2024-04-23

The Saint-Venant (shallow water) and Euler equations

Surprisingly often, a system of conservation laws is most conveniently formulated in terms of *non-conserved* quantities. Changing the variables to get it into the standard form $u_t + f(u)_x = 0$ may lead to a more complicated equation, and lead to unnecessary complications in analyzing the system.

This little note takes a look at conservation laws – with the Saint-Venant (shallow water) and Euler equations as prime examples – on the form

$$\psi(u)_t + \varphi(u)_x = 0, \tag{$*$}$$

where ψ is a diffeomorphism and $\psi(u)$ represents the conserved quanities. Here, the conserved quantities are of course $U = \psi(u)$, satisfying

$$U_t + f(U)_x = 0, \qquad f = \varphi \circ \psi^{-1} \tag{\#}$$

While we develop the general theory for equations on the form (#), the form ($*$) is often more convenient for the practical calculation of rarefaction waves and shocks. This is certainly the case for the two equations considered here.

The modifications to the standard machinery needed to work with equations on the form ($*$) are trivial:

A *rarefaction wave* on the form $u(x,t) = w(x/t)$ must satisfy the ODE (where $\xi = x/t$)

$$\big(\varphi'(w(\xi)) - \xi\psi'(w(\xi))\big)w'(\xi) = 0,$$

so $w'(\xi)$ must be an eigenvector of $(\psi')^{-1}\varphi'$ with eigenvalue ξ. The normalization of the eigenvectors works the same way as the general theory states: If $\xi = \lambda_j\big(w(\xi)\big)$ for some eigenvalue λ_j, differentiation leads to $w'(\xi) \cdot \nabla\lambda_j\big(w(\xi)\big) = 1$.

If needed, we can map the eigenvector $r_j = w'(\xi)$ into the space of conserved quantities U: The resulting vector $R_j(U) = R_j(\psi(u)) = \psi'(u)r_j(u)$ is an eigenvector of $f'(U) = \varphi'(u)(\psi'(u))^{-1}$.

The *Rankine–Hugoniot* condition is even simpler: It is simply

$$[\![\varphi(u)]\!] = s[\![\psi(u)]\!].$$

And that is all there is to it. We proceed with some examples.

Rarefaction waves for the shallow water equations

The shallow water equations are most conveniently written in the form

$$\left.\begin{aligned} h_t + (hv)_x &= 0, \\ (hv)_t + (hv^2 + \tfrac{1}{2}h^2)_x &= 0. \end{aligned}\right\} \tag{SW}$$

Here, $u = (h, v)$, $\psi(u) = \psi(h, v) = (h, hv)$, and $\varphi(u) = \varphi(h, v) = (hv, hv^2\frac{1}{2}h^2)$, so that

$$\psi'(u) = \begin{pmatrix} 1 & 0 \\ v & h \end{pmatrix}, \quad \varphi'(u) = \begin{pmatrix} v & h \\ v^2 + h & 2hv \end{pmatrix},$$

and

$$(\psi')^{-1}\varphi' = \frac{1}{h}\begin{pmatrix} \psi & 0 \\ -v & 1 \end{pmatrix}\begin{pmatrix} v & h \\ v^2 + h & 2hv \end{pmatrix} = \begin{pmatrix} v & h \\ 1 & v \end{pmatrix},$$

which clearly has the eigenvalues $\lambda_\pm = v \pm \sqrt{h}$ with eigenvectors $(\pm\sqrt{h}, 1)$. We find $(\pm\sqrt{h}, 1) \cdot \nabla\lambda_\pm = \frac{3}{2}$, so the properly normalized eigenvectors are $r_\pm = \frac{2}{3}(\pm\sqrt{h}, 1)$.

Thus the rarefaction curve satisfies $h' = \pm\frac{2}{3}\sqrt{h}$ and $v' = \frac{2}{3}$, and in particular $dv/dh = \pm 1/\sqrt{h}$, with the general solution $v = v_0 \pm 2\sqrt{h}$, where v_0 is constant.

Stated differently, $v \mp 2\sqrt{h}$ is constant along each rarefaction curve. We call these functions *Riemann invariants*. More generally, a Riemann invariant for wave family j is a smooth function which is constant along each rarefaction curve of family j. Equivalently, a Riemann invariant R satisfies $r_j(u) \cdot \nabla R(u) = 0$.

Rarefaction waves for the Euler equations

The Euler equations for an ideal gas can be written in variables (ρ, v, p) as

$$\left.\begin{aligned} \rho_t + (\rho v)_x &= 0, \\ (\rho v)_t + (\rho v^2 + p)_x &= 0, \\ \Big(\frac{p}{\gamma - 1} + \frac{\rho v^2}{2}\Big)_t + \Big(v\Big(\frac{\gamma p}{\gamma - 1} + \frac{\rho v^2}{2}\Big)\Big)_x &= 0 \end{aligned}\right\} \tag{Eu}$$

where the terms in the last equation represent energy density and pressure:

$$E = \frac{p}{\gamma - 1} + \frac{\rho v^2}{2}, \quad E + p = \frac{\gamma p}{\gamma - 1} + \frac{\rho v^2}{2}.$$

The first term in E represents the internal energy of the gas per unit volume, while the second term is the kinetic energy, again per unit volume. The constant γ is about 1.4 for a mostly diatomic gas like air.

Equations (Eu) have the form (∗) with

$$\psi(\rho,v,p)=\begin{pmatrix}\rho\\ \rho v\\ \dfrac{p}{\gamma-1}+\dfrac{\rho v^2}{2}\end{pmatrix},\quad \varphi(\rho,v,p)=\begin{pmatrix}\rho v\\ \rho v^2+p\\ \dfrac{\gamma p v}{\gamma-1}+\dfrac{\rho v^3}{2}\end{pmatrix},$$

leading to

$$\psi'=\begin{pmatrix}1 & 0 & 0\\ v & \rho & 0\\ \frac12 v^2 & \rho v & \dfrac{1}{\gamma-1}\end{pmatrix},\quad \varphi'=\begin{pmatrix}v & \rho & 0\\ v^2 & 2\rho v & 1\\ \frac12 v^3 & \dfrac{\gamma p}{\gamma-1}+\dfrac32\rho v^2 & \dfrac{\gamma v}{\gamma-1}\end{pmatrix},$$

and further[1] (an apparent miracle of cancellations)

$$A:=(\psi')^{-1}\varphi'=\begin{pmatrix}v & \rho & 0\\ 0 & v & 1/\rho\\ 0 & \gamma p & v\end{pmatrix}.$$

We immediately find the eigenvalues and eigenvectors of A:

$$\lambda_0:=v,\qquad \lambda_\pm:=v\pm c\qquad \text{where}\quad c=\sqrt{\frac{\gamma p}{\rho}}.$$
$$r_0:=(1,0,0),\qquad \tilde r_\pm:=(\rho,\pm c,\gamma p).$$

(The eigenvectors $\tilde r_\pm$ are not properly normalized, hence the added tilde.)

The "0" family has Riemann invariants v and p. Since the eigenvalue v is one of them, this family is *linearly degenerate*. Only ρ changes along the wave curves.

From the first and last components of $\tilde r_\pm$, the corresponding wave curves satisfy $dp/d\rho=\gamma p/\rho$. This separable equation has general solution $p=C\rho^\gamma$, so p/ρ^γ *is a Riemann invariant* for both families. Conventionally, one often uses instead the *entropy density*[2]

$$S:=\ln\Big(\frac{p}{\rho^\gamma}\Big).$$

[1]The calculation is easily done if you remember your introductory linear algebra course: Perform the same row operations on ψ' and φ', transforming ψ' to the identity and hence φ' to $(\psi')^{-1}\varphi'$. *Be sure* not to compute $(\psi')^{-1}$ and multiply!

A further note: The matrix $f'=\varphi'(\psi')^{-1}$ is much uglier. Another benefit of working with the chosen variables!

[2]Apart from a constant factor, this is the physical entropy density. We should multiply by -1 to get mathematical entropy in the sense used by the theory of conservation laws.

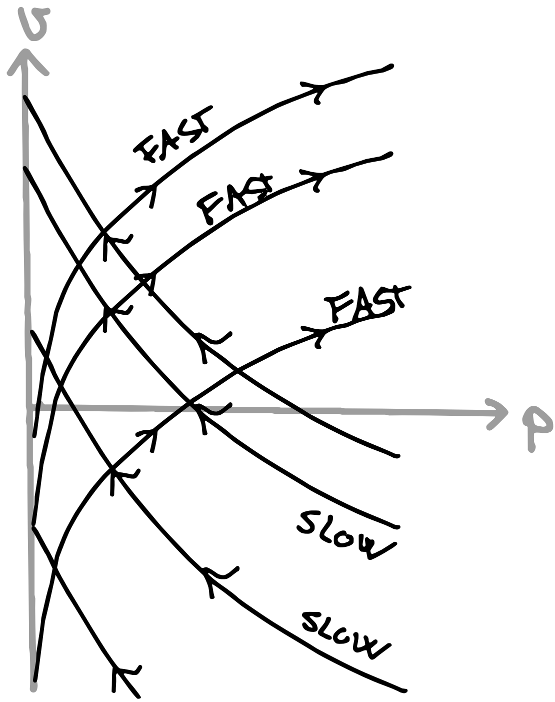

Figure 1: A rough sketch of rarefaction curves for the Euler equations, projected to the (v, p) plane. This is oversimplified: Since the ρ coordinate is not shown, curves from the same family might intersect in the projection, but not of course in (ρ, v, p) space.

Now we have the consequence $c = C\rho^{(\gamma-1)/2}$ along any rarefaction curve; and so we get $dv/d\rho = \pm c/\rho = \pm C\rho^{(\gamma-3)/2}$, which integrates to $v = v_0 \pm 2C/(\gamma-1)\rho^{(\gamma-1)/2} = v_0 \pm 2c/(\gamma-1)$. So we have now collected *two Riemann invariants* for each of the "±" wave curves:

$$\frac{p}{\rho^\gamma} \quad \text{and} \quad v \mp \frac{2c}{\gamma-1}.$$

To normalize the " ± " eigenvectors, note

$$\nabla c = \Big(-\frac{c}{2\rho}, 0, \frac{c}{2p}\Big), \quad \nabla\lambda_\pm = \nabla v \pm \nabla c = \Big(\mp\frac{c}{2\rho}, 1, \pm\frac{c}{2p}\Big), \quad \tilde{r}_\pm \cdot \nabla\lambda_\pm = \pm\frac{\gamma+1}{2}c,$$

and so the properly normalized eigenvectors will be

$$r_\pm = \frac{\pm 2}{(\gamma+1)c}(\rho, \pm c, \gamma p).$$

Note, in particular, that the positive direction along the wave curve is in the direction of *increasing* v for both families.[3]

Summary: A look at the Riemann invariants reveals that ρ, p, and c increase in the same direction along a rarefaction curve; for c, that requires $\gamma > 1$. For the "fast" $(+)$ wave, v also increases in that direction, while for the "slow" $(-)$ wave, v decreases instead. Furthermore ρ, p, and c all approach ∞ together in one direction, and zero in the other direction. Thus the forward direction of a slow wave, and the backward direction of a fast wave, terminate in a vacuum $(\rho = p = c = 0)$ at a finite velocity. This has consequences for the Riemann problem, whilch cannot always be solved without a vacuum state.

Our next task is to study the Hugoniot locus for the Saint-Venant and Euler equations: But first, we develop some more tools.

[3] I have departed from tradition here and labeled the wave families $-$, 0, and $+$ instead of the conventional 1, 2, 3. Note that we always pick the *top* sign for the $+$ family and the *bottom* sign for the $-$ family.

Galilean invariance

Physical systems typically satisfy a criterion of Galilean invariance, meaning that their governing equations are unchanged when transformed to a moving coordinate system.

Consider a moving coordinate system moving with a speed $-\bar{v}$, i.e., $\hat{x} = x + \bar{v}t$ and $\hat{t} = t$, the point of the latter being that we write $z_{\hat{t}}$ as the partial derivative with $\hat{x}$ fixed. So for any function z, $z_x = z_{\hat{x}}$ and $z_t = z_{\hat{t}} + \bar{v}z_{\hat{x}}$.

A velocity v in the original (x,t) coordinate system transforms to $\hat{v} = v + \bar{v}$ in the moving coordinate system. For a function z, we immediately find

$$z_t + (zv)_x = z_{\hat{t}} + (z\hat{v})_{\hat{x}}, \tag{Ga}$$

and, more importantly, the weak formulation of (Ga):

$$\iint (z\varphi_t + zv\varphi_x)\,dx\,dt = \iint \big(z(\varphi_{\hat{t}} + \bar{v}\varphi_{\hat{x}}) + zv\varphi_{\hat{x}}\big)\,dx\,dt = \iint (z\varphi_{\hat{t}} + z\hat{v}\varphi_{\hat{x}})\,d\hat{x}\,dt. \tag{wGa}$$

(Note that it makes no difference whether we integrate over x or $\hat{x}$. We chose the latter on the right hand side to make the invariance of the weak formulation more obvious.)

Product rules for jumps

We now establish some useful identities for dealing with jumps of quantities across a discontinuity. For any quantity u with left and right values u_L abd u_R, we define the *jump* and *average* as

$$[\![u]\!] = u_R - u_L, \quad \langle u \rangle = \tfrac{1}{2}(u_R + u_L).$$

Then we have product rules

$[\![uv]\!] = u_R[\![v]\!] + v_L[\![u]\!]$	two terms $u_R v_L$ cancel
$= u_L[\![v]\!] + v_R[\![u]\!]$	interchange u and v in the above
$= \langle u\rangle[\![v]\!] + \langle v\rangle[\![u]\!]$	average the two above,
$[\![u^2]\!] = 2\langle u\rangle[\![u]\!]$	special case
$\langle uv\rangle = \langle u\rangle\langle v\rangle + \tfrac{1}{4}[\![u]\!][\![v]\!]$	expand the right hand side
$\langle u^2\rangle = \langle u\rangle^2 + \tfrac{1}{4}[\![u]\!]^2$	special case

Hugoniot locus for the shallow water equations

In the shallow water equations (SW), the first equation is clearly Galilean invariant, by (wGa) with $z = h$. Next, with $z = hv$,

$$\begin{aligned}(hv)_t + (hv^2)_x &= (hv)_{\hat t} + (hv\hat v)_{\hat x} \\ &= (h\hat v - h\bar v)_{\hat t} + (h\hat v^2 - h\bar v\hat v)_{\hat x} \\ &= (h\hat v)_{\hat t} + (h\hat v^2)_{\hat x} - \big(h_t + (h\hat v)_x\big)\bar v,\end{aligned}$$

so $(hv)_t + (hv^2)_x$ is Galilean invariant given the first equation of (SW). The term $\frac{1}{2}(h^2)_x$ is trivially invariant, and hence the whole system (SW) is Galilean invariant.

Thanks to this invariance, we may consider *stationary* shocks, i.e., shocks with speed $s = 0$, and then use the Galilei transform to obtain general shocks.

The Rankine–Hugoniot conditions for a stationary shock for (SW) are

$$[\![hv]\!] = 0, \qquad [\![hv^2 + \frac{1}{2}h^2]\!] = 0.$$

From the first of these, we can write $q = h_L v_L = h_r v_R$. Then the second condition becomes $q[\![v]\!] + \langle h\rangle[\![h]\!] = 0$. We multiply this by $[\![h]\!]$, and use

$$q[\![h]\!] = qh_R - qh_L = h_L h_R(v_L - v_R) = -h_L h_R[\![v]\!]$$

to obtain $h_L h_R[\![v]\!]^2 = \langle h\rangle[\![h]\!]^2$, which is Galilean invariant and hence valid for all shocks, whether stationary or not. Dividing by $h_L h_R$ and taking square roots, we write this as

$$[\![v]\!] = \pm\sqrt{\langle h^{-1}\rangle}[\![h]\!].$$

To obtain the shock speed, consider the first component of the general Rankine–Hugoniot condition, $[\![vh]\!] = s[\![h]\!]$, from which we get $(s - \langle v\rangle)[\![h]\!] = \langle h\rangle[\![v]\!]$, that is,

$$s = \langle v\rangle + \frac{\langle h\rangle[\![v]\!]}{[\![h]\!]} = \langle v\rangle \pm \langle h\rangle\sqrt{\langle h^{-1}\rangle}.$$

To do: Finish this. I needed to get the Euler equations done first.

Hugoniot locus for the Euler equations

The Euler equations are Galilean invariant. I leave the details to the reader; they are similar to what we did for the shallow water equations, but obviously a bit more involved. For the third equation, you will need to use the invariance of both the first and the second equation.

The Rankine–Hugoniot conditions for a stationary shock for (Eu) are

$$[\![\rho v]\!] = 0, \quad [\![\rho v^2 + p]\!] = 0, \quad \left[\!\!\left[\frac{\gamma v p}{\gamma - 1} + \frac{\rho v^3}{2}\right]\!\!\right] = 0.$$

From the first equation we can write $q = \rho_L v_L = \rho_R v_R$. With $q = 0$ and no vacuum we get $v_L = v_R = 0$ and $[\![p]\!] = 0$. This is the (unsurprising) *contact discontiuity*. So let us assume $q \neq 0$. The other two equations become

$$q[\![v]\!] + [\![p]\!] = 0, \quad \gamma[\![vp]\!] + \tfrac{1}{2}(\gamma - 1)q[\![v^2]\!] = 0.$$

We must have $[\![p]\!] \neq 0$ and $[\![v]\!] \neq 0$, or the solution becomes trivial. The second equation above becomes

$$\gamma(\langle v\rangle[\![p]\!] + \langle p\rangle[\![v]\!]) + (\gamma - 1)q\langle v\rangle[\![v]\!] = 0,$$

and further substituting $[\![p]\!] = -q[\![v]\!]$ and dividing by $[\![v]\!]$ we are left with $\gamma\langle p\rangle = q\langle v\rangle$. Next, substitute $q = -[\![p]\!]/[\![v]\!]$ to arrive at the first inequality below:

$$\frac{[\![p]\!]}{\gamma\langle p\rangle} = -\frac{[\![v]\!]}{\langle v\rangle} = \frac{[\![\rho]\!]}{\langle \rho\rangle}. \tag{1}$$

The second equality comes directly from the expansion of $[\![\rho v]\!] = 0$.

To better understand the implications of (1), consider ρ:

$$\frac{[\![\rho]\!]}{2\langle \rho\rangle} = \frac{\rho_R - \rho_L}{\rho_R + \rho_L} = \frac{\rho_* - 1}{\rho_* + 1} \qquad \text{where } \rho_* = \frac{\rho_R}{\rho_L} \text{ is the \textit{density jump ratio}}$$

and similarly for the other terms. As ρ_* increases from 0 to ∞, the ratio $[\![p]\!]/(2\langle p\rangle)$ increases from -1 to $+1$, and so $[\![v]\!]/(2\langle v\rangle)$ decreases from $+1$ to -1. Actually, the relation $[\![\rho v]\!]$ is written more simply $\rho_L v_L = \rho_R v_R$, implying that $v_* = 1/\rho_*$.

However, the jump ratios p_* and v_* are further constrained, since $|[\![p]\!]/(2\langle p\rangle)| < 1$ implies $|[\![\rho]\!]/(2\langle \rho\rangle)| < 1/\gamma$, and so

$$\frac{\gamma - 1}{\gamma + 1} < \rho_* < \frac{\gamma + 1}{\gamma - 1} \quad \text{and} \quad \frac{\gamma - 1}{\gamma + 1} < v_* < \frac{\gamma + 1}{\gamma - 1} \tag{2}$$

We can now give the full description of stationary shocks, not counting the contact discontinuities and disregarding entropy conditions:

Fix ρ_L, p_L and p_R, and hence also the ratios in (1), and therefore also p_*, ρ_* and $v_* = \rho_*^{-1}$. Thus also ρ_R is determined, and only v_L and v_R remain. Return to the relation $[\![\rho v^2 + p]\!] = 0$ above. Write it as $\rho_R v_L^2 v_*^2 + p_R = \rho_L v_L^2 + p_L$ and note that only v_L is unknown. Write this as $(\rho_* v_*^2 - 1)\rho_L v_L^2 = (1 - p_*)p_L$, and note that $\rho_* v_*^2 = \rho_*^{-1}$, and that $\rho_*^{-1} < 1 \Leftrightarrow \rho_* > 1 \Leftrightarrow [\![\rho]\!] > 0 \Leftrightarrow [\![p]\!] > 0 \Leftrightarrow p_* > 1$. Thus there are two real solutions v_L. The solution with $v_L < 0$ is the *fast* wave (the shock moves to the right relative to the gas), and $v_L > 0$ yields a *slow* wave.

It remains to translate this to a general non-stationary shock. To this end, fix a left state (ρ_L, v_L, p_L) and a pressure ratio $p_* = p_R/p_L$. The stationary shock solution then provides the remaining data for a stationary shock. We will write $\hat{v}_L$ and $\hat{v}_R$ for the resulting gas velocities *relative to the shock*. If s is the shock speed, that means $\hat{v}_L = v_L - s$, and so the shock speed is given, and our solution is complete.

To reiterate, the right state is given by the pressure ratio and the shock family (fast or slow). We don't bother to write down the exact solution here, but it is only a question of chasing through the calculations provided.

However, we do wish to obtain the general shape of the Hugoniot locus. To this end, we return to the stationary shock solutions and rewrite the equation for v_L as

$$(1 - \rho_*^{-1})\rho_L \hat{v}_L^2 = (1 - p_*)p_L,$$

where I added the hat for clarity (these are velocities relative to the shock). What happens as we vary the pressure ratio p_*? First, note that $p_* = 1$ gives $\hat{v}_L = 0$. This is the zero strength "shock". Now keeping ρ_L and p_L constant and remembering the bounds (2), we see that $|\hat{v}_L| \to \infty$ when $p_* \to \infty$, but that $\hat{v}_L$ remains bounded when $p_* \to 0$.

The Riemann problem for the Euler equations

Consider the slow family first. That is, $\hat{v}_L > 0$. We have $s = v_L - \hat{v}_L$ and $v_R = s + \hat{v}_R = v_L + \hat{v}_R - \hat{v}_L = v_L + (v_* - 1)\hat{v}_L$. Recall that v increases in the positive direction along a rarefaction curve. Thus an admissible shock (at least a small one) should have $v_R < v_L$, equivalently $\hat{v}_R < \hat{v}_L$ For the slow family, where these speeds are positive, that means $v_* < 1$, and so $p_* > 1$. If we now let $p_* \to \infty$, we also get $v_L \to \infty$. At the same time, $v_* \to (\gamma - 1)/(\gamma + 1) < 1$, so $v_R \to -\infty$.

Next, consider the fast family, but now turn things around so we keep the *right* state fixed. The analysis is identical to what we did, but with everything turned around. The general picture looks like a mirror image of the above.

To solve the Riemann problem, work in the (v, p) plane. Draw the "$-$" forward wave curve for the left state, and the "$+$" backward wave curve for the right state. Either they meet in a common point (v_m, p_m), or they don't. If they don't, there will be a vacuum in the solution. If they do, take the densities into account: The wave from the left state will have terminated in a state (ρ_1, v_m, p_m) while the wave from the right state ended up in (ρ_2, v_m, p_m) instead. Connect the two with a contact discontinuity.

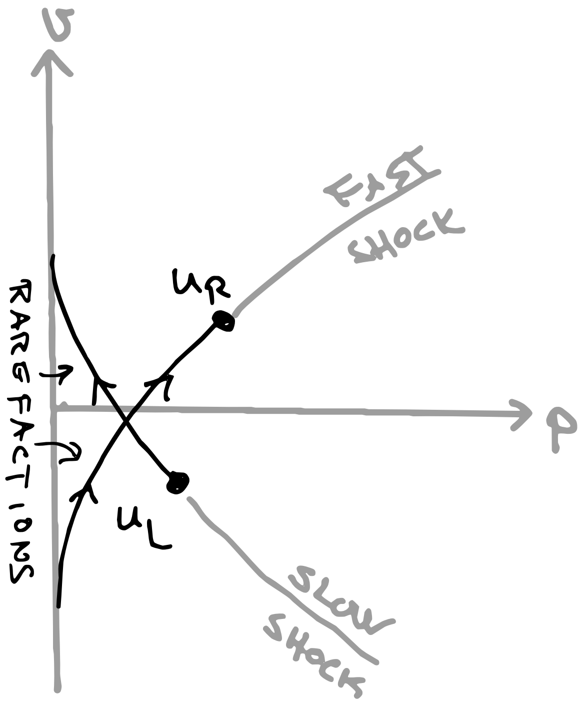

Figure 2: Forward wave curves from a left state and backward curve from a right state, projected in the (v, ρ) plane. In the shown configuration, the Riemann problem is solved using two rarefaction waves plus a contact discontinuity connecting the points of the two wave curves having the same projection in (v, p). Clearly, any combination is possible. A vacuum occurs if you shove the right state u_R in the figure up far enough (the fast wave curve will move with it.)