$$

which concludes the proof.

As a corollary of Lemma 5.21 and Lemma 5.22 we get a bound on $\mathbb{E}[u] - E_N[u_h]$.

thm:mcapproxerror

Theorem 5.23 *Assume that we are given approximations $\{u_j\}_{h>0}$ such that (5.12) holds (for all sufficiently small h), and that (5.9) holds for some $r > 1$. Then, for any $1 \leq p \leq \infty$ and for all $0 \leq t \leq T$ and for $q = \min\{2, r\}$, we have the error estimate*

$$\begin{aligned}
&\|\mathbb{E}[u] - E_N[u_h]\|_{L^q(\Omega;L^p(D))} \\
&\qquad \leq C\Big(N^{\frac{1-q}{q}} \, \|u_0\|_{L^q(\Omega;L^\infty(D))} + h^{\frac{s}{p}} \, \|u_0\|_{L^q(\Omega;L^\infty(D))}^{1-\frac{1}{p}} \Big),
\end{aligned}$$

where the constant C depends on D, p, q and T.

Proof By the triangle inequality,

$$\begin{aligned}
\|\mathbb{E}[u] - E_N[u_h]\|_{L^q(\Omega;L^p(D))} \leq \; &\|\mathbb{E}[u] - E_N[u]\|_{L^q(\Omega;L^p(D))} \\
&+ \|E_N[u] - E_N[u_h]\|_{L^q(\Omega;L^p(D))}.
\end{aligned}$$

Use Lemma 5.21 to bound the first term and Lemma 5.22 to bound the second.

It is natural to consider $p = 1$ since scalar conservation laws are well posed in L^1, therefore we define the error of the Monte-Carlo method as

$$\mathcal{E}_{\mathrm{MC}}(h, N) = \|\mathbb{E}[u] - E_N[u_h]\|_{L^q(\Omega;L^1(D))}.$$

For general $p > 1$ the ensuing analysis is similar, but one must keep track of p. To minimize the error we must choose N and h such that both terms on the right scale identically, i.e.,

$$h^s = \mathcal{O}\left(N^{\frac{1-q}{q}} \right) = \mathcal{O}(\varepsilon). \tag{5.13}$$

eq:scalemc

then $\mathcal{E}_{\mathrm{MC}}(h,N) \lesssim \varepsilon$.

In practice it is important to know how many operations (multiplications, additions etc.) are needed to ensure that $\mathcal{E}_{\mathrm{MC}} \lesssim \varepsilon$. Let $\mathcal{W}_{\mathrm{MC}}(h,N)$ denote the number of operations needed to compute $E_N[u_h]$. If $w(h)$ denotes the number of operations used to compute u_h, then $\mathcal{W}_{\mathrm{MC}}(h,N) = Nw(h)$. Therefore we estimate $w_{\Delta t}$ and w_δ. Thus the computational cost needed to compute $E_N[u_h]$ such that the error $\mathcal{E}_{\mathrm{MC}} \leq \varepsilon$ scales as

$$\mathcal{C}_\varepsilon = \Big(\frac{1}{\varepsilon}\Big)^{\frac{q}{q-1}} w(\varepsilon^{1/s}). \tag{5.14}$$

eq:mccost

Bounding $w(\Delta t)$ for the finite volume method: We need to compute $u_{\Delta t}(x_j, t^n)$ for j and n such that $|j| \leq |D|/\Delta x$ and $n \leq T/\Delta t$. This means that the work needed to compute $u_{\Delta t}$ scales as $1/\Delta t^2$ since $\Delta x \lesssim \Delta t$ by the CFL-condition $\Delta t/\Delta x \lesssim 1$. Thus

$$w(\Delta t) \sim \frac{1}{\Delta t^2}. \tag{5.15}$$

eq:wdtbnd

Bounding $w(\delta)$ for the front tracking method: The computational work needed to calculate u_δ is proportional to the number of collisions, and for each collision we must solve a Riemann problem, calculate the speeds of the resulting new fronts and calculate two new collision times.

If f is convex, the number of fronts is strictly decreasing at each collision. Initially there are at most $|u_0|_{BV}/\delta$ fronts, and therefore we have at most $|u_0|_{BV}/\delta$ collisions. For general non-convex flux functions we have that the functional $T(t)$ (defined on page ??) is strictly decreasing at each collision. Thus the total number of collisions is bounded by

$$T(0) \leq \frac{(\sup_D u_0 - \inf_D u_0)}{\delta} \frac{|u_0|_{BV}}{\delta} + \frac{|u_0|_{BV}}{\delta} \leq \frac{C}{\delta^2},$$

for small δ.

Next we calculate the number of operations needed to solve a Riemann problem with left state u_l and right state u_r. This means finding the convex envelope of $f_\delta(u)$ for u between u_l and u_r. Using Graham's algorithm this has a computational cost of

$$G \log G, \quad \text{where} \quad G = \frac{|u_l - u_r|}{\delta}.$$

Finally we have to compute the speeds of each of the resulting fronts, and there are at most G of these. If f is convex, the result of a collision is always one front. Thus the computational cost of resolving one collision is 1 if f is convex, and

$$\frac{|u_0|_{BV}}{\delta} \log\Big(\frac{|u_0|_{BV}}{\delta}\Big) \quad \text{if } f \text{ is not convex.}$$

We conclude that

$$w(\delta) \sim \begin{cases} \frac{1}{\delta} & \text{if } f \text{ is convex,} \\ \frac{1}{\delta^3}\log\left(\frac{1}{\delta}\right) & \text{otherwise.} \end{cases} \tag{5.16}$$

eq:wdeltabnd

Using (5.15) in (5.14) with $s = 1/2$ we find the cost of the finite volume Monte-Carlo method;

$$\mathcal{C}_\varepsilon^{\text{MCFV}} = \varepsilon^{\frac{q}{1-q}}\varepsilon^{-4} = \left(\frac{1}{\varepsilon}\right)^{\frac{5q-4}{q-1}}, \tag{5.17}$$

eq:fvcost

so that even in the "best" case with $q = 2$, the cost of an error of size ε scales as ε^{-6}.

For the front tracking Monte-Carlo method, we find the corresponding cost inserting (5.16) in (5.14) with $s = 1$,

$$\mathcal{C}_\varepsilon^{\text{MCFT}} = \varepsilon^{\frac{q}{1-q}} \begin{cases} \varepsilon^{-1} \\ \varepsilon^{-3}\log(\frac{1}{\varepsilon}) \end{cases} = \begin{cases} \left(\frac{1}{\varepsilon}\right)^{\frac{2q-1}{q-1}} & \text{if } f \text{ is convex,} \\ \left(\frac{1}{\varepsilon}\right)^{\frac{4q-3}{q-1}}\log\left(\frac{1}{\varepsilon}\right) & \text{otherwise.} \end{cases} \tag{5.18}$$

eq:ftcost

In case $q = 2$, the cost scales as ε^{-3} for convex f and $\varepsilon^{-5}\log(\varepsilon^{-1})$ for non-convex f.

♢ ***Example*** Let $\{c_k\}_{k=1}^{60}$ be a sequence of random variables such that $c_k = 1$ if $k = 1 + 3j$ for some $j \in \{0, 1, 2, 3, \ldots\}$, and otherwise $\text{Prob}\{c_k = 1\} = \text{Prob}\{c_k = -1\} = 1/2$. Define the random initial data by

$$u_0(x) = \frac{2}{\pi}\sum_{k=1}^{60}\frac{c_k}{k}\sin(k\pi x).$$

Let u be the random entropy solution to the initial value problem

$$\begin{cases} u_t + \frac{1}{2}(u^2)_x = 0, & t > 0, \\ u(x, 0) = u_0(x, \omega). \end{cases}$$

In Figure 5.2 we show the Monte-Carlo approximation to $\mathbb{E}[u(\cdot, 1.5)]$ calculated by using a finite volume scheme with the Engquist-Osher flux (see exercise 3) and front tracking. These averages are shown by the red lines. For the finite volume method we used the parameters $\Delta x = 1/1000$ and $N = 1000$. Since $\|u_0(\cdot, \omega)\|_\infty < \infty$ and $|u_0(\cdot, \omega)|_{BV} < \infty$ for all ω, we can use $q = 2$ in the estimates. Therefore the error should be (or scale as) $1/33$. In order to get a comparable error with the front tracking approximations, we used $\delta = 1/33$ and $N = 1000$ when calculating $E_N[u_\delta]$. The gray areas in the figures is bounded by $E_N[v] + \sigma_N[v]$ and $E_N[v] - \sigma_N[v]$, where σ_N denotes the Monte-Carlo estimate for the standard deviation,

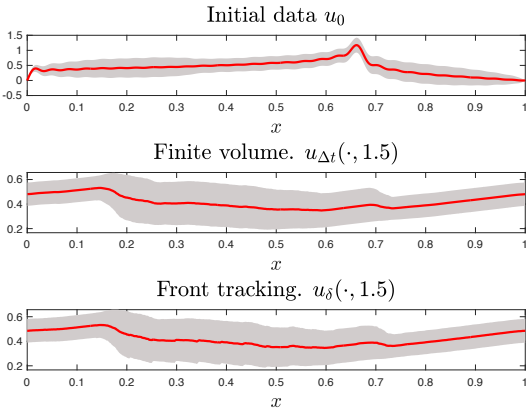

Fig. 5.2: The initial data (top), $E_{1000}[u_{\Delta t}]$ for the finite volume approximation (middle) and $E_{1000}[u_\delta]$ for the front tracking approximation (bottom).

fig:mcexample

$$\sigma_N[v] = \left(E_N\left(E_N[v] - v\right)^2\right)^{1/2},$$

where v is $u_{\Delta x}$ or u_δ. This gray area gives an indication of how uncertain the random solutions are, roughly 70% of the random solutions should lie in the gray area. □

5.2.3 The multilevel Monte-Carlo method

subsec:MLMC

We have seen that in order to make reasonably accurate assertions about the mean of random entropy solutions, one is forced to compute many approximations. The multilevel Monte-Carlo method approximates the mean differently, which in many cases reduces the work to accuracy ratio compared with the Monte-Carlo method.

As in the previous section, u_h is a monotone approximation such that the deterministic error estimate (5.12) holds Let $U^L = \{u_{h_l}\}_{l=0}^L$ be a collection of approximations with $h_0 > h_1 > h_2 > \cdots > h_L$. For simplicity we assume

that there is a fixed constant k such that $kh_l \geq h_{l-1}$. The rationale behind the multilevel Monte-Carlo method is the following observation:

$$\mathbb{E}\left[u_{h_L}\right] = \mathbb{E}\Big[\sum_{l=0}^{L}\left(u_{h_l} - u_{h_{l-1}}\right)\Big] = \sum_{l=0}^{L}\mathbb{E}\left[u_{h_l} - u_{h_{l-1}}\right],$$

where we have defined $u_{h_{-1}} = 0$. Based on this we define the multilevel Monte-Carlo estimator as

$$E^L\left[U^L\right] = \sum_{l=0}^{L} E_{N_l}\left[u_{h_l} - u_{h_{l-1}}\right]. \tag{5.19}$$

eq:mlmcdef

This gives us the freedom to choose the numbers $\{N_l\}_{l=0}^L$ and $\{h_l\}_{l=0}^L$ such that (hopefully) E^L is a more efficient estimator than E_N.

Before we consider the efficiency of E^L we must prove that E^L approximates $\mathbb{E}$. This is ensured by the following error estimate:

lem:mlmcerror

Lemma 5.25 *Let $q = \min\{2, r\}$, for any sequence $\{N_l\}_{l=0}^L$ and for any sequence $\{h_l\}_{l=0}^L$ as specified above we have that*

$$\begin{aligned}
&\left\|\mathbb{E}[u](\cdot,t) - E^L[U^L]\right\|_{L^q(\Omega;L^p(D))} \\
&\quad\leq C\Big(\|u_0\|_{L^1(\Omega;L^\infty(D))}^{1-\frac{1}{\tilde{p}}} h_L^{\frac{s}{\tilde{p}}} + \|u_0\|_{L^q(\Omega;L^\infty(D))} N_0^{\frac{1-q}{q}} \\
&\qquad + \|u_0\|_{L^q(\Omega;L^\infty(D))}^{1-\frac{1}{\tilde{p}}} h_0^{\frac{s}{\tilde{p}}} N_0^{\frac{1-q}{q}} + \|u_0\|_{L^q(\Omega;L^\infty(D))}^{1-\frac{1}{\tilde{p}}} \sum_{l=1}^{L} N_l^{\frac{1-q}{q}} h_{l-1}^{\frac{s}{\tilde{p}}}\Big),
\end{aligned} \tag{5.20}$$

eq:mlmcerror

where $\tilde{p} = \max\{p, q\}$, and the constant C depends on p, q and D. In particular,

$$E^L[U^L] \overset{L^q(\Omega;L^p(D))}{\longrightarrow} \mathbb{E}[u] \quad \text{as } N_l \to \infty \text{ and } h_l \to 0 \text{ for } l = 0, \ldots, L.$$

Proof By the triangle inequality

$$\begin{aligned}
\left\|\mathbb{E}[u] - E^L[U^L]\right\|_{L^q(\Omega;L^p(D))} &\leq \left\|\mathbb{E}[u] - \mathbb{E}[u_{h_L}]\right\|_{L^q(\Omega;L^p(D))} \\
&\quad + \left\|\mathbb{E}[u_{h_L}] - E^L[U^L]\right\|_{L^q(\Omega;L^p(D))} \\
&\leq \underbrace{\left\|\mathbb{E}[u] - \mathbb{E}[u_{h_L}]\right\|_{L^q(\Omega;L^p(D))}}_{A} \\
&\quad + \sum_{l=0}^{L} \underbrace{\left\|\mathbb{E}\left[u_{h_l} - u_{h_{l-1}}\right] - E_{N_l}\left[u_{h_l} - u_{h_{l-1}}\right]\right\|_{L^q(\Omega;L^p(D))}}_{B_l}
\end{aligned}$$

Since $\mathbb{E}[u] - \mathbb{E}[u_{h_L}]$ does not depend on ω, we can use L^1/L^∞ interpolation and the deterministic error estimate to bound A;

$$\begin{aligned} A &\leq \|u - u_{h_L}\|_{L^1(\Omega;L^p(D))} \\ &\leq \|u - u_{h_L}\|^{1-\frac{1}{p}}_{L^1(\Omega;L^\infty(D))} \|u - u_{h_L}\|^{\frac{1}{p}}_{L^1(\Omega;L^1(D))} \\ &\leq C \|u_0\|^{1-\frac{1}{p}}_{L^1(\Omega;L^\infty(D))} h_L^{\frac{s}{p}}. \end{aligned}$$

Next, we bound B_l for $l > 0$. We use the notation $\Delta_l = u_{h_l} - u_{h_{l-1}}$. First assume that $p \geq q$. By the triangle inequality and the fact that $\|u_h\|_{L^p(D)} \leq C \|u_h\|_{L^\infty(D)}$, we get

$$\|u_h\|_{L^r(\Omega;L^p(D))} \leq C \|u_h\|_{L^r(\Omega;L^\infty(D))} \leq C \|u_0\|_{L^r(\Omega;L^\infty(D))},$$

which implies that $\Delta_l \in L^r(\Omega; L^p(D))$. Therefore we can use Lemma 5.20 to get

$$\|\mathbb{E}[\Delta_l] - E_{N_l}[\Delta_l]\|_{L^q(\Omega;L^p(D))} \leq C N_l^{\frac{1-q}{q}} \|\Delta_l\|_{L^q(\Omega;L^p(D))}.$$

If $p < q$ then we can apply Hölder's inequality and get

$$\|\mathbb{E}[\Delta_l] - E_{N_l}[\Delta_l]\|_{L^q(\Omega;L^p(D))} \leq C \|\mathbb{E}[\Delta_l] - E_{N_l}[\Delta_l]\|_{L^q(\Omega;L^q(D))}.$$

We can therefore use the same arguments as in the proof of Lemma 5.20 to conclude that $\Delta_l \in L^q(\Omega; L^q(D))$. Then we invoke Lemma 5.20 to deduce that

$$\|\mathbb{E}[\Delta_l] - E_{N_l}[\Delta_l]\|_{L^q(\Omega;L^p(D))} \leq C N_l^{\frac{1-q}{q}} \|\Delta_l\|_{L^q(\Omega;L^q(D))},$$

for $p < q$. Therefore, for any p,

$$\|\mathbb{E}[\Delta_l] - E_{N_l}[\Delta_l]\|_{L^q(\Omega;L^p(D))} \leq C N_l^{\frac{1-q}{q}} \|\Delta_l\|_{L^q(\Omega;L^{\tilde{p}}(D))},$$

where $\tilde{p} = \max\{p, q\}$. Next we again use L^1/L^∞ interpolation, the deterministic error estimate for u_h and the triangle inequality to deduce

$$\begin{aligned} \|\Delta_l\|_{L^q(\Omega;L^{\tilde{p}}(D))} &\leq C \|\Delta_l\|^{\frac{1}{\tilde{p}}}_{L^q(\Omega;L^1(D))} \|\Delta_l\|^{1-\frac{1}{\tilde{p}}}_{L^q(\Omega;L^\infty(D))} \\ &\leq C \left(h_l^{\frac{s}{\tilde{p}}} + h_{l-1}^{\frac{s}{\tilde{p}}}\right) \|u_0\|^{1-\frac{1}{\tilde{p}}}_{L^q(\Omega;L^\infty(D))} \\ &\leq C h_{l-1}^{\frac{s}{\tilde{p}}} \|u_0\|^{1-\frac{1}{\tilde{p}}}_{L^q(\Omega;L^\infty(D))}. \end{aligned}$$

This means that

$$B_l \leq C N_l^{\frac{1-q}{q}} h_{l-1}^{\frac{s}{\tilde{p}}} \|u_0\|^{1-\frac{1}{\tilde{p}}}_{L^q(\Omega;L^\infty(D))},$$

for $l > 0$.

For $l = 0$ we use the Monte-Carlo error estimate (Theorem 5.23), the triangle inequality, and the deterministic error estimate (5.12)

$$\begin{aligned} B_0 &\leq C N_0^{\frac{1-q}{q}} \|u_{h_0}\|_{L^q(\Omega;L^p(D))} \\ &\leq C N_0^{\frac{1-q}{q}} \left(\|u - u_{h_0}\|_{L^q(\Omega;L^p(D))} + \|u\|_{L^q(\Omega;L^p(D))} \right) \\ &\leq C N_0^{\frac{1-q}{q}} \left(h_0^{\frac{s}{p}} \|u_0\|^{1-\frac{1}{p}}_{L^q(\Omega;L^\infty(D))} + \|u_0\|_{L^q(\Omega;L^\infty(D))} \right). \end{aligned}$$

To complete the proof we collect the bounds on A, B_l for $l > 0$ and B_0.

Notes

sec:UQ.notes

Exercises

5.1