Chapter 5
Uncertainty Quantification for conservation laws

It is not certain that everything is uncertain.
Blaise Pascal, 1857 (find better!)

Whenever a (partial) differential equation arises as a model of something outside mathematics, it is dependent on certain external components. Typical examples are boundary or initial values and coefficients in the PDE, such as heat conductivity, equation of state, the speed of light in a vacuum, Boltzmann’s constant etc. Very often, these quantities are obtained from measurements, and cannot be known exactly. In other words, they are uncertain, but whose uncertainty can be quantified. In this book, uncertainty quantification is the attempt to determine how this uncertainty influences and manifests itself in an uncertainty in solutions of the (partial) differential equation.

If we want to study uncertainty in solutions to hyperbolic conservation laws we must resort to studying uncertainty in numerical approximations. If we have error estimates we can try to glean information on how uncertain the exact solutions are by computing a lot of “typical” numerical solutions and various statistical estimates on these. Ideally, we also would like to have statistical bounds on how accurate these statistical estimates are.

To model uncertainty mathematically, it is common to use so-called random variables

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“Random” means unpredictable, with no specific pattern.
Chat GPT 3.5

To define a random variable we start by having a set \( \Omega \) of all possible outcomes (of an experiment), and a \( \sigma \)-algebra \( \mathcal{B} \) of subsets of \( \Omega \). Members of
are then assigned a probability, this means that we have a measure \( \mathbb{P} \) on the measure space \((\Omega, \mathcal{B})\) such that \( \mathbb{P}(D) \geq 0 \) for all \( D \in \mathcal{B} \) and \( \mathbb{P}(\Omega) = 1 \). The measure \( \mathbb{P} \) is called a probability measure.

\( \diamond \) **Example**

The simplest and perhaps most familiar situation where one can model randomness using these concepts is a single toss of a coin. Let “H” and “T” denote the outcome that the coin comes up heads or tails respectively. This means that

\[
\Omega = \{H, T\}, \quad \mathcal{B} = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}.
\]

Given two nonnegative numbers \( p_H \) and \( p_T \) such that \( p_H + p_T = 1 \), we can define a probability measure \( \mu \) by

\[
\mathbb{P}(D) = \begin{cases} 
p_H & D = \{H\}, \\
p_T & D = \{T\}.
\end{cases}
\]

If we want to model \( N \) tosses of the same coin, we can let \( \Omega \) consist of all sequences of \( N \) letters where each letter is “H” or “T”. The \( \sigma \)-algebra \( \mathcal{B} \) will then consist of all subsets of \( \Omega \), and the probability of an elementary even is

\[
\mathbb{P}(l_1 l_2 \ldots l_N) = p_H^{(# \text{H})} p_T^{(# \text{T})},
\]

where “# H” denotes the number of “H”s among the \( l_i \)s. \( \diamond \) \( \square \)

However, we very rarely need the concrete description of \( \Omega \), it is simpler to view \( \Omega \) as an abstract set containing the elementary outcomes of an experiment.

A real valued *random variable* is defined as a measurable function \( X : \Omega \to \mathbb{R} \), where \( \mathbb{R} \) is equipped with the Lebesgue measure, so that the set \( X^{-1}(A) \in \mathcal{B} \) for all Lebesgue measurable \( A \subset \mathbb{R} \). We then define the *expectation* (also called *mean*) of \( X \) as the integral of \( X \) over \( \Omega \) with respect to \( \mathbb{P} \), i.e.,

\[
\mathbb{E}[X] = \int_{\Omega} X(\omega) \, d\mathbb{P}(\omega).
\]

Once we have a notion of an integral, we can define the spaces \( L^p(\Omega; \mathbb{R}) \) in the usual manner, and Cauchy-Schwartz’s, Hölder’s and Minkowski’s inequalities hold. In particular Jenssen’s inequality holds and in this formulation reads

\[
\Theta(\mathbb{E}[X]) \leq \mathbb{E}[\Theta(X)],
\]

for any convex function \( \Theta \). Let \( p \geq q \) and use Jenssen’s inequality for the convex function \( \Theta(u) = |u|^{p/q} \) to find that

\[
\|X\|_{L^p(\Omega; \mathbb{R})}^p = \mathbb{E}[\Theta(|X|^q)] \geq \Theta(\mathbb{E}[|X|^q]) = \|X\|_{L^q(\Omega; \mathbb{R})}^p.
\]
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Thus for \( p \geq q \), \( L^p(\Omega; \mathbb{R}) \subset L^q(\Omega; \mathbb{R}) \). If a random variable is in \( L^2(\Omega; \mathbb{R}) \) we define its variance as

\[
\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2.
\]

For any random variable \( X \) we can define a probability measure \( \mu_X \) on \( \mathbb{R} \) by the relation

\[
\mu_X(A) = \mathbb{P}(X^{-1}(A)).
\]

the measure \( \mu_X \) is called the law of \( X \), and intuitively \( \mu_X(A) \) is the probability that \( X(\omega) \) takes values in \( A \). We shall use the notation \( \mathbb{P}(X \in A) = \mathbb{P}(X^{-1}(A)) = \mu_X(A) \). We can also integrate (in \( \mathbb{R} \)) with respect to \( \mu_X \) using the definition

\[
\int_{\mathbb{R}} f(x) d\mu_X(x) = \int_{\Omega} f(X(\omega)) d\mathbb{P}(\omega).
\]

A real valued random variable is called Gaussian if its probability density \( \mu_X \) is

\[
d\mu_X(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx,
\]

for a positive constant \( \sigma \), and it is called standard if \( \sigma = 1 \). We define the Fourier transform of a real random variable by the formula

\[
\hat{X}(\xi) := \mathbb{E}[e^{-i\xi X}] = \int_{\mathbb{R}} e^{-i\xi x} d\mu_X(x),
\]

where \( i = \sqrt{-1} \). A random variable \( X \) is standard Gaussian if and only if \( \hat{X} \) is given by

\[
\hat{X}(\xi) = e^{-\xi^2/2}.
\]

The following inequalities give bounds on the probability that random variables are “large”.

**Lemma 5.2 (Chebyshev’s inequalities)** Let \( X \) be a random variable.

- If \( \mathbb{E}[X] < \infty \), then for any \( t > 0 \)

  \[
  \mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}.
  \]

- If \( \text{Var}[X] < \infty \), then for any \( t > 0 \)

  \[
  \mathbb{P}(|X - \mathbb{E}[X]| \geq t) \leq \frac{\text{Var}[X]}{t^2}.
  \]

**Proof** Observe that for \( \omega \in X^{-1}([t, \infty)) \)

\[
1 = \chi_{X^{-1}([t, \infty))}(\omega) \leq \frac{X(\omega)}{t}
\]
where $\chi_A$ denotes the indicator function of the set $A$. Hence

$$\text{Prob}(X \geq t) = \mathbb{E}\left[\chi_{[t,\infty)}(X)\right] \leq \mathbb{E}\left[\frac{X}{t}\right] = \frac{\mathbb{E}[X]}{t}.$$  

This proves the first inequality. To prove the second, observe that

$$\text{Prob}(|X - \mathbb{E}[X]| \geq t) = \text{Prob}((|X - \mathbb{E}[X]|^2 \geq t^2),$$

and apply the first inequality to the random variable $|X - \mathbb{E}[X]|^2$.

If we have a collection $X_1, \ldots, X_N$ of random variables defined on $(\Omega, \mathcal{B}, \mathbb{P})$ we say that these are independent if

$$\text{Prob}(X_1 \in A_1, X_2 \in A_2, \ldots, X_N \in A_N) = \prod_{i=1}^{N} \text{Prob}(X_i \in A_i) \quad \text{(5.1)}$$

for all measurable sets $A_i \subset \mathbb{R}$. It follows immediately that if $f_1, \ldots, f_N$ are real measurable functions and $X_1, \ldots, X_N$ are independent, then so are the random variables $f_1(X_1), \ldots, f_N(X_N)$. Conversely, if $f_1(X_1), \ldots, f_N(X_N)$ are independent for any collection of measurable functions $f_1, \ldots, f_N$, then $X_1, \ldots, X_N$ are independent.

Regarding the expectation of products of independent random variables we have the very useful lemma:

**Lemma 5.3** Let $X$ and $Y$ be random variables defined on the probability space $(\Omega, \mathcal{B}, \mathbb{P})$, then $X$ and $Y$ are independent if and only if

$$\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[g(Y)],$$

for all real measurable functions $f$ and $g$.

**Proof** Assume that $X$ and $Y$ are independent. We prove the equality for simple functions $X$ and $Y$, the general result then follows since simple functions are dense. If $X$ and $Y$ are simple we have that

$$X(\omega) = \sum_i x_i \chi_{A_i}(\omega) \quad \text{and} \quad Y(\omega) = \sum_j y_j \chi_{B_j}(\omega),$$

where $A_i$ and $B_j$ are in $\mathcal{B}$. Then

$$\mathbb{E}[XY] = \int_{\Omega} X(\omega)Y(\omega) d\mathbb{P}(\omega)$$

$$= \sum_{i,j} x_i y_j \int_{\Omega} \chi_{A_i}(\omega) \chi_{B_j}(\omega) d\mathbb{P}(\omega)$$

$$= \sum_{i,j} x_i y_j \text{Prob}(X \in A_i, Y \in B_j)$$
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\[
= \sum_{i,j} x_i y_j \Prob(X \in A_i) \Prob(Y \in B_j)
\]
\[
= \left( \sum_i x_i \int_{\Omega} \chi_{A_i}(\omega) \, d\Prob(\omega) \right) \left( \sum_j y_j \int_{\Omega} \chi_{B_j}(\omega) \, d\Prob(\omega) \right)
\]
\[
= \E[X] \E[Y].
\]

Assume now that the equality holds and choose \( f = \chi_A \) and \( g = \chi_B \) for measurable sets \( A \) and \( B \) in \( \mathbb{R} \). Then

\[
\Prob(X \in A, Y \in B) = \int_{\Omega} f(X(\omega)) g(Y(\omega)) \, d\Prob(\omega) = \E[f(X)g(Y)]
\]
\[
= \E[f(X)] \E[g(Y)] = \int_{\Omega} f(X(\omega)) \, d\Prob(\omega) \int_{\Omega} g(Y(\omega)) \, d\Prob(\omega)
\]
\[
= \Prob(X \in A) \Prob(Y \in B).
\]

If we have infinitely many random variables \( \{X_i\}_{i=1}^{\infty} \) we say that these are independent if \( \{X_k\}_{k=1}^{N} \) are independent for all finite \( N \) and for all sequences \( j_1 < j_2 < \cdots < j_N \).

5.1.1 The Monte-Carlo method

Regression to the mean has an explanation but does not have a cause.

*Daniel Kahneman, Thinking, fast and slow*

In our experience, especially outside academia, we rarely encounter expectations (though others seem to have them of us), but we see averages all the time. As an example, one often reads statements like “the average sugar consumption was 26 kilograms per Norwegian in 2020”\(^1\) When encountering a Norwegian, are we wrong to assume that this person ate 26 kilograms of sugar in 2020? In other words, can we replace the (mathematical) mean with the average? And if we do so, what error do we make?

To answer this we must first define the average. To this end, let \( X \) be a random variable on the probability space \((\Omega, \mathcal{B}, \Prob)\) and define the average as a random variable on the product probability space \((\Omega, \mathcal{B}, \Prob)^{\mathbb{N}}\)

\[
E_N(\omega_1, \ldots, \omega_N) = \frac{1}{N} \sum_{i=1}^{N} X(\omega_i).
\]

The random variable \( E_N \) is called the *average* of \( X \). To determine the relation between the mean and the average we define random variables \( \tilde{X}_i \) on the

\(^1\) Or even “the average Norwegian consumed 26 kg of sugar in 2020”!
Observe that using this notation

\[ E_N = \frac{1}{N} \left( \tilde{X}_1 + \cdots + \tilde{X}_N \right). \]

We claim that for \( i \neq j \), \( \tilde{X}_i \) and \( \tilde{X}_j \) are independent. To see this, first calculate

\[ \mathbb{E}[\tilde{X}_i] = \int_{\Omega^N} \tilde{X}_i(\omega_1, \ldots, \omega_N) \, d\mathbb{P}(\omega_1) \cdots d\mathbb{P}(\omega_N) = \int_{\Omega^N} X(\omega_i) \, d\mathbb{P}(\omega_1) \cdots d\mathbb{P}(\omega_N) = \mathbb{E}[X], \]

and then

\[ \mathbb{E}[\tilde{X}_i \tilde{X}_j] = \int_{\Omega^N} \tilde{X}_i(\omega_1, \ldots, \omega_N) \tilde{X}_j(\omega_1, \ldots, \omega_N) \, d\mathbb{P}(\omega_1) \cdots d\mathbb{P}(\omega_N) = \int_{\Omega^N} X(\omega_i)X(\omega_j) \, d\mathbb{P}(\omega_1) \cdots d\mathbb{P}(\omega_N) = \mathbb{E}[X]^2 = \mathbb{E}[\tilde{X}_i] \mathbb{E}[\tilde{X}_j]. \]

For \( i = j \), a similar calculation reveals that \( \mathbb{E}[\tilde{X}_i^2] = \mathbb{E}[X^2] \). Incidentally, the above calculations imply that \( \mathbb{E}[E_N] = \mathbb{E}[X] \), so we can expect that the average equals the expectation\(^2\). However, we can quantify the discrepancy further by calculating \( \|E_N - \mathbb{E}[X]\|_{L^2(\Omega^N)} \),

\[ \mathbb{E} \left[ (E_N - \mathbb{E}[X])^2 \right] = \mathbb{E} \left[ \frac{1}{N^2} \left( \sum_{i=1}^{N} \left( \tilde{X}_i - \mathbb{E}[X] \right) \right)^2 \right] 
= \frac{1}{N^2} \mathbb{E} \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} \left( \tilde{X}_i - \mathbb{E}[X] \right) \left( \tilde{X}_j - \mathbb{E}[X] \right) \right] 
= \frac{1}{N^2} \mathbb{E} \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} \left( \tilde{X}_i \tilde{X}_j - \mathbb{E}[X] \left( \tilde{X}_i + \tilde{X}_j \right) + \mathbb{E}[X]^2 \right) \right] 
= \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left( \mathbb{E}[\tilde{X}_i \tilde{X}_j] - 2\mathbb{E}[X]^2 + \mathbb{E}[X]^2 \right) 
= \frac{1}{N^2} \left( N\mathbb{E}[X^2] + N(N-1)\mathbb{E}[X]^2 - N^2\mathbb{E}[X]^2 \right) 
= \frac{1}{N} \left( \mathbb{E}[X^2] - \mathbb{E}[X]^2 \right) = \frac{\text{Var}[X]}{N}. \]  
\[ (5.2) \quad \text{eq:mcerrorest} \]

\(^2\) Whatever this means.
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If \( \text{Var}[X] < \infty \) we see that \( E_N \) converges to \( E[X] \) in some sense.

In order to facilitate the formulation of convergence of random variables, it is convenient to have \( X \) and \( E_N \) defined on the same probability space. To this end, let

\[
\omega = (\omega_0, \omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots) \in \Omega \times \Omega \times \cdots \times \Omega \times \cdots =: \Omega^\infty,
\]

and define \( B^\infty = B \otimes B \otimes \cdots \otimes B \otimes \cdots \) and \( \mathbb{P}^\infty = \mathbb{P} \otimes \mathbb{P} \otimes \cdots \otimes \mathbb{P} \otimes \cdots \). Then we define \( X^\infty \) and \( E_N^\infty \) as random variables on the probability space \((\Omega^\infty, B^\infty, \mathbb{P}^\infty)\) by

\[
X^\infty(\omega) = X(\omega_0) \quad \text{and} \quad E_N^\infty(\omega) = E_N(\omega_1, \ldots, \omega_N).
\]

To confuse matters we shall drop the superscript \( \infty \), this saves us some typing, but also makes the expectations appearing in the calculations above the same. So with this new notation (with the invisible \( \infty \)), \( \{E_N\}_{N=1}^\infty \) is a sequence of random variables in the probability space \((\Omega, B, \mathbb{P})\).

Let \( \{X_N\}_{N=1}^\infty \) be a sequence of real random variables defined on a probability space \((\Omega, B, \mathbb{P})\). This sequence can converge to a random variable \( X \) in different ways. We say that \( X_N \) converges to \( X \) in \( L^p \) if

\[
\lim_{N \to \infty} E[|X_N - X|^p] = 0.
\]

We say that \( X_N \) converges to \( X \) \textit{almost surely} if there is a set \( \mathcal{N} \subset \Omega \) with \( \mathbb{P}(\mathcal{N}) = 0 \), such that for \( \omega \notin \mathcal{N} \),

\[
X_N(\omega) \to X(\omega).
\]

We say that \( X_N \) converges to \( X \) \textit{in probability} if for any \( \delta > 0 \),

\[
\lim_{N \to \infty} \text{Prob}(|X_N - X| > \delta) = 0.
\]

We say that \( X_N \) converges to \( X \) \textit{in law} if \( \mu_{X_N} \) converges weakly to \( \mu_X \), meaning that

\[
\lim_{N \to \infty} \int_{\mathbb{R}} f(x) \, d\mu_{X_N}(x) = \int_{\mathbb{R}} f(x) \, d\mu_X(x),
\]

for all bounded continuous functions \( f \). We write these types of convergence as

\[
X_N \xrightarrow{L^p} X, \quad X_N \xrightarrow{a.s.} X, \quad X_N \xrightarrow{p} X \quad \text{and} \quad X_N \xrightarrow{\mu} X
\]

respectively. These modes of convergence are not independent, and we have

\[
\underbrace{\xrightarrow{L^p}}_{a.s.} \Rightarrow \xrightarrow{p} \Rightarrow \xrightarrow{\mu}.
\]
Furthermore, if \( X_N \overset{P}{\to} X \), then there exists a subsequence \( \{N_k\}_{k=1}^{\infty} \) such that \( X_{N_k} \xrightarrow{a.s.} X \).

Applying this to \( E_N \), we see that \( E_N \overset{L^2}{\to} \mathbb{E}[X] \), and that there is a subsequence such that \( E_{N_k} \xrightarrow{a.s.} \mathbb{E}[X] \). In practice, one has to stop at some finite \( N \), one would also like to be fairly sure that the estimate \( E_N \) is close to \( \mathbb{E}[X] \). So, given \( \varepsilon > 0 \) and \( \delta > 0 \), find \( N \) such that

\[
\text{Prob} \left( |E_N - \mathbb{E}[X]| > \delta \right) < \varepsilon.
\]

We can use Chebyshev’s second inequality for the random variable \( Y = E_N - \mathbb{E}[X] \), we have \( \mathbb{E}[Y] = 0 \) and therefore

\[
\text{Prob} \left( |Y| > \delta \right) \leq \frac{\text{Var}[Y]}{\delta^2} = \frac{\mathbb{E}[|E_N - \mathbb{E}[X]|^2]}{\delta^2} \leq \frac{\text{Var}[X]}{\delta^2 N}.
\]

To ensure that this is less than \( \varepsilon \) we can choose \( N > \text{Var}[X]/(\varepsilon \delta^2) \). As an example, to be 99% sure that the error is less than 1% \( (\delta = \varepsilon = 10^{-2}) \), we can choose \( N > 10^6 \text{Var}[X] \).

\[\text{Example}\]

The Monte-Carlo method is often used to numerically calculate integrals over high dimensional domains. We shall use it to estimate the volume of the unit ball \( B_d \) in \( \mathbb{R}^d \). This volume is then written as an integral

\[V_d = 2^d \int_{[0,1]^d} \chi_{B_d}(x) \, dx\]

To recast this as an expectation, let \( \Omega = [0,1]^d \), and let \( \mathbb{P} \) be the \( d \)-dimensional Lebesgue measure and \( \mathcal{B} \) the corresponding \( \sigma \)-algebra. Set \( f(x) = \chi_{B_d}(x) \). If \( X \) is a random variable in this probability space, then \( 2^d \mathbb{E}[f(X)] = V_d \). Note that we can overestimate \( \text{Var}[f(X)] \) by \( (\max_x f(x) - \min_x f(x))^2 = 1 \). We want to approximate \( V_d \) using \( E_N \). If we are to be 99% sure that the error is less than 1% for any \( d \), we need to take \( N > 2^d \times 10^6 \). For large \( d \) this is impractical, so we relax our demands; we want to be 90% sure that the error is less than 10%. This is then ensured if \( N > 1000 \times 2^d \). There is also a recursive formula for \( V_d \),

\[V_d = \begin{cases} 
1 & d = 0, \\
2 & d = 1, \\
\frac{2}{d} V_{d-2} & d > 1,
\end{cases}
\]

so we can compare \( E_N \) with the exact integral. In Figure 5.1 we plotted the relative errors; \( 100 \times |2^d E_N - V_d|/V_d, \) for \( d = 1, \ldots, 17 \). Most of the computed errors are well under the 10% mark, but for \( d = 17 \) the error is almost 16%. This is not surprising, since we can only bound the probability
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![Fig. 5.1: The relative error when using the Monte-Carlo method to compute the volume of the unit ball in \( d \) dimensions.]

that the error is large. The Monte-Carlo estimator \( E_N \) is a random variable, and in applications we must accept whatever value it gives us.

A possible source of errors is that the “random” numbers given to us by a computer\(^3\) are not truly random. In order not to stray too far from the topic of this book, we shall henceforth blissfully ignore this.

\[ \diamond \]

5.1.2 Random variables in Banach spaces

In order to study how uncertainty manifests itself in solutions of partial differential equations, in particular in weak solutions of conservation laws, we need to work with random variables taking values in Banach spaces.

As before let \((\Omega, \mathcal{B}, \mathbb{P})\) be a probability space, and let \( X \) be a function \( X : \Omega \rightarrow B \), where \( B \) is a Banach space. We first need to define \( X \). The generic method of defining the integral is via simple functions. We say that \( X : \Omega \rightarrow B \) is simple if \( X(\omega) \) can be written as

\[ X(\omega) = \sum_{i=1}^{N} x_i \mathbb{1}_{A_i}(\omega), \]

where \( A_i \in \mathcal{B} \) and \( x_i \in B \) for \( i = 1, \ldots, N \). For a simple function we define the integral

\[ \int_{\Omega} X(\omega) \, d\mathbb{P}(\omega) = \sum_{i=1}^{N} x_i \mathbb{P}(A_i). \]

Note that for simple functions the triangle inequality holds,

\[ \left\| \int_{\Omega} X \, d\mathbb{P} \right\|_B \leq \int_{\Omega} \| X \|_B \, d\mathbb{P}, \]

\(^3\) Except perhaps by a quantum computer.
where \( \| \cdot \|_B \) denotes the norm in \( B \).

For Banach space valued functions there are several types of measurability.

A function \( X \) is said to be **strongly measurable** if there exists a sequence of simple functions \( \{ X_j \} \) such that for all \( \omega \in \Omega \)

\[
\lim_{j \to \infty} X_j(\omega) = X(\omega),
\]

i.e., \( X_j \to X \) pointwise.

The norm in \( B \) induces a topology on \( B \). Let \( \mathcal{L}(B) \) denote the smallest \( \sigma \)-algebra of subsets of \( B \) containing all the open sets in this topology. A function \( X \) is called **Borel measurable** if

\[
X^{-1}(D) \in \mathcal{B} \quad \text{for all} \quad D \in \mathcal{L}(B).
\]

A function \( X : \Omega \to B \) is **weakly measurable** if the real valued random variable \( \omega \mapsto \xi(X(\omega)) \) is measurable for all \( \xi \in B^* \) (the dual of \( B \)).

Recall that by the Hahn-Banach extension theorem

\[
\| x \|_B = \sup_{\| \xi \|_{B^*} \leq 1} | \langle x, \xi \rangle |,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the pairing between \( B \) and \( B^* \). A linear subspace \( F \) of \( B^* \) is called **norming** if we have

\[
\| x \|_B = \sup_{\| \xi \|_{B^*} \leq 1} | \langle x, \xi \rangle |,
\]

for all \( \xi \in B^* \).

In order to establish whether a function \( X : \Omega \to B \) is strongly measurable we need the next lemma.

**Lemma 5.5** If \( B_0 \) is a separable\(^4 \) linear subspace of \( B \) and \( F \) is a norming linear subspace of \( B^* \), then \( F \) contains a sequence of unit vectors which is norming for \( B_0 \).

**Proof** Choose a dense sequence \( \{ x_i \}_{i=1}^{\infty} \) in \( B_0 \), and choose a sequence of unit vectors \( \{ \xi_i \}_{i=1}^{\infty} \) in \( F \) such that

\[
| \langle x_i, \xi_i \rangle | \geq (1 - \varepsilon_i) \| x_i \|_B ,
\]

where the numbers \( \varepsilon_i \) are positive and \( 1 > \varepsilon_i \to 0 \) as \( i \to \infty \). We claim that \( \{ \xi_i \} \) is norming for \( B_0 \). To verify this choose \( x \in B_0 \), a number \( \delta > 0 \) and an integer \( k \) such that \( \| x - x_k \|_B \leq \delta \) and \( 0 < \varepsilon_k \leq \delta \). Now we have

\[
(1 - \delta) \| x \|_B \leq (1 - \varepsilon_k) \| x \|_B \leq (1 - \varepsilon_k) \| x_k \|_B + (1 - \varepsilon_k) \| x - x_k \|_B \\
\leq | \langle x_k, \xi_k \rangle | + (1 - \varepsilon_k) \delta
\]

\(^4\) \( B_0 \) contains a dense countable subset.
\[ \begin{align*} &\leq |\langle x, \xi_k \rangle| + |\langle x_k - x, \xi_k \rangle| + \delta \\ &\leq |\langle x, \xi_k \rangle| + 2\delta. \end{align*} \]

Since we can choose \( \delta \) arbitrary small, we have that
\[ \|x\|_B \leq \sup_i |\langle x, \xi_k \rangle|, \]
and the sequence \( \{\xi_i\} \) is norming.

We call a function \( X : \Omega \to B \) separately valued if it takes its values in a closed separable subspace \( B_0 \subset B \).

**Theorem 5.6 (Petti’s measurability theorem, I)** Let \( F \) be a norming subspace of \( B^* \). For a function \( X : \Omega \to B \) the following are equivalent:

1. \( X \) is strongly measurable;
2. \( X \) is separately valued and \( \langle X, \xi \rangle \) is measurable for all \( \xi \in B^* \);
3. \( X \) is separately valued and \( \langle X, \xi \rangle \) is measurable for all \( \xi \in F \).

**Proof** Let \( \{X_i\} \) be a sequence of simple functions converging pointwise to \( X \), and let \( B_0 \) be the linear subspace of \( B \) spanned by this sequence. Then \( B_0 \) is separable and \( X \) takes its values in \( B_0 \). Furthermore the function \( f(\omega) = \langle X(\omega), \xi \rangle \) is \( \mathbb{P} \)-measurable for all \( \xi \in B^* \) since it is the pointwise limit of the simple functions \( f_i(\omega) = \langle X(\omega), \xi_i \rangle \). This shows that 1 \( \Rightarrow \) 2. It is obvious that 2 \( \Rightarrow \) 3, and it remains to show that 3 \( \Rightarrow \) 1.

We shall construct a sequence of simple functions approximating \( X \) and to this end we use Lemma 5.5 and choose a sequence \( \{\xi_i\} \) of unit vectors in \( F \) that is norming for a separable subspace \( B_0 \). We also have that \( X \) takes its values in \( B_0 \). Since for each \( \xi \in F \) the map \( \omega \mapsto \langle X, \xi \rangle \) is measurable, also the map
\[ \omega \mapsto \langle X(\omega) - x, \xi_i \rangle \]
is \( \mathbb{P} \)-measurable for all \( x \in B_0 \). Let \( \{x_i\}_{i=1}^\infty \) be a dense sequence in \( B_0 \). For each \( y \in B_0 \) define \( k = k(i,y) \) to be the smallest integer such that
\[ \|y - x_k\|_B = \min_{1 \leq j \leq i} \|y - x_j\|_B \]
and define \( z_i(y) = x_{k(i,y)} \).

Since \( \{x_i\} \) is dense, \( \|z_i(y) - y\|_B \to 0 \) as \( i \to \infty \), in other words, \( \{z_i(y)\} \) is a subsequence of \( \{x_i\} \) which approximates \( y \). Now define
\[ X_i(\omega) = z_i(\langle X(\omega) \rangle). \]

It is clear that \( X_i \) approximates \( X \), so we must show that \( X_i \) is simple. It is also clear that \( X_i \) takes values in the finite set \( \{x_1, \ldots, x_i\} \) and it remains to show that for \( 1 \leq j \leq i \) the set \( \{\omega \mid X_i(\omega) = x_k\} \) is \( \mathbb{P} \)-measurable. We have that \( X_i = x_k \) if the distance from \( x_k \) to \( X \) is smaller or equal to \( \min_{1 \leq j \leq i} \|X - x_j\|_B \), and there is no \( l \) smaller than \( k \) such that \( \|X - x_l\|_B \leq \|X - x_k\|_B \). Thus
\[ \{ \omega \mid X_i(\omega) = x_k \} = \left\{ \omega \mid \|X(\omega) - x_j\|_B = \min_{1 \leq j \leq i} \|X(\omega) - x_j\|_B \right\} \]

Since the real functions \( kX(\omega) = \min_{1 \leq j \leq i} \|X(\omega) - x_j\|_B \) are \( P \)-measurable, \( \{ \omega \mid X_i(\omega) = x_k \} \in B \) and hence \( 3 \Rightarrow 1 \).

**Corollary 5.7** The pointwise limit of a sequence of strongly measurable functions is strongly measurable.

**Proof** Each strongly measurable function \( X_i \) takes values in a separable subspace \( B_{0,i} \) of \( B \). This means that \( X \) takes its values in the closure of the linear span of these spaces, which is separable. Now for all \( \xi \in B^* \) the real functions \( f(\omega) = \langle X(\omega), \xi \rangle \) are measurable since each of is the pointwise limit of the measurable functions \( f_i(\omega) = \langle X(\omega), \xi \rangle \).

**Corollary 5.8** Let \( X : \Omega \to B \) be a strongly measurable function, \( G \) a Banach space and \( f : B \to G \) a continuous function. Then the composite mapping \( \omega \mapsto f(X(\omega)) \) is strongly measurable.

**Proof** Choose a sequence of simple functions \( X_i \) converging to \( X \) pointwise, then \( \{ f(X_i) \} \) is a sequence of simple functions converging to \( f(X) \) which is strongly measurable by Corollary 5.7.

**Proposition 5.9** For a function \( X : \Omega \to B \) the following is equivalent:

1. \( X \) is strongly measurable;
2. \( X \) is separately valued and for all \( D \in \mathcal{L}(B) \) we have \( X^{-1}(D) \in B \).

**Proof** Assume that \( X \) is strongly measurable, and thus separately valued. To prove that 2 holds, it suffices to prove that \( X^{-1}(D) \in B \) for all open sets \( D \). Assume that \( D \) is open and let \( \{ X_i \} \) be a sequence of simple functions converging to \( X \). For \( x \in D \), let \( D_r = \{ x \in D \mid \text{dist}(x, D^C) > r \} \). Since \( X_i \) is simple \( X_i^{-1}(D_r) \in B \), in particular

\[ X_i^{-1}(D_{1/j}) \in B \quad \text{for all } i \text{ and } j > 0. \]

After some pondering, we realized that

\[ X^{-1}(D) = \bigcup_{j \geq 1} \bigcup_{i \geq 1} X_i^{-1}(D_{1/j}). \]

That \( 2 \Rightarrow 1 \) follows from Petti's measurability theorem 5.6.

Now we can extend the definition of the integral to strongly measurable functions with values in \( B \). A function is called **Bochner integrable** if there exists a sequence of simple functions \( \{ X_i \}_{i=1}^\infty \) such that
5.1 A quick introduction to Random variables and the Monte-Carlo method

- \( \lim_{i \to \infty} X_i(\omega) = X(\omega) \) for all \( \omega \notin \mathcal{N} \), where \( \mathbb{P}(\mathcal{N}) = 0 \);
- \( \lim_{i \to \infty} \int_{\Omega} \|X - X_i\|_B \, d\mathbb{P} = 0 \).

Observe that this makes sense since \( \|X_i - X\|_B \) is measurable by Corollary 5.8. It is also straightforward to show that the integral does not depend on the approximating sequence and that the triangle inequality holds for the Bochner integral. Furthermore we have the identity

\[
\left\langle \int_{\Omega} X \, d\mathbb{P}, \xi \right\rangle = \int_{\Omega} \langle X, \xi \rangle \, d\mathbb{P},
\]

for all \( \xi \in B^* \).

**Proposition 5.10** A strongly measurable function \( X : \Omega \to B \) is Bochner integrable if and only if

\[
\int_{\Omega} \|X\|_B \, d\mathbb{P} < \infty.
\]

We remark that a “dominated convergence theorem” holds: If \( \{X_i\} \) is a sequence of Bochner integrable functions such that \( X_i \to X \) almost everywhere and \( Y \) is another Bochner integrable function with \( \|X_i\|_B \leq \|Y\|_B \) for all \( i \), then \( X \) is Bochner integrable.

Equipped with this integral, we can define the expectation of the Banach space valued random variable \( X \),

\[
\mathbb{E}[X] = \int_{\Omega} X \, d\mathbb{P}.
\]

As with real random variables, Banach space valued random variables induces a measure \( \mu_X \) on \( B \) by the formula

\[
\mu_X(D) = \mathbb{P}(X^{-1}(D))
\]

for \( D \in \mathcal{L}(B) \). The measure \( \mu_X \) is called the distribution of \( X \). If two random variables have the same distribution they are said to be identically distributed. With this definition two identically distributed random variables can be defined on different probability spaces. A random variable is called symmetric if \( X \) and \( -X \) are identically distributed.

Independence of random variables is defined as in the real case, cf. 5.1. The different notions of convergence of sequences of random variables are also defined as for real random variables, see the definitions above (5.3), with \( |\cdot| \) replaced by \( \|\cdot\|_B \).

**Proposition 5.11** Let \( X_i : \Omega_i \to B_i \) be random variables, then \( X_1, \ldots, X_N \) are independent if and only if

\[
\mu_{(X_1, \ldots, X_N)} = \mu_{X_1} \times \cdots \times \mu_{X_N}.
\]
Remark (The Monte-Carlo method for Hilbert spaces) If $B$ is a separable Hilbert space $H$ (with inner product $\langle \cdot , \cdot \rangle$) and $X$ is a $H$ valued random variable, it is straightforward to obtain an error estimate for the Monte-Carlo method. With the necessary modifications, we can repeat the calculations leading up to (5.2) to get

$$E\left[\left\| E_N - E[X] \right\|^2_H\right] = \frac{1}{N} \left( E\left[\left\| X \right\|^2_H\right] - \left\| E[X] \right\|^2_H\right).$$

(5.4) \text{eq:hilmcerrest}

In the context of conservation laws, $B$ is often an $L^1$ space (which is not Hilbert!), and we must find a replacement for this error estimate.

To bound the error of the Monte-Carlo algorithm, we must know how to estimate sums of independent random variables taking values in a Banach space.

Theorem 5.13 (Itô-Nisio theorem) Let $X_i : \Omega \rightarrow B$, $i \geq 1$, be independent symmetric random variables and let $S : \Omega \rightarrow B$ be a random variable. Define the partial sums

$$S_n = \sum_{i=1}^{n} X_i.$$  \text{thm:ito-nisio}

Then the following statements are equivalent:

1. For all $\xi \in B^*$ we have $\langle S_n, \xi \rangle \overset{a.s.}{\rightarrow} \langle S, \xi \rangle$ as $n \rightarrow \infty$;
2. For all $\xi \in B^*$ we have $\langle S_n, \xi \rangle \overset{p}{\rightarrow} \langle S, \xi \rangle$ as $n \rightarrow \infty$;
3. $S_n \overset{a.s.}{\rightarrow} S$ as $n \rightarrow \infty$;
4. $S_n \overset{p}{\rightarrow} S$ as $n \rightarrow \infty$.

If these equivalent statements hold and if $E[\|S\|^p_B] < \infty$ for some $1 \leq p < \infty$ then

$$\lim_{n \rightarrow \infty} E[\|S_n - S\|^p_B] = 0.$$  \text{thm:ito-nisio}

For a proof of this, see [?].

We now show some estimates on sums of random variables, and start with the so-called Kahane contraction principle:

Theorem 5.14 Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent, symmetric random variables. Then for all $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$ and all $1 \leq p < \infty$

$$E\left[\left\| \sum_{i=1}^{N} a_i X_i \right\|^p_B\right] \leq \max_i |a_i| E\left[\left\| \sum_{i=1}^{N} X_i \right\|^p_B\right].$$

\text{thm:kahanecontract}

\textbf{Proof} First we assume that $a_i \in \{-1, 1\}$ for all $i$. Since $X_i$ is symmetric, $E[a_i X_i] = E[X_i]$ and therefore

$$E\left[\left\| \sum_{i=1}^{N} a_i X_i \right\|^p_B\right] = E\left[\left\| \sum_{i=1}^{N} X_i \right\|^p_B\right].$$
Secondly, we observe that to prove the general case suffices to consider the case when \( \max_i |a_i| \leq 1 \), in which case \( a \) is in the hypercube in \( \mathbb{R}^N \) with \( 2^N \) corners \( \{c_j \}_{j=1}^{2^N} \) with components \( c_{ji} \in \{-1, 1\} \). Therefore \( a \) is a convex combination of the vectors \( c_j \), i.e.,

\[
a = \sum_{j=1}^{2^N} \alpha_j c_j,
\]

where \( \alpha_j \geq 0 \) and \( \sum_j \alpha_j = 1 \). Then we calculate

\[
E \left[ \left\| \sum_{i=1}^{N} a_i X_i \right\|^p_B \right] = E \left[ \left\| \sum_{i=1}^{N} \sum_{j=1}^{2^N} \alpha_j c_{ji} X_i \right\|^p_B \right] \\
= E \left[ \left\| \sum_{j=1}^{2^N} \sum_{i=1}^{N} \alpha_j c_{ji} X_i \right\|^p_B \right] \\
\leq E \left[ \left( \sum_{j=1}^{2^N} \alpha_j \right) \left\| \sum_{i=1}^{N} c_{ji} X_i \right\|^p_B \right] \\
\leq E \left[ \sum_{j=1}^{2^N} \alpha_j \left\| \sum_{i=1}^{N} c_{ji} X_i \right\|^p_B \right] \\
= \sum_{j=1}^{2^N} \alpha_j E \left[ \left\| \sum_{i=1}^{N} c_{ji} X_i \right\|^p_B \right] = E \left[ \left\| \sum_{i=1}^{N} X_i \right\|^p_B \right],
\]

where we used Jensen's inequality to get from the third line to the fourth.

We shall have use for so-called Rademacher (random) variables. A \( \{-1, 1\} \) valued random variable \( r \) is called a Rademacher variable if

\[
\text{Prob} \{ r = 1 \} = \text{Prob} \{ r = -1 \} = \frac{1}{2}.
\]

A Rademacher sequence is a sequence \( \{r_i\} \) of independent Rademacher variables. If \( \{\varphi_i\}_{i=1}^{\infty} \) is an independent sequence of symmetric real random variables, and \( \{r_i\}_{i=1}^{\infty} \) is a Rademacher sequence which is independent of \( \{\varphi_i\} \), then we claim that the sequences \( \{r_i | \varphi_i | \}_{i=1}^{\infty} \) and \( \{\varphi_i \}_{i=1}^{\infty} \) are identically distributed. To show this claim we use independence to compute

\[
\text{Prob} \{ r \in A \} = \text{Prob} \{ r = 1, \ | \varphi | \in A \} + \text{Prob} \{ r = -1, \ | \varphi | \in -A \} \\
= \frac{1}{2} \text{Prob} \{ | \varphi | \in A \} + \frac{1}{2} \text{Prob} \{ | \varphi | \in -A \} \\
= \frac{1}{2} \text{Prob} \{ \varphi \geq 0, \ \varphi \in A \} + \frac{1}{2} \text{Prob} \{ \varphi < 0, \ \varphi \in -A \}.
\]
+ \frac{1}{2} \text{Prob} \{ \varphi \geq 0, \varphi \in -A \} + \frac{1}{2} \text{Prob} \{ \varphi < 0, \varphi \in A \} \\
= \frac{1}{2} \text{Prob} \{ \varphi \in A \} + \frac{1}{2} \text{Prob} \{ \varphi \in -A \} = \text{Prob} \{ \varphi \in A \},

since \varphi is symmetric. With the use of this claim we can prove the following theorem.

**Theorem 5.15** Let \( \{ r_i \}_{i=1}^{\infty} \) be a Rademacher sequence, and let \( \{ \varphi_i \}_{i=1}^{\infty} \) be a sequence of real symmetric random variables with \( \text{E}[|\varphi_i|] \geq 1 \) for all \( i \). If \( \{ \varphi_i \} \) and \( \{ r_i \} \) are independent then for all \( x_1, \ldots, x_N \in B \) and all \( 1 \leq p < \infty \) we have

\[
\text{E} \left[ \left\| \sum_{i=1}^{N} r_i x_i \right\|_B^p \right] \leq \text{E} \left[ \left\| \sum_{i=1}^{N} \varphi_i x_i \right\|_B^p \right].
\]

**Proof** Since \( r_i \) and \( \varphi_i \) are independent, we can think of these as defined on different probability spaces, and for clarity we denote by \( \text{E}_r \) and \( \text{E}_\varphi \) the expectations on these different probability spaces.

Set \( Y_i = \text{E}_\varphi [|\varphi_i| r_i x_i] \) and \( a_i = 1/\text{E}_\varphi [|\varphi_i|] \) in the Kahane comparison principle to get

\[
\text{E}_r \left[ \left\| \sum_{i=1}^{N} r_i x_i \right\|_B^p \right] = \text{E}_r \left[ \left\| \sum_{i=1}^{N} a_i Y_i \right\|_B^p \right] \leq \max_i |a_i| \text{E}_r \left[ \left\| \sum_{i=1}^{N} Y_i \right\|_B^p \right] \\
= \frac{1}{\min_i \text{E}_\varphi [|\varphi_i|]} \text{E}_r \left[ \left\| \sum_{i=1}^{N} \text{E}_\varphi [|\varphi_i|] r_i x_i \right\|_B^p \right] \\
\leq \text{E}_r \left[ \left\| \text{E}_\varphi \left[ \sum_{i=1}^{N} |\varphi_i| r_i x_i \right] \right\|_B^p \right] \\
\leq \text{E}_r \left[ \text{E}_\varphi \left[ \left\| \sum_{i=1}^{N} |\varphi_i| x_i \right\|_B^p \right] \right] \quad \text{(Jenssen’s ineq.)} \\
= \text{E}_\varphi \left[ \left\| \sum_{i=1}^{N} \varphi_i x_i \right\|_B^p \right] \quad \text{(above claim)}.
\]

To prove the convergence of the Monte-Carlo method in Banach spaces, it is useful to be able to “measure” how far a Banach space is from being a Hilbert space. This is indicated by its *type*, which is defined as follows:

**Definition 5.16** We say that a Banach space \( B \) has type \( q \), where \( 1 \leq q \leq 2 \), if there is a constant \( \kappa \) such that for any finite sequence \( \{ x_i \}_{i=1}^{N} \subset B \) and any Rademacher sequence \( \{ r_i \} \), the inequality

\[
\text{E} \left[ \left\| \sum_{i=1}^{N} r_i x_i \right\|_B^q \right] \leq \kappa^q \sum_{i=1}^{N} \| x_i \|_B^q.
\]

(5.5) \( \text{eq:typedef} \)

The smallest constant \( \kappa \) such that (5.5) holds is called the *type constant*, and we shall henceforth refer to the type constant as \( \kappa \). Every Banach space
has type 1, and every Hilbert space has Banach space type 2. Moreover, $L^p$ spaces have type $q = \min \{2, p\}$. For Lebesgue spaces we have the following lemma:

**Lemma 5.17** Let $\{\Omega, \mathcal{B}\}$ be a measure space, $1 \leq r \leq \infty$ and $B$ a Banach space of type $q$. Then $L^r(\Omega, B)$ is a Banach space of type $\min \{r, q\}$.

Observe that every Hilbert space has type $q = 2$, and crucially for our application $L^1(\mathbb{R})$ has type 1.

In order to deduce some consequences of $B$ having a type, we prove a useful inequality. Let $F$ be a real convex function on $[0, 1]$ be $B$ valued zero mean random variables, and in addition assume that $Y$ is symmetric. Now

$$E[F(||X||_B)] = E[H((X + Y)/2 + (X - Y)/2)]$$

$$\leq \frac{1}{2}E[H(X + Y)] + \frac{1}{2}E[H(X - Y)]$$

$$= E[H(X + Y)] = E[F(||X + Y||_B)].$$

With this observation we can prove the following lemma:

**Lemma 5.18** Let $F : [0, \infty) \to [0, \infty]$ be a convex function, and let $\{r_i\}$ be a Rademacher sequence. Then for any finite sequence $\{X_i\}_{i=1}^N$ of independent zero mean random variables such that $E[F(||X_i||_B)] < \infty$ for all $i$,

$$E\left[F\left(\frac{1}{2}\left\|\sum_{i=1}^N r_i X_i\right\|_B\right)\right] \leq E\left[F\left(\sum_{i=1}^N ||X_i||_B\right)\right] \leq E\left[F\left(2\sum_{i=1}^N r_i X_i\right)\right].$$

**Proof** Let $\tilde{X}_i = X_i - X_i'$, where $X_i'$ is a copy of $X_i$ defined on $\Omega'$. By the above observation we have

$$E\left[F\left(\sum_{i=1}^N X_i\right)\right] \leq E\left[F\left(\sum_{i=1}^N \tilde{X}_i\right)\right]$$

$$= E\left[F\left(\sum_{i=1}^N r_i \tilde{X}_i\right)\right] \leq E\left[F\left(2\sum_{i=1}^N r_i X_i\right)\right],$$

and conversely

$$E\left[F\left(\frac{1}{2}\left\|\sum_{i=1}^N r_i X_i\right\|_B\right)\right] \leq E\left[F\left(\frac{1}{2}\left\|\sum_{i=1}^N r_i \tilde{X}_i\right\|_B\right)\right]$$

$$= E\left[F\left(\frac{1}{2}\sum_{i=1}^N \tilde{X}_i\right)\right] \leq E\left[F\left(\sum_{i=1}^N \tilde{X}_i\right)\right].$$

Now we are ready to prove the important result by which we can gain information about the error of the Monte-Carlo method.
Lemma 5.19 Let $B$ be a Banach space of type $q$ with type constant $\kappa$. Then for every finite sequence $\{X_i\}_{i=1}^N$ of $B$ valued independent and zero mean random variables in $L^q(\Omega;B)$

$$
\mathbb{E}\left[\left\|\sum_{i=1}^N X_i\right\|^q_B\right] \leq (2\kappa)^q \sum_{i=1}^N \mathbb{E}[\|X_i\|^q_B].
$$

**Proof** Using Lemma 5.18 and the definition of the type

$$
\mathbb{E}\left[\left\|\sum_{i=1}^N X_i\right\|^q\right] \leq 2^q \mathbb{E}\left[\left\|\sum_{i=1}^N r_i X_i\right\|^q\right] \leq (2\kappa)^q \sum_{i=1}^N \mathbb{E}[\|X_i\|^q_B].
$$

Now we are ready to estimate how close the average $E_N$ is to the mean $\mathbb{E}[X]$.

Let $X_i, i = 1, \ldots, N$ be $N$ independent copies of $X$, set $Y_i = (X_i - \mathbb{E}[X])/N$. Then $\{Y_i\}_{i=1}^N$ are independent zero mean random variables, and

$$
E_N - \mathbb{E}[X] = \sum_{i=1}^N Y_i.
$$

Lemma 5.19 says that

$$
\mathbb{E}\left[\|E_N - \mathbb{E}[X]\|^q_B\right] \leq (2\kappa)^q N^{q-1} \sum_{i=1}^N \mathbb{E}[\|X_i - \mathbb{E}[X]\|^q_B],
$$

or

$$
\mathbb{E}[\|E_N - \mathbb{E}[X]\|^q_B] \leq (2\kappa)^q N^{1-q} \mathbb{E}[\|X - \mathbb{E}[X]\|^q_B],
$$

where $q$ is the type of $B$. Unfortunately, if $B = L^1(\mathbb{R}^d)$, which has type 1, this estimate fails to imply that $E_N$ is close to $\mathbb{E}[X]$ for large $N$.

In the special case where $X \in L^r(\Omega;L^p(\mathbb{R}^d))$ we can partially circumvent this difficulty, and we have the following lemma:

Lemma 5.20 Assume that $1 \leq r, p \leq \infty$ and let $X \in L^r(\Omega;L^p(\mathbb{R}^d))$. For $q = \min\{2, r, p\}$ we have the bound

$$
\|\mathbb{E}[X] - E_N[X]\|_{L^q(\Omega;L^p(\mathbb{R}^d))} \leq C N^{q-1/q} \|X\|_{L^q(\Omega;L^p(\mathbb{R}^d))},
$$

where the constant $C$ depends on $q$ and $\kappa$.

**Proof** In general, $L^p(\mathbb{R}^d)$ is a Banach space of type $\min\{2, p\}$, thus $L^q(\Omega;L^p(\mathbb{R}^d))$ is of type $q$. By (5.6) and Jensen's inequality

$$
\mathbb{E}\left[\|\mathbb{E}[X] - E_N[X]\|_{L^p(\mathbb{R}^d)}^q\right] \leq (2\kappa)^q N^{1-q} \mathbb{E}\left[\|\mathbb{E}[X] - X\|_{L^p(\mathbb{R}^d)}^q\right]
$$

$$
\leq C(2\kappa)^q N^{1-q} \mathbb{E}\left[\left(\|\mathbb{E}[X]\|_{L^p(\mathbb{R}^d)} + \|X\|_{L^p(\mathbb{R}^d)}\right)^q\right]
$$

$$
\leq C 2^{q+1} \kappa^q N^{1-q} \left(\mathbb{E}[\|\mathbb{E}[X]\|_{L^p(\mathbb{R}^d)}^q] + \mathbb{E}[\|X\|_{L^p(\mathbb{R}^d)}^q]\right)
$$
5.2 Random solutions of conservation laws

A random entropy solution of the scalar conservation law
\[ u_t + \operatorname{div} f(u) = 0, \]
is a random variable \( u \), defined on some probability space \((\Omega, \mathcal{B}, \mathbb{P})\) and taking values in \( C([0,T]; L^1(\mathbb{R}^d)) \) or some subset thereof, such that \( u \) is almost surely an entropy solution, i.e., for all convex functions \( \eta \) the inequality
\[ \eta(u) + \operatorname{div} q(u) \leq 0, \]
holds weakly for \( \omega \in \Omega \setminus \mathcal{N} \), where \( P(\mathcal{N}) = 0 \). Here \( q \) is the entropy flux associated to \( \eta \), defined by \( q_i(u) = \eta'(u)f_i(u) \) for \( i = 1, \ldots, d \). There are many different ways to construct random solutions depending on what one wants to model. We shall focus on two examples; random initial data and random flux function.

5.2.1 Existence of random entropy solutions

In order to show the existence of random entropy solutions we have to show that the map \( \omega \mapsto u(\cdot, t, \omega) \) is strongly measurable.

Random initial data

One can envisage many situations in which the time evolution, i.e., the conservation law, is known but the initial data is uncertain. The uncertainty is modeled by a random variable \( u_0 \). In order to keep matters simple, we assume that \( u_0 \) takes values in the set
\[ V = \left\{ v \in L^1(\mathbb{R}^d) \mid |v|_{BV(\mathbb{R}^d)} < \infty \right\}, \]
and that
\[ \|u_0\|_{L^r(\Omega; L^\infty(\mathbb{R}^n))} < \infty, \]
for some \( r > 1 \). Define \( u : \mathbb{R}^d \times [0,T] \times \Omega \to \mathbb{R} \) to be the entropy solution of
\[
\begin{align*}
\begin{cases}
  u_t(x,t) + \text{div} \, f(u(x,t)) &= 0, \quad (x,t) \in \mathbb{R}^d \times (0,T], \\
  u(x,0) &= u_0(x).
\end{cases}
\end{align*}
\]

We see that \( \{u(\cdot,\cdot,\omega)\}_{\omega \in \Omega} \) is a family of entropy solutions to the scalar conservation law (5.7), in particular the continuity with respect to the initial data estimate (4.26) holds. In the present context this reads:

\[
\|u(\cdot,t,\omega) - u(\cdot,t,\omega')\|_{L^1(\mathbb{R}^d)} \leq \|u_0(\cdot,\omega) - u_0(\cdot,\omega')\|_{L^1(\mathbb{R}^d)},
\]

for \( \omega \) and \( \omega' \) in \( \Omega \) and \( t \geq 0 \). This means that the composite map

\[
\Omega \ni \omega \mapsto u_0(\cdot,\omega) \mapsto u(\cdot,t,\omega) \in L^1(\mathbb{R}^d)
\]

is the composition of a continuous and a measurable map. Therefore, by Corollary 5.8 this composite map \( \omega \mapsto u(\cdot,t,\omega) \) is strongly measurable. Furthermore, (4.24) states that \( |u(\cdot,t,\omega)|_{BV(\mathbb{R}^d)} \leq |u_0(\cdot,\omega)|_{BV(\mathbb{R}^d)} < \infty \). Hence \( u(\cdot,t,\cdot) \) is a (well defined) random variable taking values in \( V \).

**Random flux functions**

For some phenomena described by conservation laws the flux function is uncertain. One may know \( f(0) \) and \( f(1) \), but not \( f(u) \) for \( u \) between 0 and 1. Similarly, it happens that one only knows the “shape” of the flux function, such as monotonicity, convexity and so on. In such cases it is natural to describe the flux function as a random variable. For simplicity, in this section we assume that \( d = 1 \), the extension to \( d > 1 \) is straightforward.

Let \( f = f(u,\omega) \) be a random variable taking values in the Sobolev space \( W^{1,\infty} \). Since there is no loss of generality in assuming \( f(0,\cdot) = 0 \), we can assume that \( f \) takes values in

\[
V = \{g \in W^{1,\infty}(\mathbb{R}) \mid g(0) = 0\}
\]

Since the initial data is not random, by scaling if suffices to consider \( u \in [-1,1] \). Therefore we can use

\[
\|g\|_V = \sup_{u,v \in [-1,1], u \neq v} \frac{|g(u) - g(v)|}{|u - v|} = \|g\|_{\text{Lip}},
\]

as a norm for \( V \). We shall also assume that \( \|u_0\|_{L^1(\mathbb{R})} \leq M \) and \( |u_0|_{BV(\mathbb{R})} \leq M \), which clearly implies that \( \|u_0\|_{L^\infty(\mathbb{R})} < \infty \).

In this case we define a random solution \( u = u(x,t,\omega) \) as the entropy solution to
Lemma 5.21 \textit{Now we can prove that the Monte-Carlo estimate approximates the mean.}

\[
\begin{aligned}
\left\{u_t(x,t,\omega) + f_x(u(x,t,\omega),\omega) = 0, \quad (x,t) \in \mathbb{R} \times (0,T],
\right.
\left.u(x,0,\omega) = u_0(x). \right. 
\end{aligned}
\tag{5.8}
\]

Since \( \{u(\cdot,\omega)\}_{\omega \in \Omega} \) is a parameterised family of entropy solutions, we can invoke the stability estimate (4.26) for each pair \( (\omega,\omega') \) to get

\[
\|u(\cdot,t,\omega) - u(\cdot,t,\omega')\|_{L^1(\mathbb{R})} \leq tM\|f(\cdot,\omega) - f(\cdot,\omega')\|_{Lip}.
\]

Thus the map from \( \Omega \) to \( L^1(\mathbb{R}) \) defined by \( \omega \mapsto u(\cdot,t,\omega) \) is the composition of the measurable map from \( \Omega \) to \( V; \omega \mapsto f(\cdot,\omega) \), and the continuous map from \( V \) to \( L^1(\mathbb{R}) \) given by \( f \mapsto u(\cdot,t) \). Hence by Corollary 5.8 the random variable \( u \) is strongly measurable.

5.2.2 Computing approximations to the mean

Se start by bounding the difference between the mean and the average. For simplicity we hereafter assume that \( d = 1 \). We shall also assume that \( u_0 \) has bounded support or is periodic, either of which implies that there exists a finite interval \( D \) such that all relevant norms are in \( L^p(D) \) rather than \( L^p(\mathbb{R}) \).

In addition we assume that

\[
\|u_0\|_{L^r(\Omega; L^\infty(D))} < \infty, \tag{5.9}
\]

for some \( 1 < r \leq \infty \). This implies that also

\[
\|u(\cdot,t)\|_{L^r(\Omega; L^\infty(D))} < \infty \quad \text{and} \quad \|u_h(\cdot,t)\|_{L^r(\Omega; L^\infty(D))} < \infty
\]

as well. We shall use this simplifying assumption as follows: Assume that \( u_0 \in L^r(\Omega; L^\infty(D)) \) for some \( r > 1 \). For any \( 1 \leq p \leq \infty \) we have

\[
\|u(\cdot,t,\cdot)\|_{L^r(\Omega; L^p(D))}^{r/p} = \int_{\Omega} \|u(\cdot,t,\omega)\|_{L^p(D)}^{r/p} d\mathbb{P}(\omega)
\]

\[
\leq |D|^{r/p} \int_{\Omega} \|u(\cdot,t,\omega)\|_{L^\infty(D)}^{r/p} d\mathbb{P}(\omega)
\]

\[
\leq |D|^{r/p} \int_{\Omega} \|u_0(\cdot,\omega)\|_{L^\infty(D)}^{r/p} d\mathbb{P}(\omega)
\]

\[
= |D|^{r/p} \|u_0\|_{L^r(\Omega; L^\infty(D))}^{r/p}. \tag{5.10}
\]

Now we can prove that the Monte-Carlo estimate approximates the mean.

Lemma 5.21 \textit{For each} \( 1 \leq p \leq \infty \) \textit{and} \( 0 \leq t \leq T < \infty \) \textit{and} \( q = \min \{2,r\} > 1 \), \textit{we have that}

\[
\|\mathbb{E}[u] - E_N[u]\|_{L^q(\Omega; L^p(D))} \leq CN^{\frac{1-s}{r}} \|u_0\|_{L^r(\Omega; L^\infty(D))},
\]
for some constant $C$ depending on $D$, $p$, $q$ and $T$.

**Proof** First consider the case $p \geq q$. By (5.10) and Lemma 5.20

$$
\|\mathbb{E}[u] - E_N[u]\|_{L^p(\Omega;L^p(D))} \leq CN^{\frac{1}{p} - \frac{1}{q}} \|u(\cdot,t)\|_{L^q(\Omega;L^p(D))} \\
\leq CN^{\frac{1}{p} - \frac{1}{q}} \|u(\cdot,t)\|_{L^r(\Omega;L^p(D))} \\
\leq CN^{\frac{1}{p} - \frac{1}{q}} \|u_0\|_{L^r(\Omega;L^\infty(D))}.
$$

If $p < q$ we can use Hölder’s inequality to get

$$
\|\mathbb{E}[u] - E_N[u]\|_{L^p(\Omega;L^p(D))} \leq C \|\mathbb{E}[u] - E_N[u]\|_{L^q(\Omega;L^q(D))}.
$$

Since $q \leq r$

$$
\|u(\cdot,t)\|_{L^q(\Omega;L^q(D))} \leq C \|u_0\|_{L^q(\Omega;L^\infty(D))} \leq C \|u_0\|_{L^r(\Omega;L^\infty(D))},
$$

and thus $u(\cdot,t) \in L^q(\Omega;L^r(D))$ and we can invoke Lemma 5.20 to get

$$
\|\mathbb{E}[u] - E_N[u]\|_{L^q(\Omega;L^q(D))} \leq CN^{\frac{1}{q} - \frac{1}{r}} \|u(\cdot,t)\|_{L^r(\Omega;L^q(D))} \\
\leq CN^{\frac{1}{q} - \frac{1}{r}} \|u_0\|_{L^r(\Omega;L^\infty(D))}.
$$

Hence we have proved the lemma for any $1 \leq p \leq \infty$.

In general we cannot compute the entropy solutions to conservation laws, therefore we do not know the averages $E_N[u]$. What we can compute are approximations to entropy solutions and we can use the approximations to approximate the Monte-Carlo averages $E_N[u]$.

To generate approximations to random entropy solutions we can use any convergent numerical method, we shall consider front tracking and finite volume methods. Let $u_\delta$ and $u_{\Delta t}$ denote the front tracking approximation and the finite volume approximation respectively. These depend on the initial data $u_0$ and the flux functions $f$. See Section 2.3 regarding how to compute $u_\delta$. For convenience we give a brief summary of how to construct $u_{\Delta t}$.

Let $\Delta t$ and $\Delta x$ be (small) positive numbers, $t^n = n \Delta t$ for $n = 0, 1, 2, \ldots$, $x_j = j \Delta x$ for $j \in \mathbb{Z}/2 \cap D/\Delta x$, $I^n = [t^n, t^{n+1})$, $I_j = [x_{j-1/2}, x_{j+1/2})$ and define the piecewise constant function

$$
u_{\Delta t}(x, t) = \sum_{j,n} u^n_j \chi_{I_j}(x) \chi_{I^n}(t),$$

where the sum is over integers $j$ such that $j \Delta x \in D$ and $n$ such that $t^n \leq T$. The numbers $u^n_j$ are computed by a conservative, consistent and monotone finite volume method.
\[ u_j^{n+1} = u_j^n - \lambda \left( f_{j+1/2}^n - f_{j-1/2}^n \right), \quad n \geq 0, \]
\[ u_0^0 = \frac{1}{\Delta x} \int_{\mathcal{I}_j} u_0(x) \, dx, \]

where \( \lambda = \Delta t / \Delta x \), and recall that in order for the method to be monotone, \( \lambda \leq C \) for some constant \( C \) depending on \( \| f \|_{\text{Lip}} \). Then we can apply Kuznetsov’s lemma (Theorem 3.14) and the calculations leading up to (3.71) to get the error estimate

\[ \| u(\cdot, t) - u_{\Delta t}(\cdot, t) \|_{L^1(D)} \leq C \| u_0 \|_{BV(D)} (1 + \| f \|_{\text{Lip}}) \sqrt{\Delta t}, \]

(5.11) \hspace{1cm} \text{eq: detailed rate}

where the constant \( C \) depends on \( T \) but not on the initial data \( u_0 \) or on the flux function \( f \). Recall that this estimate also holds for the front tracking approximation with \( \delta \) replacing \( \sqrt{\Delta t} \), viz.

\[ \| u(\cdot, t) - u_{\delta}(\cdot, t) \|_{L^1(D)} \leq C \| u_0 \|_{BV(D)} (1 + \| f \|_{\text{Lip}}) \delta. \]

In order to unify the presentation we assume that we have a family of approximations \( \{ u_h \}_{h > 0} \) satisfying the error estimate

\[ \| u(\cdot, t, \omega) - u_h(\cdot, t, \omega) \|_{L^1(D)} \leq C h^s, \]

(5.12) \hspace{1cm} \text{eq: det stability}

where \( h = \Delta t \) and \( s = 1/2 \) for the finite volume approximation, and \( h = \delta \) and \( s = 1 \) for the front tracking approximation.

Now we can bound how well we can estimate how well we can approximate averages of exact solutions by averages of numerical approximations.

\textbf{Lemma 5.22} For each \( 1 \leq p \leq \infty \) and \( 0 \leq t \leq T < \infty \) and for \( q = \min \{ 2, r \} > 1 \), we have that

\[ \| E_N[u] - E_N[u_h] \|_{L^\infty(\Omega; L^p(D))} \leq C h^s \| u_0 \|_{L^r(\Omega; L^\infty(D))}, \]

for some constant \( C \) depending on \( D, p, q \) and \( T \).

\textbf{Proof} Let \( u^i = u(\cdot, \cdot, \omega_i) \) and \( u_h^i = u_h(\cdot, \cdot, \omega_i) \) where \( \{ \omega_i \}_{i=1}^\infty \) are independent and identically distributed. By definition

\[ \| E[u] - E_N[u_h] \|_{L^\infty(\Omega; L^p(D))} = \frac{1}{N} \left\| \sum_{i=1}^N u^i - u_h^i \right\|_{L^\infty(\Omega; L^p(D))} \]
\[ \leq \frac{1}{N} \sum_{i=1}^N \| u - u_h \|_{L^\infty(\Omega; L^p(D))} = \| u - u_h \|_{L^\infty(\Omega; L^p(D))}. \]

By the simplifying assumption (5.10), both \( u(\cdot, t, \omega) \) and \( u_h(\cdot, t, \omega) \) are in \( L^\infty(D) \cap L^1(D) \) \( \omega \) almost surely. We can then use \( L^1 / L^\infty \) interpolation, which says that if \( g \in L^p \cap L^\infty \) then
\[ \| g \|_{L^p} \leq \| g \|_{L^1}^{\frac{1}{p}} \| g \|_{L^\infty}^{1 - \frac{1}{p}}, \]

for any \( 1 \leq p \leq \infty \). This inequality gives us

\[
\| u - u_h \|_{L^q(\Omega; L^p(D))} = \left( \int_{\Omega} \| u - u_h \|_{L^p(D)}^q \, dP \right)^{\frac{1}{q}} \\
\leq \left( \int_{\Omega} \| u - u_h \|_{L^1(D)}^{\frac{q}{2}} \| u - u_h \|_{L^\infty(D)}^{\frac{q(1 - \frac{1}{p})}{2}} \, dP \right)^{\frac{1}{q}} \\
\leq \| u - u_h \|_{L^q(\Omega; L^1(D))} \| u - u_h \|_{L^q(\Omega; L^\infty(D))}^{\frac{1}{2}} \\
\leq C h^{\frac{q}{2}} \| u_0 \|_{L^q(\Omega; L^1(D))},
\]

which concludes the proof.

As a corollary of Lemma 5.21 and Lemma 5.22 we get a bound on \( \mathbb{E}[u] - E_N[u_h] \).

**Theorem 5.23** Assume that we are given approximations \( \{u_j\}_{h>0} \) such that (5.12) holds (for all sufficiently small \( h \)), and that (5.9) holds for some \( r > 1 \). Then, for any \( 1 \leq p \leq \infty \) and for all \( 0 \leq t \leq T \) and for \( q = \min \{2, r\} \), we have the error estimate

\[
\| \mathbb{E}[u] - E_N[u_h] \|_{L^q(\Omega; L^p(D))} \\
\leq C \left( N^{\frac{1-p}{2}} \| u_0 \|_{L^q(\Omega; L^\infty(D))} + h^{\frac{q}{2}} \| u_0 \|_{L^q(\Omega; L^1(D))}^{\frac{1}{2}} \right),
\]

where the constant \( C \) depends on \( D, p, q \) and \( T \).

**Proof** By the triangle inequality,

\[
\| \mathbb{E}[u] - E_N[u_h] \|_{L^q(\Omega; L^p(D))} \leq \| \mathbb{E}[u] - E_N[u_h] \|_{L^q(\Omega; L^p(D))} + \| E_N[u] - E_N[u_h] \|_{L^q(\Omega; L^p(D))}.
\]

Use Lemma 5.21 to bound the first term and Lemma 5.22 to bound the second.

It is natural to consider \( p = 1 \) since scalar conservation laws are well posed in \( L^1 \), therefore we define the error of the Monte-Carlo method as

\[
\mathcal{E}_{MC}(h, N) = \| \mathbb{E}[u] - E_N[u_h] \|_{L^q(\Omega; L^1(D))}.
\]

For general \( p > 1 \) the ensuing analysis is similar, but one must keep track of \( p \). To minimize the error we must choose \( N \) and \( h \) such that both terms on the right scale identically, i.e.,

\[
h^s = O \left( N^{\frac{1-p}{2}} \right) = O(\varepsilon). \tag{5.13}
\]
then $E_{MC}(h, N) \leq \varepsilon$.

In practice it is important to know how many operations (multiplications, additions etc.) are needed to ensure that $E_{MC} \leq \varepsilon$. Let $W_{MC}(h, N)$ denote the number of operations needed to compute $E_N [u_h]$. If $w(h)$ denotes the number of operations used to compute $u_h$, then $W_{MC}(h, N) = Nw(h)$. Therefore we estimate $w_\Delta$ and $w_5$. Thus the computational cost needed to compute $E_N [u_h]$ such that the error $E_{MC} \leq \varepsilon$ scales as

$$C_\varepsilon = \left( \frac{1}{\varepsilon} \right)^{\frac{3}{2}} w(\varepsilon^{1/3}).$$

(5.14)  

Bounding $w(\Delta t)$ for the finite volume method: We need to compute $u_\Delta (x_j, t^n)$ for $j$ and $n$ such that $|j| \leq |D| / \Delta x$ and $n \leq T / \Delta t$. This means that the work needed to compute $u_\Delta$ scales as $1/\Delta t^2$ since $\Delta x \lesssim \Delta t$ by the CFL-condition $\Delta t / \Delta x \lesssim 1$. Thus

$$w(\Delta t) \approx \frac{1}{\Delta t^2}. \quad (5.15)$$

Bounding $w(\delta)$ for the front tracking method: The computational work needed to calculate $u_\delta$ is proportional to the number of collisions, and for each collision we must solve a Riemann problem, calculate the speeds of the resulting new fronts and calculate two new collision times.

If $f$ is convex, the number of fronts is strictly decreasing at each collision. Initially there are at most $|u_0|_{BV} / \delta$ fronts, and therefore we have at most $|u_0|_{BV} / \delta$ collisions. For general non-convex flux functions we have that the functional $T(t)$ (defined on page ??) is strictly decreasing at each collision. Thus the total number of collisions is bounded by

$$T(0) \leq \frac{(\sup_D u_0 - \inf_D u_0)}{\delta} |u_0|_{BV} + \frac{|u_0|_{BV}}{\delta} \leq \frac{C}{\delta^2},$$

for small $\delta$.

Next we calculate the number of operations needed to solve a Riemann problem with left state $u_l$ and right state $u_r$. This means finding the convex envelope of $f_\delta(u)$ for $u$ between $u_l$ and $u_r$. Using Graham’s algorithm this has a computational cost of

$$G \log G, \quad \text{where} \quad G = \frac{|u_l - u_r|}{\delta}.$$

Finally we have to compute the speeds of each of the resulting fronts, and there are at most $G$ of these. If $f$ is convex, the result of a collision is always one front. Thus the computational cost of resolving one collision is $1$ if $f$ is convex, and

$$\frac{|u_0|_{BV}}{\delta} \log \left( \frac{|u_0|_{BV}}{\delta} \right) \quad \text{if } f \text{ is not convex.}$$
We conclude that
\[ w(\delta) \sim \begin{cases} \frac{1}{\delta} & \text{if } f \text{ is convex}, \\ \frac{1}{\pi \delta} \log \left( \frac{1}{\delta} \right) & \text{otherwise}. \end{cases} \] (5.16) \text{eq:wdeltabnd}

Using (5.15) in (5.14) with \( s = 1/2 \) we find the cost of the finite volume Monte-Carlo method;
\[ C_{\varepsilon}^{MCFV} = \varepsilon^{\frac{q}{2}} \varepsilon^{-4} = \left( \frac{1}{\varepsilon} \right)^{\frac{2q-4}{q-1}}, \] (5.17) \text{eq:fvcost}

so that even in the “best” case with \( q = 2 \), the cost of an error of size \( \varepsilon \) scales as \( \varepsilon^{-6} \).

For the front tracking Monte-Carlo method, we find the corresponding cost inserting (5.16) in (5.14) with \( s = 1 \),
\[ C_{\varepsilon}^{MCFT} = \varepsilon^{\frac{-q}{2}} \varepsilon^{-3} \log \left( \frac{1}{\varepsilon} \right) \] (5.18) \text{eq:ftcost}

In case \( q = 2 \), the cost scales as \( \varepsilon^{-3} \) for convex \( f \) and \( \varepsilon^{-5} \log(\varepsilon^{-1}) \) for non-convex \( f \).

\diamond \textbf{Example} Let \( \{c_k\}_{k=1}^{60} \) be a sequence of random variables such that \( c_k = 1 \) if \( k = 1 + 3j \) for some \( j \in \{0, 1, 2, 3, \ldots\} \), and otherwise \( \text{Prob} \{c_k = 1\} = \text{Prob} \{c_k = -1\} = 1/2 \). Define the random initial data by

\[ u_0(x) = \frac{2}{\pi} \sum_{k=1}^{60} c_k \sin(k\pi x). \]

Let \( u \) be the random entropy solution to the initial value problem
\[
\begin{cases}
    u_t + \frac{1}{2}(u^2)_x = 0, \quad t > 0, \\
    u(x, 0) = u_0(x, \omega).
\end{cases}
\]

In Figure 5.2 we show the Monte-Carlo approximation to \( \mathbb{E}[u(\cdot, 1.5)] \) calculated by using a finite volume scheme with the Engquist-Osher flux (see exercise 3) and front tracking. These averages are shown by the red lines. For the finite volume method we used the parameters \( \Delta x = 1/1000 \) and \( N = 1000 \). Since \( \|u_0(\cdot, \omega)\|_\infty < \infty \) and \( |u_0(\cdot, \omega)|_{BV} < \infty \) for all \( \omega \), we can use \( q = 2 \) in the estimates. Therefore the error should be (or scale as) \( 1/33 \). In order to get a comparable error with the front tracking approximations, we used \( \delta = 1/33 \) and \( N = 1000 \) when calculating \( E_N[u_\delta] \). The gray areas in the figures is bounded by \( E_N[v] + \sigma_N[v] \) and \( E_N[v] - \sigma_N[v] \), where \( \sigma_N \) denotes the Monte-Carlo estimate for the standard deviation.
5.2 Random solutions of conservation laws

![Initial data \(u_0\)](fig:mcexample)

![Finite volume. \(u_{\Delta t}(., 1.5)\)](fig:mcexample)

![Front tracking. \(u_\delta(., 1.5)\)](fig:mcexample)

Fig. 5.2: The initial data (top), \(E_{1000}[u_{\Delta t}]\) for the finite volume approximation (middle) and \(E_{1000}[u_\delta]\) for the front tracking approximation (bottom).

\[
\sigma_N[v] = \left( E_N (E_N[v] - v)^2 \right)^{1/2},
\]

where \(v\) is \(u_{\Delta t}\) or \(u_\delta\). This gray area gives an indication of how uncertain the random solutions are, roughly 70% of the random solutions should lie in the gray area. \(\Box\)

5.2.3 The multilevel Monte-Carlo method

subsec:MLMC

We have seen that in order to make reasonably accurate assertions about the mean of random entropy solutions, one is forced to compute many approximations. The multilevel Monte-Carlo method approximates the mean differently, which in many cases reduces the work to accuracy ratio compared with the Monte-Carlo method.

As in the previous section, \(u_h\) is a monotone approximation such that the deterministic error estimate (5.12) holds. Let \(U_L = \{u_{h_i}\}_{i=0}^L\) be a collection of approximations with \(h_0 > h_1 > h_2 > \cdots > h_L\). For simplicity we assume
that there is a fixed constant \( k \) such that \( kh_l \geq h_{l-1} \). The rationale behind the multilevel Monte-Carlo method is the following observation:

\[
E[u_{hl}] = E\left[ \sum_{l=0}^{L} (u_{hl} - u_{h_{l-1}}) \right] = \sum_{l=0}^{L} E[u_{hl} - u_{h_{l-1}}],
\]

where we have defined \( u_{h_{l-1}} = 0 \). Based on this we define the multilevel Monte-Carlo estimator as

\[
E^L[U^L] = \sum_{l=0}^{L} E_{N_l} [u_{hl} - u_{h_{l-1}}]. \tag{5.19}
\]

This gives us the freedom to choose the numbers \( \{N_l\}_{l=0}^{L} \) and \( \{h_l\}_{l=0}^{L} \) such that (hopefully) \( E^L \) is a more efficient estimator than \( E_N \).

Before we consider the efficiency of \( E^L \) we must prove that \( E^L \) approximates \( E \). This is ensured by the following error estimate:

\[ \tag{5.20} \]

**Lemma 5.25** Let \( q = \min\{2,r\} \), for any sequence \( \{N_l\}_{l=0}^{L} \) and for any sequence \( \{h_l\}_{l=0}^{L} \) as specified above we have that

\[
\left\| E[u](\cdot,t) - E^L[U^L] \right\|_{L^q(\Omega;L^p(D))} 
\leq C \left( \left\| u_0 \right\|_{L^q(\Omega;L^\infty(D))} h_0^{\frac{1}{p}} + \left\| u_0 \right\|_{L^q(\Omega;L^\infty(D))} N_0^{\frac{1}{r}} \right. 
\left. + \left\| u_0 \right\|_{L^q(\Omega;L^\infty(D))} h_0^{\frac{1}{p}} N_0^{\frac{1}{r}} + \left\| u_0 \right\|_{L^q(\Omega;L^\infty(D))} \sum_{l=1}^{L} N_l^{\frac{1}{r}} h_l^{\frac{1}{p}} \right),
\]

where \( \hat{p} = \max \{p,q\} \), and the constant \( C \) depends on \( p, q \) and \( D \). In particular,

\[
E^L[U^L] \xrightarrow{L^q(\Omega;L^p(D))} E[u] \quad \text{as } N_l \to \infty \text{ and } h_l \to 0 \text{ for } l = 0, \ldots, L.
\]

**Proof** By the triangle inequality,

\[
\left\| E[u] - E^L[U^L] \right\|_{L^q(\Omega;L^p(D))} 
\leq \left\| E[u] - E[u_{hl}] \right\|_{L^q(\Omega;L^p(D))} 
+ \left\| E[u_{hl}] - E^L[U^L] \right\|_{L^q(\Omega;L^p(D))} 
\leq \left\| E[u] - E[u_{hl}] \right\|_{L^q(\Omega;L^p(D))} 
+ \sum_{l=0}^{L} \left\| E[u_{hl} - u_{h_{l-1}}] - E_{N_l} [u_{hl} - u_{h_{l-1}}] \right\|_{L^q(\Omega;L^p(D))}.
\]
Since $\mathbb{E}[u] - \mathbb{E}[u_{h,n}]$ does not depend on $\omega$, we can use $L^1/L^\infty$ interpolation and the deterministic error estimate to bound $A$:

$$A \leq \|u - u_{h,n}\|_{L^1(\Omega; L^p(D))}$$

$$\leq \|u - u_{h,n}\|_{L^1(\Omega; L^\infty(D))}^{1 - \frac{p}{q}} \|u - u_{h,n}\|_{L^1(\Omega; L^1(D))}^{\frac{p}{q}}$$

$$\leq C \|u_0\|^{1 - \frac{p}{q}} L^{1 - \frac{p}{q}}_{L^1(\Omega; L^\infty(D))} h_l^\frac{p}{q}.$$

Next, we bound $B_l$ for $l > 0$. We use the notation $\Delta_l = u_{h,l} - u_{h,l-1}$. First assume that $p \geq q$. By the triangle inequality and the fact that $\|u_h\|_{L^p(D)} \leq C \|u_h\|_{L^\infty(D)}$, we get

$$\|u_h\|_{L^r(\Omega; L^p(D))} \leq C \|u_h\|_{L^r(\Omega; L^\infty(D))} \leq C \|u_0\|_{L^r(\Omega; L^\infty(D))},$$

which implies that $\Delta_l \in L^r(\Omega; L^p(D))$. Therefore we can use Lemma 5.20 to get

$$\|\mathbb{E}[\Delta_l] - E_N[\Delta_l]\|_{L^s(\Omega; L^p(D))} \leq C N_1^{\frac{1-s}{s}} \|\Delta_l\|_{L^s(\Omega; L^p(D))}.$$

If $p < q$ then we can apply Hölder’s inequality and get

$$\|\mathbb{E}[\Delta_l] - E_N[\Delta_l]\|_{L^s(\Omega; L^p(D))} \leq C \|\mathbb{E}[\Delta_l] - E_N[\Delta_l]\|_{L^s(\Omega; L^q(D))}.$$

We can therefore use the same arguments as in the proof of Lemma 5.20 to conclude that $\Delta_l \in L^q(\Omega; L^q(D))$. Then we invoke Lemma 5.20 to deduce that

$$\|\mathbb{E}[\Delta_l] - E_N[\Delta_l]\|_{L^s(\Omega; L^p(D))} \leq C N_1^{\frac{1-s}{s}} \|\Delta_l\|_{L^s(\Omega; L^p(D))},$$

for $p < q$. Therefore, for any $p$

$$\|\mathbb{E}[\Delta_l] - E_N[\Delta_l]\|_{L^s(\Omega; L^p(D))} \leq C N_1^{\frac{1-s}{s}} \|\Delta_l\|_{L^s(\Omega; L^{\tilde{p}}(D))},$$

where $\tilde{p} = \max\{p, q\}$. Next we again use $L^1/L^\infty$ interpolation, the deterministic error estimate for $u_h$ and the triangle inequality to deduce

$$\|\Delta_l\|_{L^s(\Omega; L^{\tilde{p}}(D))} \leq C \|\Delta_l\|_{L^1(\Omega; L^{\tilde{p}}(D))}^{1 - \frac{\tilde{p}}{q}} \|\Delta_l\|_{L^1(\Omega; L^{\infty}(D))}^{\frac{\tilde{p}}{q}}$$

$$\leq C \left( h_l^{\frac{\tilde{p}}{q}} + h_{l-1}^{\frac{\tilde{p}}{q}} \right) \|u_0\|_{L^1(\Omega; L^{\infty}(D))}^{1 - \frac{\tilde{p}}{q}}$$

$$\leq C h_{l-1}^{\frac{\tilde{p}}{q}} \|u_0\|_{L^1(\Omega; L^{\infty}(D))}^{1 - \frac{\tilde{p}}{q}}.$$

This means that

$$B_l \leq C N_1^{\frac{1-s}{s}} h_{l-1}^{\frac{\tilde{p}}{q}} \|u_0\|_{L^1(\Omega; L^{\infty}(D))}^{1 - \frac{\tilde{p}}{q}},$$

for $l > 0.$
For \( l = 0 \) we use the Monte-Carlo error estimate (Theorem 5.23), the triangle inequality, and the deterministic error estimate (5.12)

\[
B_0 \leq CN_0^{\frac{1-a}{4}} \| u_{h_0} \|_{L^q(\Omega; L^p(D))} \\
\leq CN_0^{\frac{1-a}{4}} \left( \| u - u_{h_0} \|_{L^q(\Omega; L^p(D))} + \| u \|_{L^q(\Omega; L^p(D))} \right) \\
\leq CN_0^{\frac{1-a}{4}} \left( h_0^{\frac{\alpha}{4}} \| u_0 \|_{L^q(\Omega; L^\infty(D))}^{1-\frac{1}{p}} + \| u_0 \|_{L^q(\Omega; L^\infty(D))} \right).
\]

To complete the proof we collect the bounds on \( A, B_l \) for \( l > 0 \) and \( B_0 \).

Notes

Exercises

5.1