Back to $\pi_1^S \cong \mathbb{Z}/2$ wanted to compute $\pi_4(S^3)$.

Let $f \in [S^3, K(\mathbb{Z},3)] \cong H^3(S^3) \cong \mathbb{Z}$ be a generator.

(e.g. inclusion of bottom cell)

$$f_* : \pi_3(S^3) \xrightarrow[\text{Hurewicz}]{\cong} H_3(S^3) \xrightarrow[\text{UCT}]{\cong} H^3(S^3) \xrightarrow[\text{rep}]{\cong} \pi_3(K(\mathbb{Z},3))$$

convert f to fibration

$$\begin{array}{ccc} X & \longrightarrow & S^3 \\ & & \downarrow f \\ & & K(\mathbb{Z},3) \end{array}$$

and consider les in π_*

$$\cdots \to \underset{\uparrow \text{ mostly } 0}{\pi_{i+1}(K(\mathbb{Z},3))} \to \pi_i(X) \to \pi_i(S^3) \xrightarrow{f_*} \underset{0}{\pi_i(K(\mathbb{Z},3))} \to \cdots$$

$\cong$

only interesting spot $\quad \pi_3(S^3) \xrightarrow{\cong} \pi_3(K(\mathbb{Z},3))$

So $\pi_i(X) \cong \begin{cases} \pi_i(S^3) & \text{for } i > 3 \\ 0 & \text{for } i \leq 3 \end{cases}$ "peeled off bottom htpy group"

So we can replace $\pi_4(S^3)$ w/ $\pi_4(X)$

Upshot X is 3-connected so Hurewicz thm gives us

$$\begin{array}{ccc} \pi_4(X) & \xrightarrow{\cong} & H_4(X) \\ \cong & & \\ \pi_4(S^3) & & \end{array}$$

so let's compute this

We'll do that by putting X in another fiber sequence

Either

$$\begin{tikzcd} K(\mathbb{Z},3) \arrow[d] & & K(\mathbb{Z},2) \arrow[dl] \\ X \simeq \text{p.b.} \arrow[r] \arrow[d] & PK(\mathbb{Z},3) \simeq * \arrow[d] & \\ S^3 \arrow[r, "f"'] & K(\mathbb{Z},3) & \end{tikzcd}$$

or convert $X \to S^3$ to a fibration and check fiber $\simeq K(\mathbb{Z},2)$ from les

Now we'll look at Serre SS for $K(\mathbb{Z},2) \to X \to S^3$

use cohom. SS of algebras

$$E_2^{p,q} = H^p(S^3; H^q(K(\mathbb{Z},2))) \Rightarrow H^{p+q}(X)$$

want this!

$$E_2^{*,*} \cong H^*(S^3) \otimes H^*(K(\mathbb{Z},2))$$

not Ω-P fibration so not everything dies

$\Rightarrow H^*(X)$

q	0	1	2	3	4
5	0				
4	$\mathbb{Z}x^2$			$\mathbb{Z}x^2y$	
3	0				
2	$\mathbb{Z}x$ (must die)			$\mathbb{Z}xy$	
1	0				
0	$\mathbb{Z}1$	0	0	$\mathbb{Z}y$	0
					p

d_3, $\cong$; $\cdot 2$

$H^*(F) = H^*(K(\mathbb{Z},2))$

$H^*(B) = H^*(S^3)$

$E_2 \cong E_3$

X is 3-connected

$\Rightarrow H^2(X) = 0,\ H^3(X) = 0$

[not writing zeros]

$d_3(x) =: y$

$d_3(x^2) = 2x\,d_3(x) = 2xy$

could keep going but this is enough

$$E_4:\quad \begin{array}{c|ccc}
5 & 0 & & \\
2 & 0 & \mathbb{Z}/2 & \\
1 & & & \\
0 & \mathbb{Z} & 0 & 0 \\
\hline
 & 0 & 3 & 5
\end{array}$$

$\leftarrow$ total degree 5

up to total degree 5, no more possible differentials

So $\mathbb{Z}/2$ persists to E_∞

only nonzero group in total deg. 5 so

no extension problems here either

So $H^4(X) = 0$, $H^5(X) \cong \mathbb{Z}/2 \overset{UCT}{\Longrightarrow} H_4(X) \cong \mathbb{Z}/2$.

Finally, $\pi_4(S^3) \underset{\text{construction}}{\cong} \pi_4(X) \underset{\text{Hurewicz}}{\cong} H_4(X) \underset{\text{SSS}}{\cong} \mathbb{Z}/2$

and $\boxed{\pi_1^S \cong \mathbb{Z}/2}$

break

Similar techniques (+ Steenrod algebra + complicated)

can be used to compute 1st few stable stems (M+T)

<u>Next Goal</u> Stable stems are finite, π_i^S finite for $i>0$

<u>Recall</u> $\pi_i^S = \pi_{n+i}(S^n)$ for large n $(n > i+1)$

We've compute $\pi_0^S \cong \mathbb{Z}$

$\pi_1^S \cong \mathbb{Z}/2$

Def A <u>Serre class</u> of abelian groups $\mathcal{C}$ is a collection of abelian groups s.t.

1) For any ses $0 \to A \to B \to C \to 0$

$$B \in \mathcal{C} \iff A, C \in \mathcal{C}$$

2) $A, B \in \mathcal{C} \Rightarrow A \otimes B \in \mathcal{C}$, $\mathrm{Tor}(A,B) \in \mathcal{C}$

3) $A \in \mathcal{C} \Rightarrow H_n(K(A,1)) \in \mathcal{C} \quad \forall n > 0$

Ex $\mathcal{C}$ = f.g. abelian groups

$\mathcal{C}$ = torsion abelian groups

$\mathcal{C}$ = torsion abelian groups w/ elements divisible by a fixed set of primes

Non-ex $\mathcal{C}$ = torsion free abelian groups

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \to & \mathbb{Z}/2 & \to & 0 \\ & & & & \in \mathcal{C} & & \notin \mathcal{C} & & \end{array}$$

Thm If X is simply-connected then $\pi_i(X) \in \mathcal{C} \ \forall i$ iff $H_i(X) \in \mathcal{C} \ \forall i > 0$.

"mod $\mathcal{C}$"

Cor $\pi_i(S^n)$ is f.g. abelian for $n > 1$.

Thm π_i^S is finite for $i > 0$.

Proof We'll show for n odd that $\pi_i(S^n)$ finite for all $i > n$.

Then Freudenthal susp $\Rightarrow \pi_i^S$ finite $\forall i > 0$

(Proof) Fix n odd, $n > 1$.

Let $f : S^n \to K(\mathbb{Z}, n)$ s.t. $f_* : \pi_n(S^n) \to \pi_n(K(\mathbb{Z}, n))$ is an iso.

Convert f to a fibration

$$\begin{array}{ccc} X & \longrightarrow & S^n \\ & & \downarrow f \\ & & K(\mathbb{Z}, n) \end{array}$$

LES in π_* $\Rightarrow$ $\pi_i(X) \cong \begin{cases} \pi_i(S^n) & \text{for } i > n \\ 0 & \text{for } i \leq n \end{cases}$

(used $K(\mathbb{Z}, n)$ to peel off bottom homotopy group from S^n)

Take $X \to S^n$ and convert it to a fibration

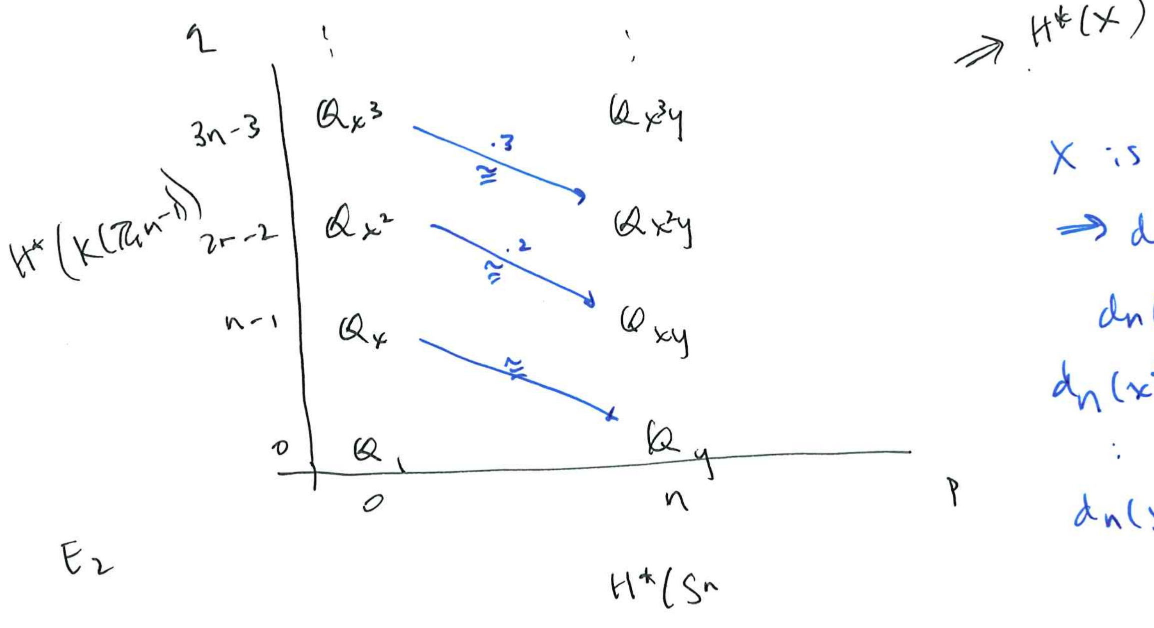
$$\begin{array}{ccc} K(\mathbb{Z}, n-1) & \longrightarrow & X \\ & & \downarrow \\ & & S^n \end{array}$$

Now use Serre SS in $\mathbb{Q}$-coefficients (field)

$$E_2^{*,*} = H^*(S^n; \mathbb{Q}) \otimes H^*(K(\mathbb{Z}, n-1); \mathbb{Q}) \Rightarrow H^*(X; \mathbb{Q})$$

From exercise: $H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q}[x], \ |x| = n & \text{if } n \text{ even} \\ \Lambda(x), \ |x| = n & \text{if } n \text{ odd} \end{cases}$

In our case, n odd so $n-1$ even



X is n-connected

$\Rightarrow d_n$ iso $n-1$ to n

$d_n(x) = y$

$d_n(x^2) = 2x\,d_n(x) = 2xy$

$\vdots$

$d_n(x^m) = m x^{m-1} d_n(x)$

$= m x^{m-1} y$

(Proof) So $\tilde{H}^*(X;\mathbb{Q}) \cong 0$

But then $\tilde{H}^i(X) \otimes \mathbb{Q} \cong \tilde{H}^i(X;\mathbb{Q}) \cong 0$.

So $\tilde{H}^i(X)$ is finite.

mod $\mathcal{C}$ thm $\Rightarrow \pi_i(X)$ is finite

and $\pi_i(X) \cong \pi_i(S^n)$ for $i > n$, n fixed odd #

Freudenthal susp $\Rightarrow \pi_i^s$ finite $\forall\, i > 0$. $\square$

So $\pi_i^s \cong \cancel{\mathbb{Z}^r} \oplus \mathbb{Z}/(p_1^{r_1}) \oplus \mathbb{Z}/(p_2^{r_2}) \oplus \cdots \oplus \mathbb{Z}/(p_k^{r_k})$

<u>Upshot</u> We can compute π_i^s one prime at a time.

Send

<u>Fun fact</u> Stable homotopy $\pi_i^s(-)$ is a reduced homology theory on pointed finite CW complexes.

$\pi_i^s = \pi_i^s(S^0)$

$\pi_i^s(X) = \operatorname{colim}_{\vec{n}} \pi_{n+i}(\Sigma^n X)$

So also $\pi_i^s(-) \otimes \mathbb{Q}$ is a homology theory

(b/c $- \otimes \mathbb{Q}$ is flat and so preserves les)

So we could define $\tilde{H}_i^{st}(X) := \pi_i^s(X) \otimes \mathbb{Q}$

Notice $\pi_i^s(S^0) \otimes \mathbb{Q} = 0$ if $i > 0$

so this defines an ordinary homology theory.

Also, $\pi_0^s(S^0) \otimes \mathbb{Q} \cong \mathbb{Q}$ so this homology theory agrees w/ $\tilde{H}_*^{sing}(-;\mathbb{Q})$!