Def^n (Homotopy fiber)

Let $f : X \longrightarrow Y$ be a map, let ~~Bla~~ P_f be the mapping path space

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\sim"'] & & Y \\ & P_f \arrow[ur, "\text{fibration}"'] & \end{tikzcd}$$

The homotopy fiber of f is the fiber of $P_f \longrightarrow Y$. (only defined up to homotopy equivalence).

Def^n: A mapp $f : (X, x_0) \longrightarrow (Y, y_0)$ is a weak equivalence if $f_* : \pi_0(X, x_0) \longrightarrow \pi_0(Y, y_0)$ is a bijection & $f_* : \pi_k(X, x_0) \longrightarrow \pi_k(Y, y_0)$ is an isomorphism.

Lemma: If $f : X \longrightarrow Y$ is a weak equivalence, then $\pi_k F(f) = 0$ for all $k \geq 0$.

Proof: Use the long exact sequence of a fibration. ∎

For a map $f : X \longrightarrow Y$, we make some observations

- A map $S^{n-1} \longrightarrow F$ corresponds to a diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "g"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "f"] \\ D^n \arrow[r, "h"'] & Y \end{tikzcd}$$

[the composite $f \circ g$ is null $\Leftrightarrow$ ~~the~~ if it extends over the disc]

- The map $\pi_n(Y) \longrightarrow \pi_{n-1}(F)$ in the long exact sequence corresponds to the map sending the class of $h : S^n \longrightarrow Y$ to the class of $\pi_{n-1}(F)$ represented by the diagram

$$\begin{tikzcd}
S^{n-1} \arrow[r, "c"] \arrow[d, hook, "i"'] & X \arrow[d] \\
D^n \arrow[r, "h"'] & Y
\end{tikzcd}$$

where $c = c_{x_0}$ is constant, & h is $D^n \longrightarrow D^n/S^{n-1} = S^n \xrightarrow{\bar{h}} Y$

• In a similar manner $\pi_{n-1}(F) \longrightarrow \pi_{n-1}(X)$ corresponds to sending a diagram

$$\begin{tikzcd}
S^{n-1} \arrow[r, "g"] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\
D^n \arrow[r, "h"'] & Y
\end{tikzcd}$$

to the class $[g]$.

<u>Lemma</u> • Suppose $f: X \longrightarrow Y$ is a map with homotopy fiber F. $\pi_{n-1}(F) = 0$ is equivalent to completing such a diagram to a diagram

$$\begin{tikzcd}
S^{n-1} \arrow[rr, "g"] \arrow[dr] \arrow[ddd, "i"'] & & X \arrow[ddd, "f"] \\
& CS^{n-1} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& CD^n \arrow[ddr, dashed, "H"] & \\
D^n \arrow[ur, dash] \arrow[rr] & & Y
\end{tikzcd}$$

Lemma. Suppose f is a map with homotopy fiber F.
Then $\pi_{n-1}(F) = 0 \Leftrightarrow$ each diagram

$$\begin{tikzcd}
S^{n-1}\times\{1\} \arrow[r, "g"] \arrow[d, hook] & X \arrow[d, "f"] \\
S^{n-1}\times I \cup D^n\times\{0\} \arrow[r, "h"] & Y
\end{tikzcd}$$

can be ~~extended~~ ~~xxx~~ completed to a diagram

$$\begin{tikzcd}
S^{n-1}\times\{1\} \arrow[rr, "g"] \arrow[dr, hook] \arrow[ddd] & & X \arrow[ddd, "f"] \\
& D^n\times\{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& D^n\times I \arrow[dr, dashed, "H"] & \\
S^{n-1}\times I \cup D^n\times\{0\} \arrow[ur, hook] \arrow[rr, "h"] & & Y
\end{tikzcd}$$

Proof: For any disc we have a homeomorphism $CD^n \cong D^n\times I$ which sends the cone point to the centre of $D^n\times\{1\}$, D^n to $S^{n-1}\times I \cup D^n\times\{0\}$, S^{n-1} to $S^{n-1}\times\{1\}$, & (S^{n-1} to $D^n\times\{1\}$. Now apply the observation.

Rem: Closed under products.

Theorem (Homotopy Extension & Lifting Property) (HELP!)

Suppose (X, A) is a relative CW-pair, & $f: Y \to Z$ is a weak equivalence. Then any diagram

$$\begin{tikzcd}
A\times\{1\} \arrow[r, "g"] \arrow[d, hook] & Y \arrow[d, "f"] \\
A\times I \cup \cancel{\emptyset}\, X\times\{0\} \arrow[r] & Z
\end{tikzcd}$$

can be completed to

$$\begin{tikzcd}
A\times\{1\} \arrow[rr, "g"] \arrow[dr, hook] \arrow[ddd] & & Y \arrow[ddd, "f"] \\
& X\times\{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& X\times I \arrow[dr, dashed, "H"] & \\
A\times I \cup X\times\{0\} \arrow[ur] \arrow[rr, "h"] & & Z
\end{tikzcd}$$

<u>Exercise</u>: Take $f = id_{Y}$, then show that this is just the homotopy extension property

<u>Proof</u>: We can build $G_n : X_n \times \{1\} \longrightarrow Y$ &

$H_n : X_n \times I \longrightarrow Z$ inductively

The base case is trivial, & for the inductive step one can apply the previous lemma to the diagram

$$\begin{tikzcd}
X_{n-1} \times \{1\} \arrow[rr, "G_{n-1}"] \arrow[dd, hook] \arrow[dr, hook] & & Y \arrow[dd, "f"] \\
& X_n \times \{1\} \arrow[ur, dashed, "G_n"] \arrow[d, hook] & \\
& X_n \times I \arrow[dr, dashed, "H_n"] & \\
X_{n-1} \times I \cup X_n \times \{0\} \arrow[rr, "H_{n-1} \cup h"'] \arrow[ur] & & Z
\end{tikzcd}$$

<u>Some theorems in homotopy</u>

~~Def (cellular approximation)~~

<u>Lem</u>: For any weak equivalence $f: Y \to Z$ & any CW complex X, the induced map $f_* : [X, Y] \to [X, Z]$ is a bijection.

<u>Proof</u>: <u>Surjectivity</u>: (X, ϕ) is a relative CW-pair. Then, for $h: X \to Z$, we have a diagram

$$\begin{tikzcd}
* \arrow[rr] \arrow[dd] \arrow[dr, hook] & & Y \arrow[dd, "f"] \\
& X \times \{1\} \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
& X \times I \arrow[dr, dashed, "H"] & \\
X \times \{0\} \arrow[rr, "h"'] \arrow[ur] & & Z
\end{tikzcd}$$

The homotopy H satisfies
$H(-, 0) = h$ & $H_1 = f \circ G$.
therefore $[h] = f_*[G]$, & f_* is surjective

Assume $g_0, g_1 \in [X,Y]$ with $f_*[g_0] = f_*[g_1]$, ie

$f g_0 \simeq f g_1$ via some homotopy $F : X \times I \longrightarrow Z$.

Consider the pair $(X \times I,\ X \times \partial I)$

$$\begin{tikzcd}
X \times \partial I \times \{1\} \arrow[rr, "g"] \arrow[dr, hook] \arrow[dd, hook] & & Y \arrow[dd, "f"] \\
& X \times I \arrow[ur, dashed, "G"] \arrow[d, hook] & \\
X \times \partial I \times I \cup X \times I \times \{0\} \arrow[ur, hook] \arrow[r, "h"'] & X \times I \times I \arrow[r, dashed, "H"] & Z
\end{tikzcd}$$

$$g : X \times \partial I \longrightarrow Y$$
$$(x, v) \longmapsto g_v(x), \quad v = 0, 1$$

$$h : X \times \partial I \times I \longrightarrow Z$$
$$(x, i, s) = f \circ g_i(s) \quad i = 0, 1$$
$$X \times I \times \{0\} \longrightarrow Z \text{ is just } F$$

The lift $G : X \times I \longrightarrow Y$ gives a homotopy between g_0 & g_1, ie $[g_0] = [g_1] \Rightarrow f_*$ is injective. $\square$

<u>Thm</u> (<u>Whitehead theorem</u>) If $f : X \to Y$ is a weak equivalence b/w CW-complexes, then f is a homotopy equivalence.

~~<u>Proof</u>: Take $X = Z$ in the previous $[X,Y] \xrightarrow[\cong]{f_*} [X,X]$~~

~~choose a lift of $id_X : X \to X$. So we find $g : X \to Y$ with $f \circ g \simeq id_X$. The $f \circ g \circ f \simeq f$, but we also have~~

~~$f_* : [X,X] \xrightarrow{\cong} [X,Y]$~~

~~$id_X \longmapsto f$~~

~~Note that~~

~~$f_*[id_X]$~~

~~$f_*[g \circ f]$~~

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Pf: $f: X \longrightarrow Y$ is a weak equivalence

$f_0 : [Y, X] \xrightarrow{\simeq} [Y, Y]$, so there exists $g: Y \longrightarrow X$ such that $f_0[g] = [f \circ g] = [id_Y]$, ie, $f \circ g \simeq id_Y$.
Then, $f \circ g \circ f \simeq f$. But,

$$\begin{aligned} f_0 : [X, X] &\xrightarrow{\simeq} [X, Y] \\ id_X &\longmapsto f \\ g \circ f &\longmapsto f \end{aligned}$$

ie $id_X \simeq g \circ f$

so $X \simeq Y$.

Cor: If X is a CW-complex with $\pi_k X = 0$ for every k, then X is contractible.

Proof: ~~Use~~ Use the unique map $X \longrightarrow *$.

Remark: We cannot drop assumptions

① We require a map inducing the weak equivalence

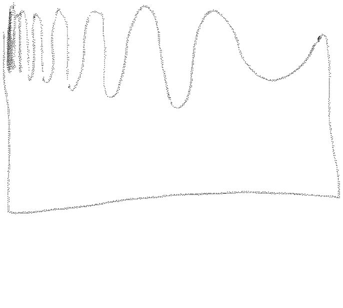
e.g. $\mathbb{R}P^2 \times S^3$, $S^2 \times \mathbb{R}P^3$.

② The topologist's sine curve

$A = \{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 1, \; y = \sin(\pi/x)\}$

$B = \{(0, y) \in \mathbb{R}^2 \mid -3/2 \leq y \leq 1$

$C = \{(0, y) \in \mathbb{R}^2 \mid y \in [-3/2, -1]\}$
$\cup \{(x, -3/2) \mid x \in [0, 1]\} \cup \{(1, y) \mid y \in [-3/2, 0]\}$



Then $\pi_k X = 0$ for all k, but X is ~~weak~~ not contractible