Eulerian / Spatial vs Lagrangian / Referential

2020 week 14
Friday part 1

References: (see the course page)

Wagner (1987)

Dafermos (1999)

also the book: Evans & Gariepy: Measure theory and the fine properties of functions

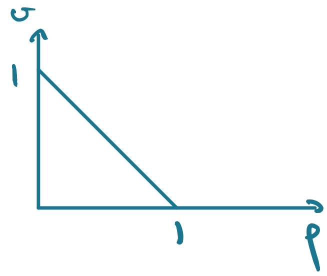
---

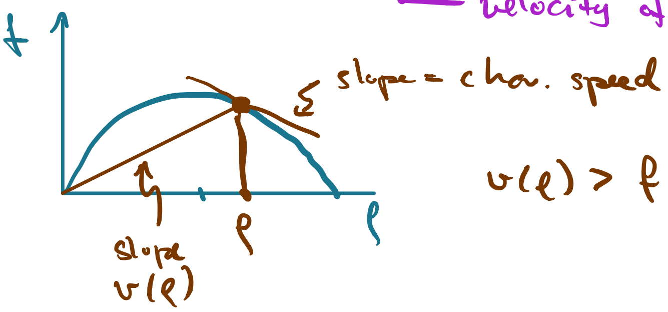
Example: Traffic flow. (LWR)

$$\rho_t + f(\rho)_x = 0$$

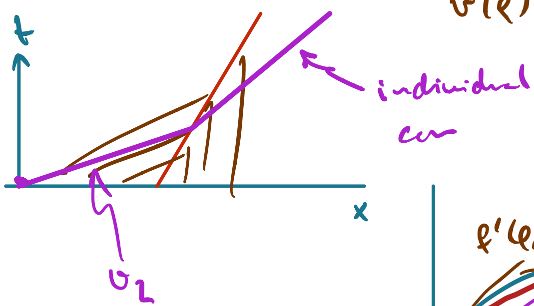
$$f(\rho) = \rho v(\rho)$$

velocity of cars!

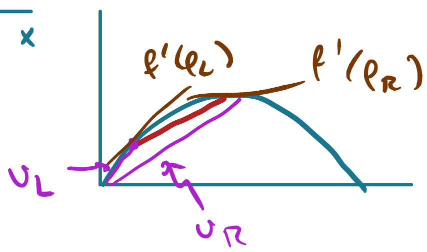


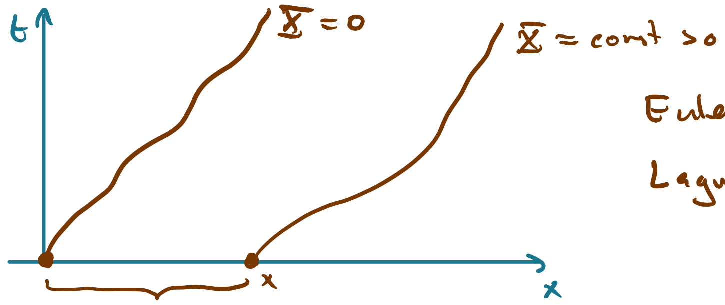


$$v(\rho) > f'(\rho)$$



$$\dot{x}(t) = v(\rho(x(t),t)) =: v(x,t)$$





$\bar X = \int_0^x \rho_0(\xi)\,d\xi$

Eulerian coord: x

Lagrangian coord: X

$$X = \hat X(x,t)$$
$$x = \hat x(X,t)$$

$$\hat X(x,t) = \int_{\hat x(0,t)}^{x} \rho(\xi,t)\,d\xi$$

$$\hat X(\hat x(X,t),t) = X$$
$$0 = \frac{\partial}{\partial t}\hat X(\hat x(X,t),t) = \hat X_x \hat x_t + \hat X_t = \rho v + \hat X_t$$

$$\hat x_t = v(\hat x, t) \quad (1)$$
$$\hat X_x = \rho(x,t) \quad (2)$$
$$\hat X_t = -\rho v \quad (3)$$

$$\hat X_{xt} = \hat X_{tx} \iff \rho_t = -(\rho v)_x \iff \rho_t + (\underbrace{\rho v}_{f(\rho)})_x = 0$$

Pick std mollifier $w(x,t)$, $w_\varepsilon(x,t) = \frac{1}{\varepsilon^2} w(\frac{x}{\varepsilon}, \frac{t}{\varepsilon})$

$$(w_\varepsilon * \rho)_t + (w_\varepsilon * (\rho v))_x = w_\varepsilon * \underbrace{(\rho_t + (\rho v)_x)}_{\text{weak deriv.} \dots = 0} = 0$$

$\Downarrow$

$\exists\, \hat X^\varepsilon(x,t)$: $\hat X^\varepsilon_x = w_\varepsilon * \rho$
$\hat X^\varepsilon_t = -w_\varepsilon * (\rho v)$

Assume $\|\rho\|_\infty \le M$, $\|\rho v\|_\infty \le M$. So $\|\hat X^\varepsilon_x\|_\infty \le M$, $\|\hat X^\varepsilon_t\|_\infty \le M$

i.e. $\hat X^\varepsilon$ is Lipschitz. Arzelà–Ascoli:

$\exists\, \varepsilon_k \to 0$, $\hat X^\varepsilon \to \hat X$ locally uniformly to a Lischitz limit.

have $\hat{X}^{\varepsilon}_x = \omega_\varepsilon * \rho$

$\hat{X}^{\varepsilon}_t = -\omega_\varepsilon * (\rho v)$

want $\hat{X}_x = \rho(x,t)$ (2)

$\hat{X}_t = -\rho v$ (3)

$$-\iint \hat{X}\,\varphi_x \,dx\,dt = -\lim_{h\to\infty}\iint \hat{X}^{\varepsilon_h}\varphi_x\,dx\,dt$$

test fn.

$$= \lim_{h\to\infty}\iint \hat{X}^{\varepsilon_h}_x\,\varphi\,dx\,dt$$

$$= \lim_{h\to\infty}\iint \underbrace{(\omega_{\varepsilon_h} * \rho)}_{\to \rho \text{ in } L^1_{loc}}\varphi\,dx\,dt$$

$$= \iint \rho\,\varphi\,dx\,dt$$

so $\hat{X}_x = \rho$ in the weak sense!

Rademacher's theorem:

A (locally) Lipschitz function $\mathbb{R}^n \to \mathbb{R}^m$ is differentiable almost everywhere!

Meaning: $f(x+y) = f(x) + \underbrace{Df(x)}_{\text{matrix}}\cdot y + o(|y|)$, $y \to 0$

for almost every x

Moreover, a Lipschitz function belongs to $W^{1,\infty}_{loc}$ (it's in L^∞_{loc}, with 1^{st} order partial derivatives in L^∞_{loc}) the weak derivative = the classical derivative.

2020 week 14
Friday part 1

Derivation of the map $(x,t) \to (\bar{X}, t)$ is:

$$\begin{pmatrix} \hat{X}_x & \hat{X}_t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \rho & -\rho v \\ 0 & 1 \end{pmatrix}$$

with inverse

$$\begin{pmatrix} \hat{x}_X & \hat{x}_t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \rho & -\rho v \\ 0 & 1 \end{pmatrix}^{-1} = \frac{1}{\rho}\begin{pmatrix} 1 & \rho v \\ 0 & \rho \end{pmatrix} = \begin{pmatrix} 1/\rho & v \\ 0 & 1 \end{pmatrix}$$

so
$$\hat{x}_X = \frac{1}{\rho} =: \tau$$
$$\hat{x}_t = v \quad \longleftarrow \text{ speed of car at } \bar{X}, t$$

$$\hat{x}_{Xt} = \hat{x}_{tX} \iff \tau_t = v_X$$

Lagrangian form of conservation $\left\{ \underbrace{\tau_t - v_X = 0}_{\text{conservation law!}} \right.$

$$\int_{X_1}^{X_2} \tau(X,t)\, dX = \int_{X_1}^{X_2} \hat{x}_X \, dX = \underbrace{\hat{x}(X_2,t) - \hat{x}(X_1,t)}_{\text{distance btw cars}}$$

$$\frac{d}{dt}\int_{X_1}^{X_2} \tau(X,t)\, dX = \int_{X_1}^{X_2} \tau_t \, dX = \int_{X_1}^{X_2} v_X \, dX = \underbrace{v(X_2,t) - v(X_1,t)}_{\text{speed diff.}}$$

Coordinate change, general setting

2020 week 14
Friday part 2

Replace $\left.\begin{array}{l}(t,x)\in\mathbb{R}^2 \\ (t,\xi)\in\mathbb{R}^2\end{array}\right\}$ by $\left\{\begin{array}{l} y\in\omega\subseteq\mathbb{R}^n \\ Y\in\Omega\subseteq\mathbb{R}^n\end{array}\right.$

$$\omega \underset{\hat{y}}{\overset{\hat{Y}}{\rightleftarrows}} \Omega$$

$\hat{y}$, $\hat{Y}=(\hat{y})^{-1}$ Lipschitz

bi-Lipschitz

Scalar Balance law: $\operatorname{div} G(Y)+H(Y)=0 \quad Y\in\Omega)$

$G:\Omega\to\mathbb{R}^n$

$H:\Omega\to\mathbb{R}$

$$\sum_j \frac{\partial G_j}{\partial Y_j}+H=0$$

Weak form: $\displaystyle\int_\Omega \Big(-\sum_j G_j\frac{\partial\Phi}{\partial Y_j}+H\Phi\Big)\,dY=0$

$G(Y)\in\mathbb{R}^n$: a column vector

$$\int_\Omega(-\nabla\Phi\cdot G+H\Phi)\,dY=0$$

$\Phi\in C_c^\infty(\Omega)$ ... better: $\Phi\in W_0^{1,1}(\Omega)$

Change variables:

$$\int_\omega(-\nabla\Phi\cdot G+H\Phi)(\hat{Y}(y))\cdot(\det D\hat{Y})\,dy=0$$

Evans & Gariepy p.99

Put $\varphi(y)=\Phi(\hat{Y}(y))$

$$\frac{\partial\varphi}{\partial y_k}=\sum_j\frac{\partial\Phi}{\partial Y_j}\frac{\partial\hat{Y}_j}{\partial y_k}=\sum_j\frac{\partial\Phi}{\partial Y_j}(D\hat{Y})_{jk}=(\nabla\Phi\cdot D\hat{Y})_k$$

$$\nabla\varphi=\nabla\Phi\cdot D\hat{Y}$$

$$\nabla\Phi=\nabla\varphi\cdot(D\hat{Y})^{-1}=\nabla\varphi\cdot D\hat{y}$$

Put $\quad g(y) = (\det D\hat{Y})\, D\hat{y} \cdot G(Y) \qquad Y = \hat{Y}(y)$

$$h(y) = (\det D\hat{Y})\, H(Y)$$

$$\int_\omega (-\nabla\varphi \cdot g + h\varphi)\, dy = 0$$

$$G(Y) = (\det D\hat{y})\, D\hat{Y} \cdot g(y)$$

$$H(Y) = (\det D\hat{y})\, h(y)$$

End of part 2. There will be a part 3, and this file will be updated when that is done.