Note: This pdf will be updated with more problem solutions with time.
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3.1 Consider the difference scheme (3.4). Show that if u^0 is given by

$$u_j^0 = \begin{cases} 0 & \text{for } j < 0, \quad j > 0 \\ 1 & \text{for } j \geq 0, \quad j \leq 0 \end{cases}$$

then $u^n = u^0$ for all n, thus indicating the solution $u(x,t) = \chi_{[0,\infty)}$ $[x \leq 0]$. Determine the weak entropy solution.

$$u_t + u u_x = 0$$

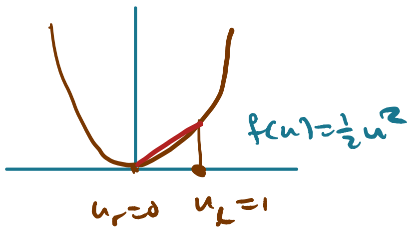
$$u_j^{n+1} = u_j^n - \lambda u_j^n \left(u_j^n - u_{j-1}^n \right), \tag{3.4}$$

$u_j^n = 0 \Rightarrow u_j^{n+1} = 0$, so $u_j^n = 0$ for all $j > 0$, $n \geq 0$ (by induction on n)

$u_j^n = u_{j-1}^n \Rightarrow u_j^{n+1} = u_j^n$, so $u_j^n = 1$ for all $j \leq 0$, $n \geq 0$ (induction on n)

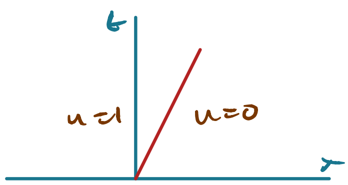
in the limit, $u(x,t) = [x \leq 0]$ (Iverson bracket)

Correct entropy solution



Shock w speed

$$\sigma = \frac{\frac{1}{2}u_r^2 - \frac{1}{2}u_\ell^2}{u_r - u_\ell} = \frac{\frac{1}{2}0^2 - \frac{1}{2}1^2}{0 - 1} = \frac{1}{2}$$



$$u(x,t) = \left[x \leq \tfrac{1}{2} t \right]$$

Problem 3.3: Engquist – Osher scheme

Generalised Upwind

$$F(u,v) = \int_0^u f'(s)\vee 0 \, ds + \int_0^v f'(s)\wedge 0 \, ds + f(0)$$

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if $f' \geq 0$ then $F(u,v) = \int_0^u f' \, ds + f(0) = f(u)$ upwind

if $f' \leq 0$ then $F(u,v) = \int_0^v f' \, ds + f(0) = f(v)$ upwind

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(a) Consistent:

F is Lipschitz since f' is bounded

$$F(u,u) = \int_0^u (f'\vee 0 + f'\wedge 0)\, ds + f(0)$$

$$= \int_0^u f' \, ds + f(0) = f(u)$$

Monotone:

$$u_j^{n+1} = G_j(u^n) = u_j^n - \lambda\left(F(u_j^n, u_{j+1}^n) - F(u_{j-1}^n, u_j^n)\right)$$

Must show $\partial_{u_i} G_j(u) \geq 0$ for all i $(i = j-1, j+1, j)$

$$\partial_{u_{j-1}} G_j(u) = \lambda\, \partial_1 F(u_{j-1}, u_j) = \lambda\, f'(u_{j-1})\vee 0 \geq 0$$

$$\partial_{u_{j+1}} G_j(u) = -\,\partial_2 F(u_j, u_{j+1}) = -\lambda\, f'(u_{j+1})\wedge 0 \geq 0$$

$$\partial_{u_j} G_j(u) = 1 - \lambda\left(\partial_1 F(u_j, u_{j+1}) - \partial_2 F(u_{j-1}, u_j)\right)$$

$$= 1 - \lambda\left(f'(u_j)\vee 0 - f'(u_j)\wedge 0 \right)$$

$$= 1 - \lambda\, |f'(u_j)|$$

$$\geq 0 \Leftarrow \text{CFL}$$

(c) $$F(u,v) = \int_0^u f'(s)\vee 0 \, ds + \int_0^v f'(s)\wedge 0 \, ds + f(0)$$

$$F(u,u) = f(u)$$

$$\partial_2 F(u,v) = f'(v)\wedge 0$$

$$F(u,v) = f(u) + \int_u^v f'(s)\wedge 0 \, ds$$

$$= f(u) + \int_u^v \tfrac{1}{2}\left(f'(s) - |f'(s)|\right) ds$$

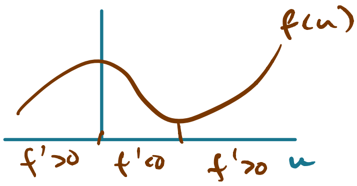
$$= f(u) + \tfrac{1}{2}\left(f(v) - f(u) - \int_u^v |f'| \, ds\right)$$

$$F(u,v) = \tfrac{1}{2}\left(f(u) + f(v) - \int_u^v |f'| \, ds\right)$$

If $f' \geq 0$ between u and v then $F(u,v) = f(u)$

If $f' \leq 0$ —— " —— then $F(u,v) = f(v)$

(b) Theorem 3.1: If F is C^3, method consistent and monotone, then the method is at most first order accurate.



Upwind schemes are 1st order

EO is at most first order!

Consider smooth sol'n $u(x,t)$

$$L_{\Delta t}(x) = \frac{1}{\Delta t}\Big(u(x,t+\Delta t) - u(x,t) + \lambda\big(F(u(x),u(x+\Delta x)) - F(u(x-\Delta x),u(x))\big)\Big)$$

$$= \overbrace{\frac{1}{\Delta t}\left(u(x,t+\Delta t) - u(x,t)\right)}^{A}$$

$$\underbrace{+ \frac{1}{\Delta x}\left(F(u(x),u(x+\Delta x)) - F(u(x-\Delta x),u(x))\right)}_{B}$$

Only possible doubt: $\left.\begin{array}{l} f'(u(x,t)) = 0 \\ u_t + f'(u)u_x = 0 \end{array}\right\} \to u_t(x,t) = 0$

$\Rightarrow A = O(\Delta t)$

$$B = \frac{1}{2\Delta x}\Big(\cancel{f(u(x))} + f(u(x+\Delta x)) - \underbrace{\int_{u(x)}^{u(x+\Delta x)} |f'(s)|\,ds}_{O(\Delta x^2)}$$

$$- f(u(x-\Delta x)) - \cancel{f(u(x))} + \underbrace{\int_{u(x-\Delta x)}^{u(x)} |f'(s)|\,ds}_{O(\Delta x^2)} \Big)$$

$f'(u(x)) = 0$
$\Downarrow$
$f'(s) = O(|s - u(x)|)$

$$= \underbrace{\frac{1}{2\Delta x}\big(f(u(x+\Delta x)) - f(u(x-\Delta x))\big)}_{\partial_x f(u(x))|_{x=\theta}} + O(\Delta x)$$

$\theta \in (x-\Delta x, x+\Delta x)$

$= O(\Delta x)$ because $\partial_x f(u) = f'(u)u_x = 0$ at x

$= O(\Delta x)$

(d) Burgers: $f(u) = \frac{1}{2}u^2 \qquad f'(u) = u$

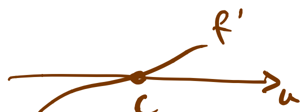
$$F(u,v) = \int_0^u s \vee 0 \, ds + \int_0^v s \wedge 0 \, ds + 0$$

$$= \frac{1}{2}(u \vee 0)^2 + \frac{1}{2}(v \wedge 0)^2$$

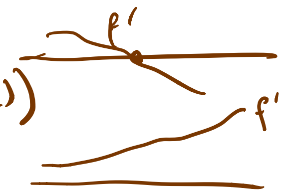
If $f'' \neq 0$ then $f'' \gtrless 0$: f' is monotone

$$|f'(u)| = \pm \operatorname{sgn}(u-c) f'(u)$$

$$\int |f'| \, du = \pm \operatorname{sgn}(u-c) f(u)$$



$c \in [-\infty, \infty]$



$$F(u,v) = \frac{1}{2}\big(f(u) + f(v) \mp \operatorname{sgn}(v-c) f(v) \pm \operatorname{sgn}(u-c) f(u)\big)$$

$$= [\pm(u-c) > 0] f(u) + [\mp(v-c) > 0] f(v)$$

$$= [u \gtrless c] f(u) + [v \lessgtr c] f(v)$$