MA 8102 - SPRING 2021

Topics for the exam

(1) Definition of measure-preserving and ergodic maps, examples.
(2) Poincaré's reccurence theorem
(3) Associated isometry U_T and its properties.
(4) Equivalent formulations for map to be measure-preserving and ergodic
(5) Maximal ergodic theorem
(6) Birkhoff ergodic theorem and its applications. Normal numbers.
(7) One-sided and two-sided shifts.
(8) Weak-mixing and strong-mixing maps, equivalent formulations, examples.
(9) Measure-preserving maps on compact groups, characters, ergodicity.
(10) Rotations and endomorphisms of compact groups.
(11) Isomorphism, conjugacy and spectral isomorphism of spaces and maps, examples.
(12) Measure preserving transformations with countable Lebesgue spectrum
(13) Spectral invariants
(14) Eigenvalues and eigenfunctions of U_T
(15) Measure-preserving transformations with discrete spectrum.
(16) Entropy of a mesure-preserving transformation
(17) Kolmogorov-Sinai theorem
(18) Entropy of rotations of a compact group

Problems and questions

Problem 1. *Let $(X, \mathcal{B}, m)$ be a measure space, where $X = [0, 1)$, $\mathcal{B}$ is Borel $\sigma-$ algebra, and m is Borel measure. Let $T : X \to X$ be a map given by $Tx = (x + a) \mod 1$ for some $a \in X$. For each $A \in \mathcal{B}$ find $T^{-1}(A)$. Prove that T is measure-preserving.*

Problem 2. *Let $T : [0, 1) \to [0, 1)$ be a map given by $Tx = 3x \mod 1$. For each $A \in \mathcal{B}$ find $T^{-1}(A)$. Prove that T preserves Borel measure.*

Problem 3. *Let $T : [0, 1) \to [0, 1)$ be a map given by $Tx = 1/x \mod 1$. For each interval $(a, b) \subset (0, 1)$ find $T^{-1}((a, b))$. Prove that T preserves the measure*

$$\mu(A) = \int_A \frac{1}{1+x} dm,$$

where m is Borel measure.

Problem 4. *Let $T : [0, 1) \to [0, 1)$ be a map given by $Tx = 1/x^2 \mod 1$. Find an atomic measure μ on $[0, 1)$ for which T is measure-preserving.*

Problem 5*. *Do exist a such measure, which is absolutely continuous with respect to Borel measure?*

Problem 6. *Let $T : [0, 1] \to [0, 1]$ be a map given by $Tx = 4x(1 - x)$. Find an absolute continuous measure μ on $[0, 1]$ for which T is μ-invariant.*

Problem 7. *Prove that maps from Problem 1-2 are ergodic.*

Problem 8*. *Prove that the map from Problem 3 is ergodic.*

Problem 9. *Let K be the set $|z| = 1$ in $\mathbb{C}$. Find all $n \in \mathbb{Z}$ for which $Tz = z^n$ is strong mixing.*

Problem 10. *Let $x = 0.a_1a_2\dots$ be a ternary presentation of a number $x \in [0, 1]$. Prove that the limit*

$$\lim_{N\to\infty} \frac{|\{n \in [1, N] : a_n = a_{n+1} = 2\}|}{N}$$

exists for almost all $x \in [0, 1]$. Find this limit.

Problem 11. *Prove that if $T : X \to X$ is a bijective measure-preserving map then U_T is surjective.*

Problem 12. *Let $\gamma : S_3 \to K$ be a character (that is a map which preserves group operations) such that $\gamma((1\,2)) = 1$. Prove that $\gamma(a) = 1$ for all $a \in S_3$.*

Problem 13. *Prove that the maps $T_1x = 3x \mod 1$ and $T_2z = z^3$ are isomorphic. Construct an isomorphism $\varphi : [0, 1) \to K$.*

Problem 14*. *Prove that maps $T_1, T_2 : K \to K$ given by $T_1z = z^2$ and $T_2z = z^3$ are spectrally isomorphic.*

Problem 15*. *Let* $T_1, T_2 : K^2 \to K^2$ *be two maps given by* $T_1(z_1, z_2) = (z_1^{m_{11}} z_2^{m_{12}}, z_1^{m_{21}} z_2^{m_{22}})$*, and* $T_2(z_1, z_2) = (z_1^{n_{11}} z_2^{n_{12}}, z_1^{n_{21}} z_2^{n_{22}})$*. When are they spectrally isomorphic?*

Problem 16. *Find a map with positive entropy which is ergodic but not weak-mixing.*

Problem 17. *Find a map with infinite entropy.*

Problem 18. *Let* $T : K^2 \to K^2$ *be a map given by* $T(z_1, z_2) = (a_1 z_1, a_2 z_2)$*, where* $a_1, a_2 \in K$*. Find* $H(T)$*.*