2020-02-08

Notes after yesterday's lecture

In the Friday lecture, I introduced a very slight variant of the Crandall–Tartar lemma (here in abbreviated form):

Lemma 1 (Crandall–Tartar)**.** *If $D \subseteq L^1(\Omega)$ is closed under the pointwise maximum operator $\vee$ and $T\colon D \to L^1(\Omega)$ satisfies*

$$\int_\Omega T(\phi) = \int_\Omega \phi + c \qquad (\phi \in D), \tag{$\bullet$}$$

then the following are equivalent:

(*i*) *for all ϕ, $\psi \in D$, $\phi \le \psi \Rightarrow T(\phi) \le T(\psi)$,*

(*ii*) *for all ϕ, $\psi \in D$, $\displaystyle\int_\omega \bigl(T(\phi) - T(\psi)\bigr)^+ \le \int_\omega (\phi - \psi)^+$,*

(*iii*) *for all ϕ, $\psi \in D$, $\displaystyle\int_\omega \bigl|T(\phi) - T(\psi)\bigr| \le \int_\omega |\phi - \psi| e$.*

The "slight variant" here is the additive constant c, which I think could be useful in some applications.

Proof. The equivalence of (*ii*) and (*iii*) follows trivially from ($\bullet$) and the identity $a^+ = \frac{1}{2}(|a| + a)$.

To show that (*i*) implies (*ii*), we rewrite (*ii*) as

$$\int_\Omega \bigl(T(\phi) \vee T(\psi) - T(\psi)\bigr) \le \int_\Omega (\phi \vee \psi - \psi).$$

Note that $\phi, \psi \le \phi \vee \psi$, so (*i*) yields $T(\phi), T(\psi) \le T(\phi \vee \psi)$, and therefore $T(\phi) \vee T(\psi) \le T(\phi \vee \psi)$, and so $T(\phi) \vee T(\psi) - T(\psi) \le T(\phi \vee \psi) - T(\psi)$, which we integrate, using ($\bullet$) on the right hand side to obtain the desired inequality.

Conversely, if (*ii*) holds and $\phi \le \psi$ then $(\phi - \psi)^+ = 0$, so (*ii*) implies $\bigl(T(\phi) - T(\psi)\bigr)^+ = 0$, and so (*i*) holds. $\square$

On second thought, we could generalise the Crandall–Tartar lemma further: Just assume $D \subseteq \mathcal{M}(\Omega)$ (the set of measurable, real-valued functions on Ω) and $T\colon D \to \mathcal{M}(\Omega)$ with the requirements that $\phi - \psi \in L^1(\Omega)$ and $T(\phi) - T(\psi) \in L^1(\Omega)$ for all ϕ, $\psi \in D$, and replace ($\bullet$) by

$$\int_\Omega \bigl(T(\phi) - T(\psi)\bigr) = \int_\Omega (\phi - \psi) \qquad (\phi, \psi \in D).$$

With this formulation, the lemma is directly applicable to the common situation where $D \subseteq L^1_{\mathrm{loc}}(\mathbb{R})$ and the functions in D are all identical for large values of $|x|$.

The proof of Lemma 2.11:

First of all, I should make clearer a remark I made at the beginning: The book says “Assume that $u_l \le u_r$; the case $u_l \ge u_r$ is similar.” My point is that you don't have to redo the proof for the second case: Just note that if u solves $u_t + f(u)_x = 0$, then $\tilde{u} = -u$ solves $\tilde{u}_t + \tilde{f}(\tilde{u})_x = 0$, where $\tilde{f}(\tilde{u}) = -f(-\tilde{u})$. This is true even for the Kružkov condition (2.22). This substitution will, of course, transform the second case to the first. (It will also transform the upper concave envelope to the lower convex envelope.)

I admit I got a bit confused by the way the proof is organised, first proving it for convex flux functions and then for the general case. What I had not noticed on my first reading is that a stronger result was proved in the convex case, and that this stronger result is needed for the general case. I am talking about the first equality in (2.17):

$$\|u(\,\cdot\,,t) - v(\,\cdot\,,t)\|_{L^1} = t\int_{u_l}^{u_r} |f' - g'|\,du = t\,\mathrm{TV}_{[u_l,u_r]}(f-g).$$

(We won't need the formulation in terms of total variation, but I thought it worth mentioning.) This is then utilised in the general case, resulting in

$$\|u(\,\cdot\,,t) - v(\,\cdot\,,t)\|_{L^1} = t\int_{u_l}^{u_r} |f'_{\smile} - g'_{\smile}|\,du \overset{*}{\le} t\int_{u_l}^{u_r} |f' - g'|\,du.$$

The final step, then, is to note that in general

$$\int_a^b |h'(u)|\,du \le (b-a)\|h\|_{\mathrm{Lip}}$$

for any piecewise differentiable Lipschitz function h, since $|h'| \le \|h\|_{\mathrm{Lip}}$.

The inequality $\overset{*}{\le}$ above came from an application of the Crandall–Tartar lemma. *I did not have time for this proof in the lecture.* Also, the proof in the book is unfortunately not correct. The mistake is minor, however, and easily corrected:

(See the next page.)

What we have to prove is the condition (i) in the Crandall–Tartar lemma, which in this case states that if f and g are continuous and piecewise linear functions with $f' \leq g'$ on $[u_l, u_r]$, then $f'_\smile \leq g'_\smile$ on $[u_l, u_r]$. Note that these convex envelopes are defined with respect to the interval $[u_l, u_r]$, so that $f_\smile(u_l) = f(u_l)$ and $f_\smile(u_r) = f(u_r)$, and the same for g.

Assume, for contradiction, that $f'_\smile > g'_\smile$ at *some* point in $[u_l, u_r]$. Since these functions are piecewise constant, we can join together one or more consecutive intervals where the inequality holds, into a *maximal* interval $[u_1, u_2] \subseteq [u_l, u_r]$ so that $f'_\smile > g'_\smile$ everywhere in (u_1, u_2) except for the breakpoints.[1]

If $u_1 > u_l$, then as we cross $u = u_1$ from the left to the right, we transition from $f'_\smile \leq g'_\smile$ to $f'_\smile > g'_\smile$. Since both functions are non-decreasing, this can only happen by $f'_\smile$ becoming larger. Thus u_1 is a breakpoint for $f_\smile$, implying $f_\smile(u_1) = f(u_1)$. If $u_1 = u_l$, we already have this equality.[2]

A similar argument yields $g_\smile(u_2) = g(u_2)$.

Integrating the inequalities $f' \leq g'$ and $f'_\smile > g'_\smile$ over $[u_1, u_2]$ yields

$$\begin{aligned} f(u_2) - f(u_1) &\leq g(u_2) - g(u_1), \\ f_\smile(u_2) - f_\smile(u_1) &> g_\smile(u_2) - g_\smile(u_1). \end{aligned}$$

Subtracting these inequalities, we get

$$f(u_2) - f_\smile(u_2) < g_\smile(u_1) - g(u_1),$$

which is a contradiction, since the left hand side is nonnegative and the right hand side is nonpositive.

[1] Here is the mistake in the book: The construction there risks including intervals in the middle where the inequality fails.

[2] The book does not consider this special case.