MA3408 Week 6

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Question 1.

Show, using the Serre spectral sequence, that if $S^k \to S^m \to S^n$ is a fibration with $n \geq 2$, then $k = n-1$ and $m = 2n-1$.

Proof. We have

$$E^2_{p,q} \cong \begin{cases} \mathbb{Z} & (p,q) = (0,0),(n,0),(0,k),(k,n) \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we know that there are only two copies of $\mathbb{Z}$ that can survive the spectral sequence, in total degree (i.e., degree $p+q$) 0 and m.

The only possible differential in the spectral sequence is a d_n-originating at $(n,0)$ and with target $(0,n-1)$. This d_n must kill the copy of $\mathbb{Z}$ in position $(0,k)$, and so $k = n-1$. Moreover, we must have $k+n = m$ for degree reasons, and so $m = n-1+n = 2n-1$. $\square$

Question 2.

Show, using spectral sequences, the folllowing result in homological algebra (the snake lemma):

Given a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

in an abelian category with exact rows, there is a long exact sequence

$$0 \to \ker(f) \to \ker(g) \to \ker(h) \to \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h) \to 0.$$

Proof. We can think of this as a double complex, and so consider the two associated spectral sequences. To get the grading to agree with our usual grading (differentials to the left) we can draw the double complex like this:

$$\begin{tikzcd} C \arrow[d, "h"'] & B \arrow[l] \arrow[d, "g"] & A \arrow[l] \arrow[d, "f"] & 0 \arrow[l] \\ C' & B' \arrow[l] & A' \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

For the first, we take homology horizontally and then vertically. Because the rows are assumed exact, after taking homology in the horiztal direction, the E_1-page is 0, and so the spectral sequence converges to 0 (there is no problem with convergence, as there are only finitely-many non-zero terms).

The second spectral sequence is where we take homology vertically, and then horizontally. Note that by the previous paragraph we know that everything must eventually die in the spectral sequence. The E_1-term of the spectral sequence is then

$$\begin{tikzcd}
\ker(h) & \ker(g) \arrow[l, "\beta"'] & \ker(f) \arrow[l, "\alpha"'] & 0 \arrow[l] \\
\operatorname{coker}(h) & \operatorname{coker}(g) \arrow[l, "\Delta"] & \operatorname{coker}(f) \arrow[l, "\gamma"] & 0 \arrow[l]
\end{tikzcd}$$

Note that we don't make any claims about these rows being exact. Taking horizontal homology we get an E_2-term that looks like this:

$$\begin{tikzcd}
{} & {} & \operatorname{coker}(\beta) & \ker(\beta)/\operatorname{im}(\alpha) & \ker(\alpha) & 0 \\
& & \operatorname{coker}(\Delta) \arrow[ull] & \ker(\Delta)/\operatorname{im}(\gamma) \arrow[ull] & \ker(\gamma) \arrow[ull, "d_2"] & 0 \arrow[ull]
\end{tikzcd}$$

This is the last page that has a possible differential, for degree reasons. In other words, the sequences

$$0 \to \ker(f) \to \ker(g) \to \ker(h) \quad \text{ and } \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h) \to 0$$

are exact, and

$$\ker(\operatorname{coker}(f) \to \operatorname{coker}(g)) \cong \operatorname{coker}(\ker(g) \to \ker(h)).$$

We can therefore paste together the two sequences, to get a long exact sequence

$$0 \to \ker(f) \to \ker(g) \to \ker(h) \to \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h) \to 0.$$

In other words, the connecting homomorphism in the snake-lemma is precisely the d_2-differential in the spectral sequence. $\square$

Question 3.

1. Suppose we have a commutative triangule

$$\begin{tikzcd}
A \arrow[rr, "i"] \arrow[dr, "f"'] & & C \arrow[dl, "q"] \\
& B &
\end{tikzcd}$$

Show using the snake lemma that

$$\ker(\operatorname{coker} f \to \operatorname{coker} q) \cong \operatorname{im}(q)/\operatorname{im}(f) \quad \text{and} \quad \operatorname{coker}(\operatorname{coker} f \to \operatorname{coker} q) = 0.$$

2. Using Part (1), prove the following 'butterfly lemma': given a commutative diagram

$$\begin{tikzcd}
A \arrow[dd, "f"'] \arrow[dr, "i"] & & D \arrow[dl, "j"'] \arrow[dd, "g"] \\
& C \arrow[dl, "q"] \arrow[dr, "p"'] & \\
B & & E
\end{tikzcd}$$

of abelian groups, in which the diagonals pi and qj are exact at C, there is an isomorphism

$$\frac{\operatorname{im} q}{\operatorname{im} f} \cong \frac{\operatorname{im} p}{\operatorname{im} g}.$$

Proof. We have an induced diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{im}(f) \arrow[r, hook] \arrow[d, "\phi"] & B \arrow[r, two heads] \arrow[d, equal] & \mathrm{coker}(f) \arrow[r] \arrow[d, "\Psi"'] & 0 \\ 0 \arrow[r] & \mathrm{im}(q) \arrow[r, hook] & B \arrow[r, two heads] & \mathrm{coker}(q) \arrow[r] & 0 \end{tikzcd}$$

The snake lemma gives that $\ker(\phi) = 0$, $\ker(\Psi) = \mathrm{coker}(\phi)$ and $\mathrm{coker}(\Psi) = 0$. Part (1) follows.

For Part (2), rewrite the diagram as follows

$$\begin{tikzcd} E \arrow[d, equal] & D \arrow[l, "g"'] \arrow[d, "j"'] & 0 \arrow[l] \arrow[d] \\ E \arrow[d] & C \arrow[l, "p"'] \arrow[d, "q"'] & A \arrow[l, "i"'] \arrow[d, "f"] \\ 0 & B \arrow[l] \arrow[r, equal] & B \end{tikzcd}$$

We take horizontal and then vertical cohomlogy:

$$\begin{tikzcd} \mathrm{coker}(g) \arrow[d] & \ker(g) \arrow[d] & 0 \arrow[d] \\ \mathrm{coker}(p) \arrow[d] & 0 \arrow[d] & \ker(i) \arrow[d] \\ 0 & 0 & 0 \end{tikzcd}$$

then

$$\begin{array}{ccc} \mathrm{im}(p)/\,\mathrm{im}(g) & \ker(g) & 0 \\ 0 & 0 & \ker(i) \\ 0 & 0 & 0 \end{array}$$

This way the differentials go down and to the right, so there are no possible higher differentials.

Taking vertical and then horizontal homology, we get

$$\begin{tikzcd} 0 & \ker(j) \arrow[l] & 0 \arrow[l] \\ 0 & 0 \arrow[l] & \ker(f) \arrow[l] \\ 0 & \mathrm{coker}(q) \arrow[l] & \mathrm{coker}(f) \arrow[l] \end{tikzcd}$$

then

$$\begin{array}{ccc} 0 & \ker(j) & 0 \\ 0 & 0 & \ker(f) \\ 0 & 0 & \mathrm{im}(q)/\,\mathrm{im}(f) \end{array}$$

This way, the differentials go left and up, so there is no room for differentials.

Comparing the diagonals, we see that $\operatorname{im}(q)/\operatorname{im}(f) \cong \operatorname{im}(p)/\operatorname{im}(g)$ as claimed.

$\square$