Equivalence of Eulerian and Lagrangian viewpoints

To be able to talk about a correspondence between the Eulerian and Lagrangian viewpoints, we need a *velocity field*, and some conserved quantity that is transported by that field. In one space dimension, that means having an equation of the form

$$\rho_t + (\rho v)_x = 0. \tag{1}$$

We know several scalar conservation laws having this form, with v a known function of ρ: But also, many systems contain one equation of this form, with the evolution of v being governed by the other equation(s).

For now, we just asssume that ρ and v are given L^∞ functions, satisfying (1) in the weak sense.

But first, we calculate formally, assuming that v is smooth: We will use x for the spatial (Eulerian) variable, and X for the referential (Lagrangian) variable. We seek a time dependent coordinate transformation between the two, of the form[1]

$$x = \hat{x}(t, X), \qquad X = \widehat{X}(t, x).$$

That X is a *Lagrangian* variable must mean that when we keep X fixed, we follow the velocity field: $\hat{x}_t = v$. This leaves a lot of freedom for scaling X, though. We shall prefer to scale it so that $\widehat{X}_x = \rho$, so that

$$\widehat{X}(t, x_2) - \widehat{X}(t, x_1) = \int_{x_1}^{x_2} \rho(t, x)\, dx$$

measures the "amount of ρ-stuff" (mass, in the typical example) between two points. Equivalently, $\hat{x}_X = \tau$ where $\tau = 1/\rho$. Still assuming smoothness, we easily deduce that $\widehat{X}_t = -\rho v$.

We now abandon our presumption of smoothness, and return to the general L^∞ setting. We are looking for a function $\widehat{X}(t, x)$ satisfying

$$\widehat{X}_x = \rho, \quad \widehat{X}_t = -\rho v. \tag{2}$$

For a solution to (2) to exist, the equations must be consistent: The equality $\widehat{X}_{xt} = \widehat{X}_{tx}$ (which must be satisfied in the sense of distributions) requires that $\rho_t = (-\rho v)_x$ which is, of course, precisely (1). The *existence* of a solution is a standard result in the smooth case. In the general case, it can proved by applying a mollifier to the vector field $(\rho, -\rho v)$ and using the Arselà–Ascoli theorem. The resulting function $\widehat{X}$ will be *Lipschitz*. (This property is crucial in the proof.) By Rademacher's theorem, $\widehat{X}$ is differentiable almost everywhere, and it follows that (2) holds pointwise almost everywhere.

[1] Here I surrender to my personal preference in putting the t variable first.

Assume now that $\rho >$ constant > 0 (i.e., no vacuum or near-vacuum). Then the function $\widehat{X}(t,\cdot)$ has an inverse $\hat{x}(t,\cdot)$, and it will also be a Lipschitz function. The derivative of $(t,x) \mapsto (t,\widehat{X}(x))$ is

$$J(t,x) = \begin{pmatrix} 1 & 0 \\ \widehat{X}_t & \widehat{X}_x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\rho v & \rho \end{pmatrix}, \tag{3}$$

with $\det J = \rho$ and inverse $j(t,X) = J(t,x(t,X))^{-1}$ given by

$$j(t,X) = \begin{pmatrix} 1 & 0 \\ v & \tau \end{pmatrix},$$

implying that $\hat{x}_t = v$ and $\hat{x}_X = \tau$, as expected from our informal calculation. From equality of mixed partials $\hat{x}_{tX} = \hat{x}_{Xt}$ (a.e., and in the sense of distributions) we then conclude that

$$\tau_t - v_X = 0. \tag{4}$$

This, then, is the *Lagrangian* equivalent of (1).

For example, in the classical Lighthill–Whitham–Richards (LWR) traffic model, $\rho_t + (\rho(1-\rho))_x = 0$ in the Eulerian formulation, we have $v = 1 - \rho = 1 - \tau^{-1}$, so the Lagrangian formulation becomes $\tau_t - (1-\tau^{-1})_X = 0$ – or more simply: $\tau_t + (\tau^{-1})_X = 0$.

Lipschitz change of variables in conservation laws. Here we generalize from (t,x) and (t,X) to more general coordinates $Z \in \Omega \subseteq \mathbb{R}^n$ and $z \in \omega \subseteq \mathbb{R}^n$, where $\widehat{Z} : \omega \to \Omega$ and $\hat{z} : \Omega \to \omega$ are Lipschitz and mutual inverses. We write $Z = \widehat{Z}(z)$ and $z = \hat{z}(Z)$.

This time, we start with a balance law[2]

$$\operatorname{div}_z g + h = 0 \tag{5}$$

where $g \in L^1_{\text{loc}}(\omega, \mathbb{R}^n)$ and $h \in L^1_{\text{loc}}(\omega)$. Here, h is a source term.[3] In a conservation law, $h = 0$. The weak formulation of (5) can be written

$$\int_\omega (-g \cdot \nabla_z \varphi + h\varphi)\, d^n z = 0 \qquad \text{for all } \varphi \in W^{1,\infty}_c(\omega). \tag{6}$$

Here, $W^{1,\infty}_c(\omega)$ is the space of all Lipschitz functions on ω with compact support. The usual weak formulation is required to hold only for $\varphi \in C^\infty_c(\omega)$, but an approximation argument easily extends it to all of $W^{1,\infty}_c(\omega)$.

Now we employ a change of variables $z = \hat{z}(Z)$, which transforms (6) into

$$\int_\Omega (-g \cdot \nabla_z \varphi + h\varphi) \circ \hat{z} \det \hat{z}'\, d^n Z = 0$$

[2]Or *system* of balance laws. But for the sake of the variable change, we can deal with just one component at a time.

[3]More appropriately, we should call it a *sink*, as it appears on the left hand side with a plus sign.

where the matrix $\hat{z}'$ is the derivative of $\hat{z}$ [4, section 3.3]. Now write $\Phi = \varphi \circ \hat{z}$ and $\widehat{Z}' = (\hat{z}')^{-1}$ for the derivative of $\widehat{Z}$:

$$G = (\det \hat{z}')\,\widehat{Z}' g \circ \hat{z}, \qquad H = (\det \hat{z}')\,h \circ \hat{z}. \tag{7}$$

Since $\varphi = \Phi \circ \widehat{Z}$, we find $\nabla\varphi = \nabla\Phi \cdot \widehat{Z}'$. Here we adopt an agnostic position as to whether vectors are column or row vectors. Accordingly we can write $g \cdot \nabla\varphi = \nabla\varphi \cdot g = \nabla\Phi \cdot \widehat{Z}' \cdot G$, and so we find in the end that (6) is transformed into

$$\int_\Omega (-G \cdot \nabla_Z \Phi + H\Phi)\, d^n X = 0$$

which is the weak formulation of

$$\operatorname{div}_Z G + H = 0. \tag{8}$$

The situation is entirely symmetric in Z, z, of course, and so we conclude that (5) and (8) are equivalent in the weak formulation.

Example: The shallow water equations. The shallow water equations can be written

$$\begin{aligned} h_t + (hv)_x &= 0, \\ (hv)_t + \Big(hv^2 + \frac{h^2}{2}\Big)_x &= 0. \end{aligned}$$

For the Lagrangian formulation of the first equation, we turn to (4), and find $(h^{-1})_t - v_X = 0$.

The second equation is of the form (5) with $h = 0$ and[4] $g = (hv, hv^2 + \frac{1}{2}h^2)$, where $y = (t, x)$. Furthermore, we find the derivative from (3) with ρ replaced by h.Thus we should get from (7):

$$G = (\det J^{-1})\, Jg = \frac{1}{h}\begin{pmatrix} 1 & 0 \\ -hv & h \end{pmatrix}\begin{pmatrix} hv \\ hv^2 + \frac{1}{2}h^2 \end{pmatrix} = \begin{pmatrix} v \\ \frac{1}{2}h^2 \end{pmatrix},$$

and so the second equation in the Lagrangian formulation becomes $v_t + (\frac{1}{2}h^2)_X = 0$. This can be viewed as a formulation of Newton's second law: The acceleration v_t equals the force (per unit mass) $(\frac{1}{2}h^2)_X$ caused by pressure difference due to varying water depth.

Example: The *p*-system. We start with the first two components of the Euler equations:

$$\begin{aligned} \rho_t + (\rho v)_x &= 0, \\ (\rho v)_t + (\rho v^2 + p) &= 0. \end{aligned} \tag{9}$$

[4] With apologies for the name h having two different meanings in the same line of text!

In the Euler equations, these are supplemented with an equation for energy conservation, along with an equation of state relating energy to ρ, v, and the pressure p. If we assume instead that the gas undergoes only adiabatic changes, we close the system by assuming p to be a given function of ρ, and thus have equations for *isentropic* flow.

Note that these equations have *exactly* the same structure as the shallow water equations, only with h replaced by ρ and $\frac{1}{2}h^2$ (which also represents pressure!) replaced by p. So we can simply adopt that solution, and arrive at the system

$$\begin{aligned} \tau_t - v_X &= 0 \\ v_t + p(\tau)_X &= 0 \end{aligned}$$

for the Lagrangian formulation of the problem. This is known as the p-system, and more commonly written using different letters as[5]

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= 0. \end{aligned}$$

Sources. The classic paper on this topic is [1]. The present note is, however, based on [2]. The later paper [3] may also be noteworthy, but I haven't read it (yet). The book [4] is the go-to reference for many subtle topics in measure and integration theory: It is compactly written and tough going, but highly rewarding.

References

[1] D. H. Wagner, Equivalence of the Euler and the Lagrangian equations of gas dynamics for weak solutions, *J. Diff. Eqs.* **68** (1987), 118–136. doi:10.1016/0022-0396(87)90188-4

[2] C. M. Dafermos, Equivalence of referential and spatial field equations in continuum physics, in: Donato A., Oliveri F. (eds) *Nonlinear Hyperbolic Problems: Theoretical, Applied, and Computational Aspects*, 179–183 (1993). doi:10.1007/978-3-322-87871-7_21

[3] D. H. Wagner, Conservation laws, coordinate transformations, and differential forms, in *Hyperbolic problems: theory, numerics, applications* (Stony Brook, NY, 1994), 471–477 (1996).

[4] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press 1992, ISBN 0-8493-7157-0.

[5] In three space dimensions, the inverse of the density is called *specific volume*, with units m^3/kg. This explains the use of the letter v – clearly not to be confused with a velocity!