Rankine-Hugoniot locus for the shallow water equations

We first establish some useful identities for dealing with jumps of quantities across a discontinuity. For any quantity $u$ with left and right values $u_L$ and $u_R$, define the jump and average as

$$ [u] = u_R - u_L, \quad \langle u \rangle = \frac{1}{2}(u_R + u_L). $$

We have product rules

$$ [uv] = u_R [v] + v_R [u] \quad \text{two terms } u_R v_L \text{ cancel} $$

$$ = u_L [v] + v_R [u] \quad \text{interchange } u \text{ and } v \text{ in the above} $$

$$ = \langle u \rangle [v] + \langle v \rangle [u] \quad \text{average the two above,} $$

$$ [u^2] = 2\langle u \rangle [u] \quad \text{special case} $$

$$ \langle uv \rangle = \langle u \rangle \langle v \rangle + \frac{1}{4} [u] [v] \quad \text{expand the right hand side} $$

$$ \langle u^2 \rangle = \langle u \rangle^2 + \frac{1}{4} [u]^2 \quad \text{special case} $$

Consider now the shallow water equations, most conveniently (for our present purposes) written

$$ \frac{h_t}{h} + (hv)_x = 0, $$

$$ (hv)_t + (h^2 + \frac{1}{2}h^2)_x = 0. $$

The Rankine–Hugoniot condition for this system can be written

$$ s[h] = [hv], $$

$$ s[hv] = [hv^2 + \frac{1}{2}h^2]. $$

Cross-multiplying and dividing by $s$ results in

$$ [h][hv^2 + \frac{1}{2}h^2] = [hv]^2 $$

(also valid for $s = 0$). To simplify this, substitute

$$ [hv^2] = [h]\langle v^2 \rangle + (h)[v^2] = [h]\langle v \rangle^2 + \frac{1}{4}[h][v]^2 + 2(h)\langle v \rangle [v], $$

$$ [hv]^2 = ((h)\langle v \rangle + (v)\langle h \rangle)^2 = \langle h \rangle^2 \langle v \rangle^2 + [h]^2 \langle v \rangle^2 + 2(h)\langle v \rangle [h][v] $$

and $[h]^2 = 2(h)[h]$ to get

$$ [h]^2\langle v^2 \rangle + \frac{1}{4}[h]^2[v]^2 + 2(h)\langle v \rangle [h][v] + (h)[h]^2 = \langle h \rangle^2 \langle v \rangle^2 + [h]^2 \langle v \rangle^2 + 2(h)\langle v \rangle [h][v]. $$

which simplifies into

$$ (\langle h \rangle^2 - \frac{1}{4}[h]^2)[v]^2 = \langle h \rangle[h]^2. $$

But $\langle h \rangle^2 - \frac{1}{4}[h]^2 = h_R h_L$, and $(h)/(h_R h_L) = (h^{-1})$, so this simplifies further to $[v]^2 = (h^{-1})[h]^2$, or

$$ [v] = \pm \sqrt{(h^{-1})[h]}.$$

We can also find the shock speed $s$ from the first of the two R–H conditions, employing the product rule in the numerator:

$$ s = \frac{[vh]}{[h]} = \langle v \rangle + \langle h \rangle \frac{[v]}{[h]} = \langle v \rangle \pm \langle h \rangle \sqrt{(h^{-1})}. $$

Using the unsymmetric rules instead of the symmetric ones, we can also write the shock speed as

$$ s = v_L \pm h_R \sqrt{(h^{-1})} = v_R \pm h_L \sqrt{(h^{-1})}. $$