An easier path to the shock curves for a hyperbolic system

Recall: $M(u,u_L)=\int_0^1 f'((1-\alpha)u_L+\alpha u)\,d\alpha$

and so

$f(u)-f(u_L)=s(u-u_L)$

$\Leftrightarrow\ M(u,u_L)(u-u_L)=s(u,u_L)$

For u in a neighbourhood of u_L, $M(u,u_L)$ has distinct real eigenvalues

$\mu_1(u,u_L)<\cdots<\mu_n(u,u_L)$ [see next pages]

so $u-u_L$ must be a multiple of one of the corresponding eigenvectors:

$$u-u_L=\varepsilon\, v_j(u,u_L)$$

This has a solution $u=u(\varepsilon)$ by the implicit function theorem (for $|\varepsilon|$ small):

Put $\Phi(\varepsilon,u)=u-u_L-\varepsilon v_j(u,u_L)$

and note that $\Phi(0,u_L)=0$ and $\frac{\partial\Phi}{\partial u}(0,u_L)=I$

which is invertible. We could write $\Phi(\varepsilon,u_L,u)$ instead and conclude that u is a C^1 function of (ε,u_L).

Smoothness of eigenvalues, -vectors:

Given an $n \times n$ real matrix A with n real and distinct eigenvalues, there is a neighbourhood N of A so that every matrix in N has the same property.
Moreover, the eigenvalues and -vectors are smooth functions on N.

Proof: Let λ be an eigenvalue of A, with eigenvector v.
Define $\Phi : \mathbb{R}^{n\times n} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$ by

$$\Phi(B, w, \mu) = \begin{pmatrix} Bw - \mu w \\ w \cdot v - |v|^2 \end{pmatrix}$$

Then $\Phi(A, v, \lambda) = 0$, and

$$\frac{\partial \Phi}{\partial(w,\mu)}(A, v, \lambda) = \begin{pmatrix} A - \lambda I & -v \\ v^T & 0 \end{pmatrix}$$

If we can show that this matrix is invertible, the implicit function completes the proof.

To show this, take a vector $\begin{pmatrix} w \\ \mu \end{pmatrix}$ in its null space. ($w \in \mathbb{R}^n$, $\mu \in \mathbb{R}$). Thus $(A-\lambda I)w - \mu v = 0$ and $v^T w = 0$.
The range (column space) of $A - \lambda I$ is spanned by the $n-1$ eigenvectors corresponding to the $n-1$ eigenvalues $\neq \lambda$, and v is not in this space. Thus $\mu = 0$, and so w is a multiple of v — since $(A-\lambda I)w = 0$ means w is in the λ-eigenspace. But then the second equation ($v^T w = 0$) implies that $w = 0$. $\square$