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Rankine-Hugoniot locus for the shallow water equations

We first establish some useful identities for dealing with jumps of quantities across a discontinuity. For any quantity u with left and right values u_L abd u_R, define the *jump* and *average* as

$$[\![u]\!] = u_R - u_L, \quad \langle u \rangle = \tfrac{1}{2}(u_R + u_L).$$

We have product rules

$$\begin{aligned}
[\![uv]\!] &= u_R[\![v]\!] + v_L[\![u]\!] && \text{two terms } u_R v_L \text{ cancel} \\
&= u_L[\![v]\!] + v_R[\![u]\!] && \text{interchange } u \text{ and } v \text{ in the above} \\
&= \langle u \rangle [\![v]\!] + \langle v \rangle [\![u]\!] && \text{average the two above,} \\
[\![u^2]\!] &= 2\langle u \rangle [\![u]\!] && \text{special case} \\
\langle uv \rangle &= \langle u \rangle \langle v \rangle + \tfrac{1}{4}[\![u]\!][\![v]\!] && \text{expand the right hand side} \\
\langle u^2 \rangle &= \langle u \rangle^2 + \tfrac{1}{4}[\![u]\!]^2 && \text{special case}
\end{aligned}$$

Consider now the shallow water equations, most conveniently (for our present purposes) written

$$\left.\begin{aligned} h_t + (hv)_x &= 0, \\ (hv)_t + (hv^2 + \tfrac{1}{2}h^2)_x &= 0. \end{aligned}\right\}$$

The Rankine–Hugoniot condition for this system can be written

$$\left.\begin{aligned} s[\![h]\!] &= [\![hv]\!], \\ s[\![hv]\!] &= [\![hv^2 + \tfrac{1}{2}h^2]\!]. \end{aligned}\right\}$$

Cross-multiplying and dividing by s results in

$$[\![h]\!][\![hv^2 + \tfrac{1}{2}h^2]\!] = [\![hv]\!]^2$$

(also valid for $s = 0$). To simplify this, substitute

$$\begin{aligned}
[\![hv^2]\!] &= [\![h]\!]\langle v^2 \rangle + \langle h \rangle [\![v^2]\!] = [\![h]\!]\langle v \rangle^2 + \tfrac{1}{4}[\![h]\!][\![v]\!]^2 + 2\langle h \rangle \langle v \rangle [\![v]\!], \\
[\![hv]\!]^2 &= \big(\langle h \rangle [\![v]\!] + \langle v \rangle [\![h]\!]\big)^2 = \langle h \rangle^2 [\![v]\!]^2 + [\![h]\!]^2 \langle v \rangle^2 + 2\langle h \rangle \langle v \rangle [\![h]\!][\![v]\!]
\end{aligned}$$

and $[\![h]\!]^2 = 2\langle h \rangle [\![h]\!]$ to get

$$[\![h]\!]^2\langle v \rangle^2 + \tfrac{1}{4}[\![h]\!]^2[\![v]\!]^2 + 2\langle h \rangle \langle v \rangle [\![h]\!][\![v]\!] + \langle h \rangle [\![h]\!]^2 = \langle h \rangle^2 [\![v]\!]^2 + [\![h]\!]^2 \langle v \rangle^2 + 2\langle h \rangle \langle v \rangle [\![h]\!][\![v]\!],$$

which simplifies into

$$\Big(\langle h \rangle^2 - \frac{1}{4}[\![h]\!]^2\Big)[\![v]\!]^2 = \langle h \rangle [\![h]\!]^2.$$

But $\langle h \rangle^2 - \dfrac{1}{4}[\![h]\!]^2 = h_R h_L$, and $\langle h \rangle/(h_R h_L) = \langle h^{-1} \rangle$, so this simplifies further to $[\![v]\!]^2 = \langle h^{-1} \rangle [\![h]\!]^2$, or

$$[\![v]\!] = \pm\sqrt{\langle h^{-1} \rangle}[\![h]\!].$$

We can also find the shock speed s from the first of the two R–H conditions, employing the product rule in the numerator:

$$s = \frac{[\![vh]\!]}{[\![h]\!]} = \langle v \rangle + \langle h \rangle \frac{[\![v]\!]}{[\![h]\!]} = \langle v \rangle \pm \langle h \rangle \sqrt{\langle h^{-1} \rangle}.$$

Using the unsymmetric rules instead of the symmetric ones, we can also write the shock speed as

$$s = v_L \pm h_R\sqrt{\langle h^{-1} \rangle} = v_R \pm h_L\sqrt{\langle h^{-1} \rangle}.$$