MA3408 Week 12

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Question 1.

If $f\colon Y \to X$ is a continuous map and $\pi\colon E \to X$ is a $U(n)$-bundle, then $c_i(f^*\pi) = f^*c_i(\pi)$ for any i.

Remark: Note that f^* has two different meanings here, once as a vector bundle pullback, and once as the pullback of a cohomology class.

Question 2.

Let $\pi_{\mathbb{R}}$ denote the underlying real bundle of a complex bundle; note that if π has rank n as a complex bundle, then $\pi_{\mathbb{R}}$ has rank $2n$ as a real bundle. Via the map $\mathbb{Z} \to \mathbb{Z}/2$ the class $c_i(\pi) \in H^{2i}(X;\mathbb{Z})$ determines a cohomology class $\overline{c}_i(\pi) \in H^{2i}(X;\mathbb{Z}/2)$. Show that the Stiefel–Whitney classes of $\pi_{\mathbb{R}}$ are computed as follows:

1. $\omega_{2i}(\pi_{\mathbb{R}}) = \overline{c}_i(\pi)$ for $0 \le i \le n$.
2. $\omega_{2i+1}(\pi_{\mathbb{R}}) = 0$ for any integer i.

Hint: Let $\mu_n\colon U(n) \to O(2n)$ be the inclusion, then compute

$$\mu_n^*\colon H^*(O(2n);\mathbb{Z}/2) \cong \mathbb{Z}/2[c_1, c_2, \ldots, c_n] \to H^*(U(n);\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2, \ldots, w_{2n}].$$