ENERGY IN THE SWE

RECALL THE SHALLOW WATER EQUATION:

$$\boxed{\begin{aligned} h_t + (hv)_x &= 0 \quad (1a) \\ (hv)_t + \left(hv^2 + \tfrac{1}{2}h^2\right)_x &= 0 \quad (1b) \end{aligned}}$$

First, simplify (1b) using (1a). Expand it:

$$hv_t + \underbrace{h_t v + (hv)_x v}_{} + hvv_x + hh_x = 0$$

these terms cancel by (1a)

Divide the rest by h:

$$\boxed{v_t + vv_x + h_x = 0} \quad (2)$$

Now introduce ENERGY DENSITY

$\boxed{E = \frac{1}{2}hv^2 + \frac{1}{2}h^2}$. Then

$$\begin{aligned} E_t &= hvv_t + \left(\tfrac{1}{2}v^2 + h\right)h_t \\ &= -hv(vv_x + h_x) - \left(\tfrac{1}{2}v^2 + h\right)(vh_x + hv_x) \end{aligned}$$

by (1a) and (2). Rearrange:

$$\begin{aligned} -E_t &= \tfrac{3}{2}hv^2 v_x + \tfrac{1}{2}v^3 h_x + 2vhh_x + h^2 v_x \\ &= \left(\tfrac{1}{2}hv^3 + h^2 v\right)_x \end{aligned}$$

That is,

$$\boxed{E_t + Q_x = 0} \quad \text{where } Q = \tfrac{1}{2}hv^3 + h^2 v.$$

This is ENERGY CONSERVATION for smooth solutions of the SWE.

Fact of life: SWE are Galilei-invariant:

If h, v solve it, so do $\begin{cases} \tilde{h}(x,t) = h(x-\sigma t, t) \\ \tilde{v}(x,t) = v(x-\sigma t, t) + \sigma \end{cases}$

(skip the proof)

Take a stationary shock $(s=0)$:

Then $[\![hv]\!] = 0$, so $hv = q$

is constant across the shock.

$$[\![Q]\!] = [\![\tfrac{1}{2}hv^3 + h^2 v]\!] = q[\![\tfrac{1}{2}v^2 + h]\!]$$
$$= q\left(\langle v\rangle [\![v]\!] + [\![h]\!]\right)$$

Use $[\![v]\!] = \pm\sqrt{\langle h^{-1}\rangle}\,[\![h]\!]$, $\langle v\rangle = \mp\langle h\rangle\sqrt{\langle h^{-1}\rangle}$

(see the rh-sw note – this ↑ because $s=0$)

$$[\![Q]\!] = q\underbrace{\left(-\langle h\rangle\langle h^{-1}\rangle + 1\right)}_{<0 \;!\; (\text{if } [\![h]\!] \neq 0)}[\![h]\!]$$

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$[\![Q]\!] > 0$ means energy is <u>created</u>

$[\![Q]\!] < 0$ means energy is <u>lost</u>

– so we must have $q[\![h]\!] > 0$

in a stationary shock.

The condition $q[\![h]\!] > 0$ can also be written $\langle v\rangle [\![h]\!] > 0$.
Using $[\![v]\!] = \pm\sqrt{\langle h^{-1}\rangle}\,[\![h]\!]$, $\langle v\rangle = \mp\langle h\rangle\sqrt{\langle h^{-1}\rangle}$ we conclude $\boxed{[\![v]\!] < 0}$

This inequality no longer depends on the shock being stationary, so it is universally valid, for shocks of <u>both</u> families (− and +, called 1 and 2 in the book).

Proof of inequality $\langle h^{-1}\rangle\langle h\rangle > 1$ (if $h_R \neq h_L$)

Inequality: $(a-b)^2 > 0$ (if $a \neq b$)
yields $a^2+b^2 > 2ab$. Replacing a, b by $\sqrt{a}, \sqrt{b}$:

$$\frac{a+b}{2} > \sqrt{ab} \quad \text{if } a, b > 0 \text{ and } a \neq b.$$

Replace a, b by a^{-1}, b^{-1}:

$$\frac{a^{-1}+b^{-1}}{2} > \sqrt{a^{-1}b^{-1}}.$$

Multiplying these yields $\boxed{\frac{a+b}{2}\cdot\frac{a^{-1}+b^{-1}}{2} > 1}$