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Defⁿ A principal G -bundle $\pi_q: EG \rightarrow BG$ is called universal if the total space EG is (weakly) contractible.

Theorem: Let X be a CW-complex, then there is a bijection

$$\Phi: [X, BG] \xrightarrow{\cong} \mathcal{S}(X, G)$$

$$f \longmapsto f^* \pi_q$$

$$\begin{array}{ccc} f^* EG & \longrightarrow & BG \\ \downarrow \pi_q & & \downarrow \pi_q \\ X & \longrightarrow & BG \end{array}$$

Proof: We first note that Φ is well-defined, as homotopic maps induce isomorphic bundles, as we showed last week.

Φ is onto: let $\pi \in \mathcal{S}(X, G)$, $\pi: E \rightarrow X$ be a principal G -bundle. We wish to find a map $f: X \rightarrow BG$, such that $\pi \cong f^* \pi_q$, or equivalently, we wish to find a bundle map $(f, \tilde{f}): \pi \rightarrow \pi_q$. By last week this is equivalent to finding a section of the associated bundle $w: E \times_q EG \rightarrow X$ with fiber EG . Since EG is contractible, such a section exists by the following lemma.

Lemma: Let X be a CW-complex $\pi: \pi: E \rightarrow X \in \mathcal{B}(X, G, f, e)$, with $\pi_C(F) = 0$ for all $C \geq X$. If $A \subseteq X$ is a subcomplex, then every section of π over A extends to a section defined on all of X . In particular, π has a section $w: E \rightarrow E$ of π over A , which is homotopic to $w|_A$.

Proof: Given a section $\sigma_0: A \rightarrow E$ of π over A , we extend it to a section $\sigma: X \rightarrow E$ over X using induction on the dimension of the cells in $X \setminus A$.

So we can assume

$$X = A \cup_p e^n, \quad \varphi: \partial e^n \rightarrow A$$

$$\partial e^n \xrightarrow{\varphi} A$$

i.e.,

$$\begin{array}{ccc} \partial e^n & \xrightarrow{\varphi} & A \\ \downarrow & & \downarrow \\ e^n & \longrightarrow & A \cup_p e^n \end{array}$$

Since e^n is contractible, π_1 is trivial over e^n , \oplus
so we have a commutative diagram

$$\begin{array}{ccc} \pi_1^{-1}(e^n) & \xrightarrow{\cong} & e^n \times F \\ \sigma_0 \nearrow & \downarrow \pi & \swarrow p_1 \\ \delta e^n & \longrightarrow & e^n \end{array}$$

Identifying $\pi_1^{-1}(e^n)$ with $e^n \times F$, we write

$$\sigma_0(x) = (x, \tau_0(x)) \in e^n \times F$$

for some $\tau_0: \delta e^n \cong S^{n-1} \rightarrow F$

Because $\pi_{n-1}(F) = 0$, τ_0 extends over the disk to a map $\tau: e^n \rightarrow F$.

Now define $\sigma: e^n \rightarrow e^n \times F$
 $x \mapsto (x, \tau(x))$

Composing with h^{-1} gives the desired extension of σ_0 over e^n . The homotopy uniqueness follows by a similar proof.

Injectivity: Suppose $\pi_0 = f^* \pi_1 \cong g^* \pi_1 = \pi_1$, then we must show $f \simeq g$. Consider the diagrams

$$\begin{array}{ccc} f^* EG & \xrightarrow{\hat{f}} & EG \\ \pi_0 \downarrow & & \downarrow \pi_1 \\ X = \{x \in \delta\} & \xrightarrow{f} & BG \end{array} \quad \begin{array}{ccc} E_0 \cong E_1 = g^* EG & \xrightarrow{\hat{g}} & EG \\ \downarrow \pi_1 & & \downarrow \pi_1 \\ X = \{x \in \delta\} & \xrightarrow{g} & BG \end{array}$$

Putting these together we get

$$\begin{array}{ccc} E_0 \times I & \xleftarrow{\quad} & E_0 \times \{e_0, \bar{e}\} \\ \downarrow \pi_0 \times id & & \downarrow \pi \\ X \times I & \xleftarrow{\quad} & X \times \{e_0, \bar{e}\} \\ \downarrow \alpha = (f, 0) \circ (g, 1) & & \downarrow \pi \\ & & BG \end{array}$$

We wish to extend $(\alpha, \bar{\alpha})$ to a bundle map $(\alpha, \bar{\alpha}): \pi_0 \times id \rightarrow \pi_1$, so it will provide the desired homotopy.

Such a map corresponds to a section of (3)

$$\omega: (E_0 \times I) \times_{\tilde{\epsilon}} E_h \rightarrow X \times I$$

The map (x, \tilde{x}) gives a section σ_0 of the bundle

$$\omega_0: (E_0 \times S_0, I) \times_{\tilde{\epsilon}} E_h \rightarrow X \times S_0, I$$

which under the inclusion $(E_0 \times S_0, I) \times_{\tilde{\epsilon}} E_h \subseteq (E_0 \times I) \times_{\tilde{\epsilon}} E_h$ can be regarded as a section of the ~~one~~ ω over the subcomplex $X \times S_0, I$. Since E_h is contractible, the previous lemma shows such a section exists, & we are done.

Example: $B(S^n, G, F, e) \cong P(S^n, G) \cong [S^n, BG]$

$$\cong \pi_n BG$$

$\cong \pi_{n-1}(G)$, as we have already seen.

Theorem (Milnor): let G be a locally compact topological group. Then a universal principal G -bundle exists, & the construction is functorial: a continuous group homomorphism induces a bundle map $(B\mu, E\mu)$: $\pi_G \rightarrow \pi_H$. Moreover, the classifying space is unique up to homotopy.

Proof: let us show why BG is unique up to homotopy.
let $\pi_G: E_G \rightarrow BG$ & $\pi_{G'}: E_{G'} \rightarrow BG'$ be universal principal G -bundles.

$$\begin{array}{ccc} E_{G'} & \xrightarrow{f} & E_G \\ \pi_{G'} \downarrow & & \downarrow \pi_G \\ BG' & \xrightarrow{f} & BG \end{array}$$

$$\pi_G' \cong \text{pr}_1 \circ \pi_G$$

$$\pi_G \cong g^* \pi_{G'}$$

$$\begin{aligned} \Rightarrow \pi_G &\cong g^* \pi_{G'} \cong g^* f^* \pi_G \\ &= (f \circ g)^* \pi_G \\ &= (\text{id}_{BG})^* \pi_G \\ \Rightarrow f \circ g &\sim \text{id}_{BG}, \text{ sim, } g \circ f \sim \text{id}_{BG} \end{aligned}$$

That such an \mathcal{E} exists is more complicated. (2)

Recall the join of two spaces $X \times Y$ is

$$X * Y = X \times I \times Y / \sim$$

$$(x_0, y_1) \sim (x_0, y_2) \quad \text{if } y_1, y_2 \in Y$$

$$(x_1, z, y) \sim (x_2, z, y) \quad \text{if } x_1, x_2 \in X$$

Example: let y be a single point.

$$X * y = \frac{X \times I}{X \times \{y\}} = CX.$$

Example: let $y = \{y_1, y_2\}$. Then $X * Y = \Sigma X$.

In general, $\mathcal{E} X * Y \cong \Sigma(\mathcal{E} X \wedge Y)$, so

$$S^m * S^n \cong \Sigma(S^m \wedge S^n)$$

$$\# \cong S^{m+n+1} \quad (\text{but this is actually a homeomorphism}).$$

let $G^{\otimes(k+l)} = \underbrace{G \otimes \dots \otimes G}_{k+l-\text{copies of } G}$

This has a free G -action given by acting diagonally on the G -factors (trivially on the interval).

let $J(a) = \lim_{k \rightarrow \infty} G^{\otimes(k+l)}$.

Then $J(a)$ has a free G -action, & in fact

$$J(a) \rightarrow J(a)/G$$

is a universal principal G -bundle.

In practice, we will not use Milnor's construction,
 but rather construct the usual bundle in
 specific examples we are interested in.

Example We have fiber bundles

$$\mathrm{O}(n) \rightarrow V_n(\mathbb{R}^n) \rightarrow G_n(\mathbb{R}^k)$$

Working with $V_n(\mathbb{R}^n)$ $(k=n-1)$ -connected.

Letting $k \rightarrow \infty$, we have the bundle

$$\mathrm{O}(n) \rightarrow V_n(\mathbb{R}^\infty) \rightarrow G_n(\mathbb{R}^\infty).$$

with $V_n(\mathbb{R}^\infty)$ contractible. So,

$$BO(n) \cong G_n(\mathbb{R}^\infty)$$

Now $GL_n(\mathbb{R}) \cong \mathrm{O}(n)$ [Gram-Schmidt], so

$$\mathrm{Vect}_n(\mathbb{R}) \stackrel{\times}{\cong} P(\mathrm{O}(n), \times) \cong [\times, BO(n)] \cong [\times, G_n(\mathbb{R}^\infty)]$$

Similarly, we have

$$U(n) \rightarrow V_n(\mathbb{C}^\infty) \rightarrow G_n(\mathbb{C}^\infty)$$

so $BU(n) \cong G_n(\mathbb{C}^\infty)$, & this is the
 classifying space for rank n -complex vector bundles.

Recall the following: let X be an abelian group & X
 a CW-complex, then there is a bijection

$$T: [X, K(G, n)] \rightarrow H^n(X; G)$$

$$[f] \longleftrightarrow f_*([x]), \quad \text{for } x \in H^n(X; G), \quad \text{the fundamental class.}$$

Example (Classification of real line bundles) ⑥

Let $a = \mathbb{Z}/2$, & consider the principal $\mathbb{Z}/2$ bundle

$$\mathbb{Z}/2 \hookrightarrow S^{\infty} \rightarrow \mathbb{RP}^{\infty}$$

Since S^{∞} is contractible, we have $B\mathbb{Z}/2 \cong \mathbb{RP}^{\infty} = K(\mathbb{Z}/2, 1)$

So,

$$\begin{aligned} P(X, \mathbb{Z}/2) &= [X, B\mathbb{Z}/2] = [X, K(\mathbb{Z}/2, 1)] \\ &\cong H^1(X, \mathbb{Z}/2). \end{aligned}$$

Now let π be a real line bundle (= rank 1 real vector bundle) with classifying map $f_{\pi}: X \rightarrow \mathbb{RP}^{\infty}$.

Since $H^1(\mathbb{RP}^{\infty}; \mathbb{Z}/2) \cong \mathbb{Z}/2[\omega]$, we get a well-defined class

$$\omega_1(\pi) := f_{\pi}^*(\omega)$$

the first Stiefel-Whitney class. The bijection given above sends π to $\omega_1(\pi)$, so real line bundles on X are determined by their Stiefel-Whitney classes.

Example (Classification of complex line bundles) Take $a = \mathbb{C}^1$,

& consider the principal S^1 -bundle

$$S^1 \rightarrow S^{\infty} \rightarrow \mathbb{CP}^{\infty}$$

$\Rightarrow BS^1 \cong \mathbb{CP}^{\infty}$. Since $S^1 = GL_1(\mathbb{C})$, we see \mathbb{CP}^{∞} classifies complex line bundles. Since $\mathbb{CP}^{\infty} = K(\mathbb{Z}, 2)$, we

get

$$P(X, S^1) = [X, BS^1] = [X, K(\mathbb{Z}, 2)] = H^2(X; \mathbb{Z}).$$

Now let π be a complex line bundle with classifying map $f_{\pi}: X \rightarrow \mathbb{CP}^{\infty}$. Since $H^2(\mathbb{CP}^{\infty}; \mathbb{Z}) \cong \mathbb{Z}[\zeta]$, we get a class

$$c_1(\pi) := f_{\pi}^*(\zeta).$$

The bijection $P(X, S^1) \xrightarrow{\cong} H^2(X; \mathbb{Z})$, $\pi \mapsto c_1(\pi)$, so complex line bundles are classified by Chern classes.

Example: How many line bundles over $\mathbb{R}\mathbb{P}^n$ are there? ⑦

$$\text{Vect}_{\mathbb{R}}^1(\mathbb{R}\mathbb{P}^n) \cong \mathbb{Z}/2\mathbb{Z}, H^*(\mathbb{R}\mathbb{P}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

So there are 2, the trivial bundle & another one. What is the non-trivial bundle?

Let $x \in S^n$, $\llbracket x \rrbracket \in \mathbb{R}\mathbb{P}^n = S^n / \sim$ the class represented

$$\text{by } x. \text{ Let } E = \{([x], v) : [x] \in \mathbb{R}^n, v \in \mathbb{C}^{n+1}\}$$

then we have a bundle γ_1 , defined by

$$\begin{aligned} \gamma_1 : E &\longrightarrow \mathbb{R}\mathbb{P}^n \\ ([x], v) &\longmapsto [x]. \end{aligned}$$

To see this bundle is non-trivial, note that when $a=1$, this is the Möbius bundle, which is non-trivial. Then the pull-back along $\mathbb{R}\mathbb{P}^1 \hookrightarrow \mathbb{R}\mathbb{P}^n$ gives the Möbius bundle, so γ_1 could not have been trivial over $\mathbb{R}\mathbb{P}^n$.

Example: Iso classes of S^1 -bundles over S^2 are given by $[S^2, BS^1] = \pi_2(BS^1) = \pi_1(S^1) = \mathbb{Z}$. The Hopf bundle is a principal S^1 -bundle

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^3 & \longrightarrow & S^2 \\ & & \downarrow & & \downarrow \\ & & S^3 & \longrightarrow & S^2 \\ & & \downarrow & & \downarrow \\ & & S^2 & \xrightarrow{?} & \mathbb{C}P^\infty \\ & & \parallel & & \\ & & \mathbb{C}P^1 & & \end{array}$$

The map $S^2 \cong \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty$ is the inclusion map

$$\text{so } H^2(\mathbb{C}P^\infty) \xrightarrow{\cong} H^2(\mathbb{C}P^1)$$

$$w \longmapsto \omega.$$

$\Rightarrow C_1(H) \neq 0$, & $C_1(H)$ generates $H^2(\mathbb{C}P^1)$ as a cycle group.

