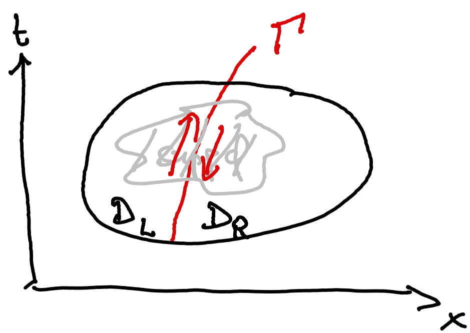
MA 8103 2022-01-18

Svak løsn av

$$u_t + f(u)_x = 0$$



Svak løsn:

$$\int_0^\infty \int_{\mathbb{R}} (u\varphi_t + f(u)\varphi_x)\,dx\,dt = \int_{\mathbb{R}} u_0(x)\varphi(x,t)\,dx$$

for alle $\varphi \in C_c^\infty(\mathbb{R}\times(0,\infty))$

$$0 = \int_0^\infty \int_{\mathbb{R}} (u\varphi_t + f(u)\varphi_x)\,dx\,dt$$

$$= \iint_D \left((u\varphi)_t + (f(u)\varphi)_x\right)\,dx\,dt$$

$$- \iint_D \underbrace{(u_t + f(u)_x)}_{=0}\varphi\,dx\,dt$$

$$= \oint_{\partial D} (-u\varphi\,dx + f(u)\varphi\,dt)$$

$$= \int_\Gamma (u_R\,dx - f(u_R)\,dt)\varphi$$

$$- \int_\Gamma (u_L\,dx - f(u_L)\,dt)\varphi$$

Green: $\iint_D (Q_x - P_t)\,dx\,dt = \oint_{\partial D} P\,dx + Q\,dy$

$$0 = \int_\Gamma (u_R\,dx - f(u_R)\,dt)\,\varphi$$
$$-\int_\Gamma (u_L\,dx - f(u_L)\,dt)\,\varphi = \cdots$$

$$\left(\begin{array}{l} [u] \approx [u_R - u_L] \\ [f(u)] = [f(u_R) - f(u_L)] \\ \\ dx = s\,dt \\ \quad\ \uparrow \text{ sjokkhastighet} \\ \qquad s(t) \end{array}\right)$$

$$\approx \int_0^\infty (\underbrace{[u]s - [f(u)]}_{\text{må være null!}})\,\varphi\,dt$$

for alle $\varphi \in C_c^\infty(\mathbb{R}\times(0,\infty))$

SJOKKBETINGELSEN RANKINE-HUGONIOT:

$$[f(u)] = s\,[u]$$

EKSEMPEL: BURGERS:

$$f(u) = \tfrac{1}{2}u^2$$

$$[f(u)] = \tfrac{1}{2}(u_R^2 - u_L^2)$$

$$= \tfrac{1}{2}(u_R + u_L)\underbrace{(u_R - u_L)}_{[u]}$$

$$s = \tfrac{1}{2}(u_R + u_L) \approx u$$

når $u_R \approx u_L \approx u$

<u>Små</u> sjokk:

$$s = \frac{[f(u)]}{[u]}$$

$$= \frac{f(u_R) - f(u_L)}{u_R - u_L}$$

$$= f'(u)$$

for en u mellom u_R og u_L

Små sjokk beveger seg med <u>ca</u> karakteristisk hastighet!

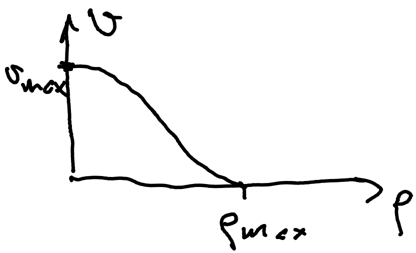
En trafikkmodell:

$$\int_a^b \rho(x,t)\,dx$$

Fluks: $v(x,t)\rho(x,t)$

$$\rho_t + (\rho v)_x = 0$$

La $v(x,t) = v(\rho(x,t))$



Vanlig: $v = v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right)$

Innfør $u = \frac{\rho}{\rho_{max}}$

$x = X x^*$, $t = T t^*$

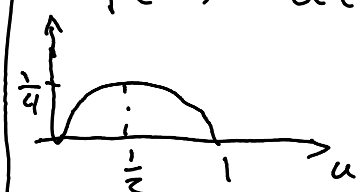
$$\frac{1}{T}\cancel{\rho_{max}} u_{t^*} + \frac{1}{X}\cancel{\rho_{max}} v_{max}\left((1-u)u\right)_{x^*} = 0$$

$\frac{X}{T} = v_{max}$

$$u_{t^*} + ((1-u)u)_{x^*} = 0$$

dropp $*$; $u_t + (u(1-u))_x = 0$

$$f(u) = u(1-u) = u - u^2$$

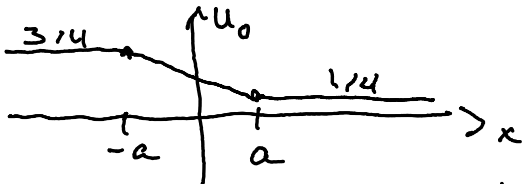


$f'(u) = 1 - 2u$

Hastighet til hver bil:

$v = 1 - u > f'(u)$

Initial data : $u(x,0)=u_0(x)$



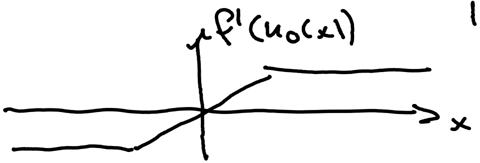
$f'(u) = 1-2u$

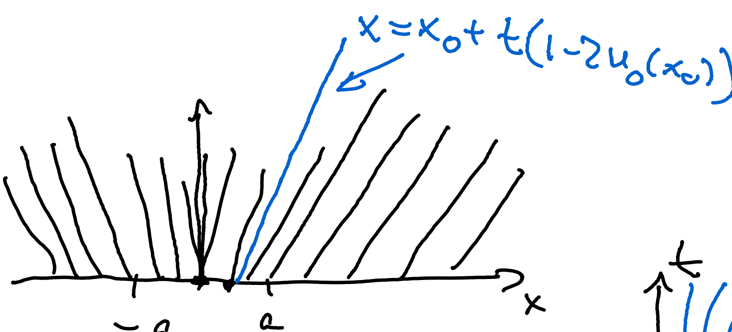
$f'(u_0) = 1-2u_0$

For $|x_0|<a$:

$u_0(x) = \frac{1}{2} - \frac{x}{4a}$

$1-2u_0(x) = \frac{x}{2a}$





$$x = x_0 + t\,\frac{x_0}{2a} = \left(1+\frac{t}{2a}\right)x_0$$

$$x_0 = \frac{x}{1+\frac{t}{2a}}$$

$$u(x,t) = u_0(x_0) = \frac{1}{2} - \frac{x}{4a+2t}$$

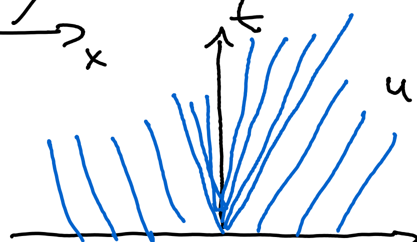
for $-a-\frac{1}{2}t < x < a + \frac{1}{2}t$

$$u(x,t) = \frac{1}{4} \qquad x > a + \frac{1}{2}t$$

$$= \frac{3}{4} \qquad x < -a - \frac{1}{2}t$$

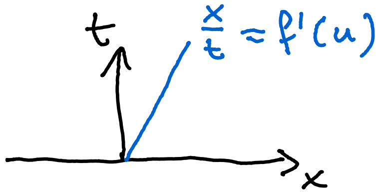
La $a \to 0$!

$$u(x,t) = \begin{cases} \frac{1}{2} - \frac{x}{2t} & \text{for } |x| < \frac{1}{2}t \\ \frac{1}{4} & \text{for } x > \frac{1}{2}t \\ \frac{3}{4} & x < -\frac{1}{2}t \end{cases}$$



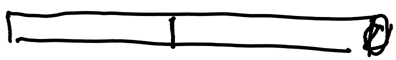
FORTYNNINGSBØLGE

$$\frac{x}{t} = 1 - 2u = f'(u)$$



$$u_0(x) = \begin{cases} u_L & x < 0 \\ u_R & x > 0 \end{cases}$$

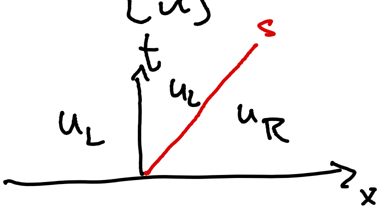
RIEMANNPROBLEM



Sjokkløsning:

Sjokk med hastighet

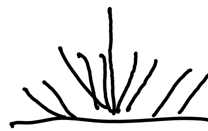
$$s = \frac{[f(u)]}{[u]}$$

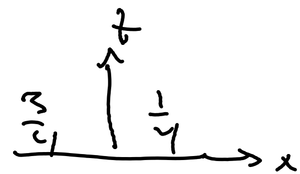


$u_L = \frac{3}{4}$, $f(u_L) = u_L(1-u_L) = \frac{3}{4}\cdot\frac{1}{4} = \frac{3}{16}$

$u_R = \frac{1}{4}$ $f(u_R) = u_R(1-u_R) = \frac{1}{4}\cdot\frac{3}{4} = \frac{3}{16}$

$[f(u)] = 0$, $s = 0$





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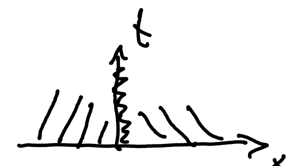
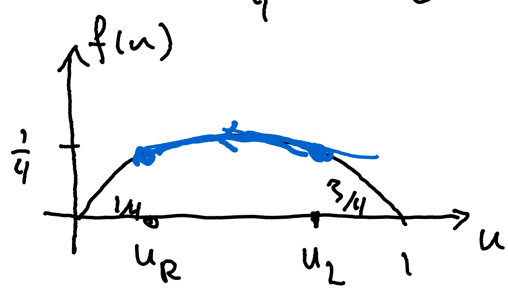
ENTROPI skiller de to!

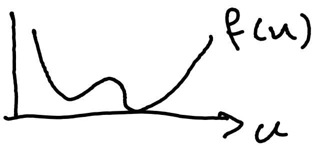
$$u_0 = \begin{cases} \frac{1}{4} & x < 0 \\ \frac{3}{4} & x > 0 \end{cases}$$





Kap 2.

$u_t + f(u)_x = 0$



Viskøs regularisering:

$u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx}$ (ε)

La $\varepsilon \to 0$, kryss fingrene

$u^\varepsilon \xrightarrow{?} u$

Riemannproblemet?

$\rightsquigarrow$ Har (ε) en bølgeløsning?

gjerne $\lim_{x\to\infty} u^\varepsilon = u_R$

$\lim_{x\to-\infty} u^\varepsilon = u_L$

Skalering: $x = \varepsilon x^*$, $t = \varepsilon t^*$

$\cancel{\frac{1}{\varepsilon}} u^\varepsilon_{t^*} + \cancel{\frac{1}{\varepsilon}} f(u^\varepsilon)_{x^*} = \cancel{\frac{\varepsilon}{\varepsilon^2}} u_{x^*x^*}$

$u^\varepsilon_{t^*} + f(u^\varepsilon)_{x^*} = u^\varepsilon_{x^*x^*}$

Prøv: $u^\varepsilon(x,t) = U(x^* - st^*)$

$$\boxed{u^\varepsilon(x,t) = U\left(\frac{x-st}{\varepsilon}\right)}$$

$-sU' + (f\circ U)' = U''$

$$\boxed{U' = (f\circ U) - sU + A}$$

Ønsker oss $U(\infty) = u_R$

$\lim\ldots \xrightarrow{?} U(-\infty) = u_L$

dvs. $f(u_R) - su_R + A = 0$

$f(u_L) - su_L + A = 0$

$[f(u)] = s[u]$

R–H.

$u^\varepsilon(x,t) = U\left(\frac{x-st}{\varepsilon}\right)$

$$\to \begin{cases} u_R & x > st \\ u_L & x < st \end{cases}$$

når $\varepsilon \to 0$

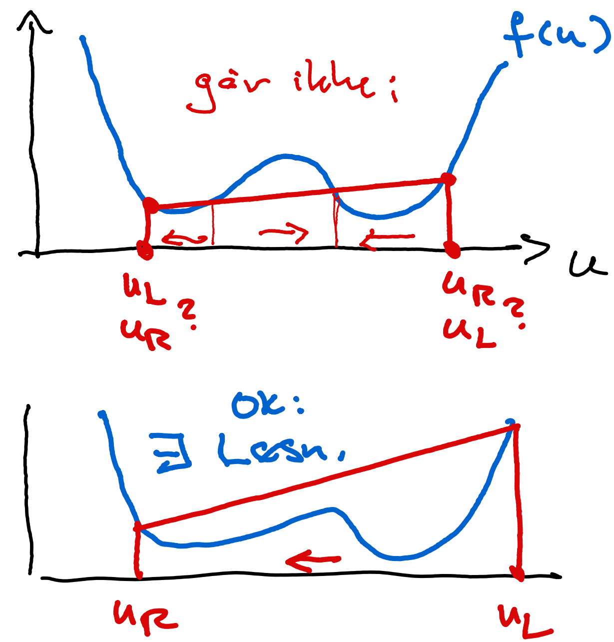
SPM: Finnes U ?

$U' = f \circ U - sU + A$

<u>der</u> $f(u_R) - su_R + A = 0$

$f(u_L) - su_L + A = 0$

$$\underbrace{f(u)}_{\text{fluks}} - \underbrace{(su - A)}_{\text{sekanten}}$$



$\xi = \frac{x-st}{\varepsilon}$

$\xi \to \pm\infty$